Applying Mellin–Barnes technique and Gröbner bases at three loops and beyond

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- Introduction. Evaluating Feynman integrals
- The method of Mellin–Barnes representation.
- Evaluating Feynman integrals four $N = 4$ SUSY.
- FIRE — an algorithm to solve reduction problems for Feynman integrals.
- Evaluating three-loop correction to the static quark potential.

Introduction

A given Feynman graph $\Gamma \to$ tensor reduction $\to$ various scalar Feynman integrals that have the same structure of the integrand with various distributions of powers of propagators.

$$F_\Gamma(a_1, a_2, \ldots) = \int \ldots \int \frac{\mathrm{d}^d k_1 \mathrm{d}^d k_2 \ldots}{(p_1^2 - m_1^2)^{a_1} (p_2^2 - m_2^2)^{a_2} \ldots}$$

$d = 4 - 2\epsilon$

The propagator as a building block

$$\frac{1}{k^2 - m^2 + i0}\,, \qquad k^2 = k_0^2 - \vec{k}^2$$

Methods to evaluate Feynman integrals: analytical, numerical, semianalytical . . .

A straightforward analytical strategy:

to evaluate, by some methods, every scalar Feynman integral generated by the given graph.

An advanced strategy:

to derive, without calculation, and then apply integration by parts (IBP) identities [K.G. Chetyrkin & F.V. Tkachov'81] between the family of given Feynman integrals as recurrence relations.

A general integral of the given family is expressed as a linear combination of some basic (master) integrals.

The whole problem of evaluation $\rightarrow$

- constructing a reduction procedure
- evaluating master integrals

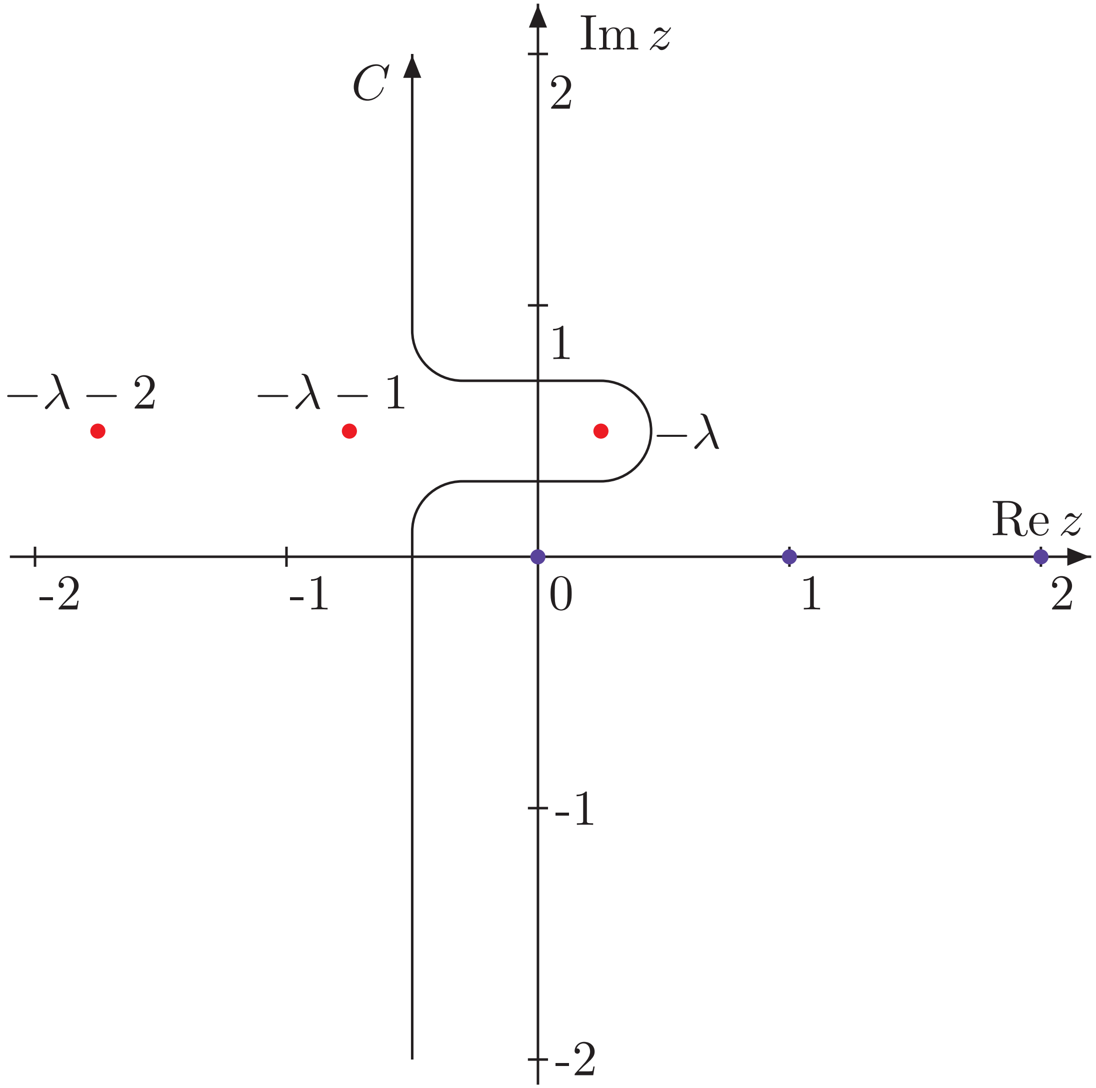
Methods to evaluate master integrals:

- Feynman/alpha parameters
- Mellin–Barnes representation
- method of differential equations

The basic formula:

$$\frac{1}{(X+Y)^\lambda} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \frac{Y^z}{X^{\lambda+z}} \Gamma(\lambda+z)\Gamma(-z) \; .$$

The poles with a $\Gamma(\ldots+z)$ dependence are to the left of the contour and the poles with a $\Gamma(\ldots-z)$ dependence are to the right


Im z
C
2
1
-1
-2
$-\lambda - 2$
$-\lambda - 1$
$-\lambda$
Re z
-2
-1
0
1
2

General recipes

- Derive a (multiple) MB representation for general powers of the propagators. (The number of MB integrations can be large (more than 10)).
  AMBRE — a package to derive MB representations automatically. [J. Gluza, K. Kajda & T. Riemann'07]
  - Use it for checks. Reducing a line to a point $\rightarrow$ tending a_i to zero $\rightarrow$ (usually) taking some residues.
    A typical situation:
    $$\frac{\Gamma(a_2+z)\Gamma(-z)}{\Gamma(a_2)}, \quad a_2 \rightarrow 0$$
    Gluing of poles of different nature. Take a (minus) residue at $z_2 = 0$, then set $a_2 = 0$.
  - Unambiguous prescriptions for choosing integration contours
  - Try to have a minimal number of MB integrations.

- Resolve the singularity structure in ϵ. The goal: to represent a given MB integral as a sum of integrals where a Laurent expansion in ϵ becomes possible.
  The basic procedure:
  take residues and shift contours
  Two strategies:
  - #1 [V.A. Smirnov'99]
    E.g., the product $\Gamma(1+z)\Gamma(-1-\epsilon-z)$ generates a pole of the type $\Gamma(-\epsilon)$.
    The general rule: $\Gamma(a+z)\Gamma(b-z)$, where a and b depend on the rest of the variables, generates a pole of the type $\Gamma(a+b)$. 'Key' gamma functions

- #2 [J.B. Tausk'99, Anastasiou'05, Czakon'05].

  Choose a domain of ϵ and Rez_i,... Rew_i in such a way that *all* the integrations over the MB variables can be performed over straight lines parallel to imaginary axis.
  Let $\epsilon \to 0$. Whenever a pole of some gamma function is crossed, take into account the corresponding residue.
  For every resulting residue, which involves one integration less, apply a similar procedure, etc.

  Two algorithmic descriptions [C. Anastasiou'05, M. Czakon'05]

  `MB` by M. Czakon

Studying cross order relations in $N = 4$ SUSY YM

[C. Anastasiou, L.J. Dixon, Z. Bern & D.A. Kosower'03,04]

[Z. Bern, L.J. Dixon & V.A. Smirnov'05]:

An exponentiation of the planar MHV n-point amplitudes in $N = 4$ SUSY YM at L loops:

$$
\begin{aligned}
\mathcal{M}_n &\equiv 1 + \sum_{L=1}^{\infty} a^L M_n^{(L)}(\epsilon) \\
&= \exp\left[\sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon) \right) \right] .
\end{aligned}
$$

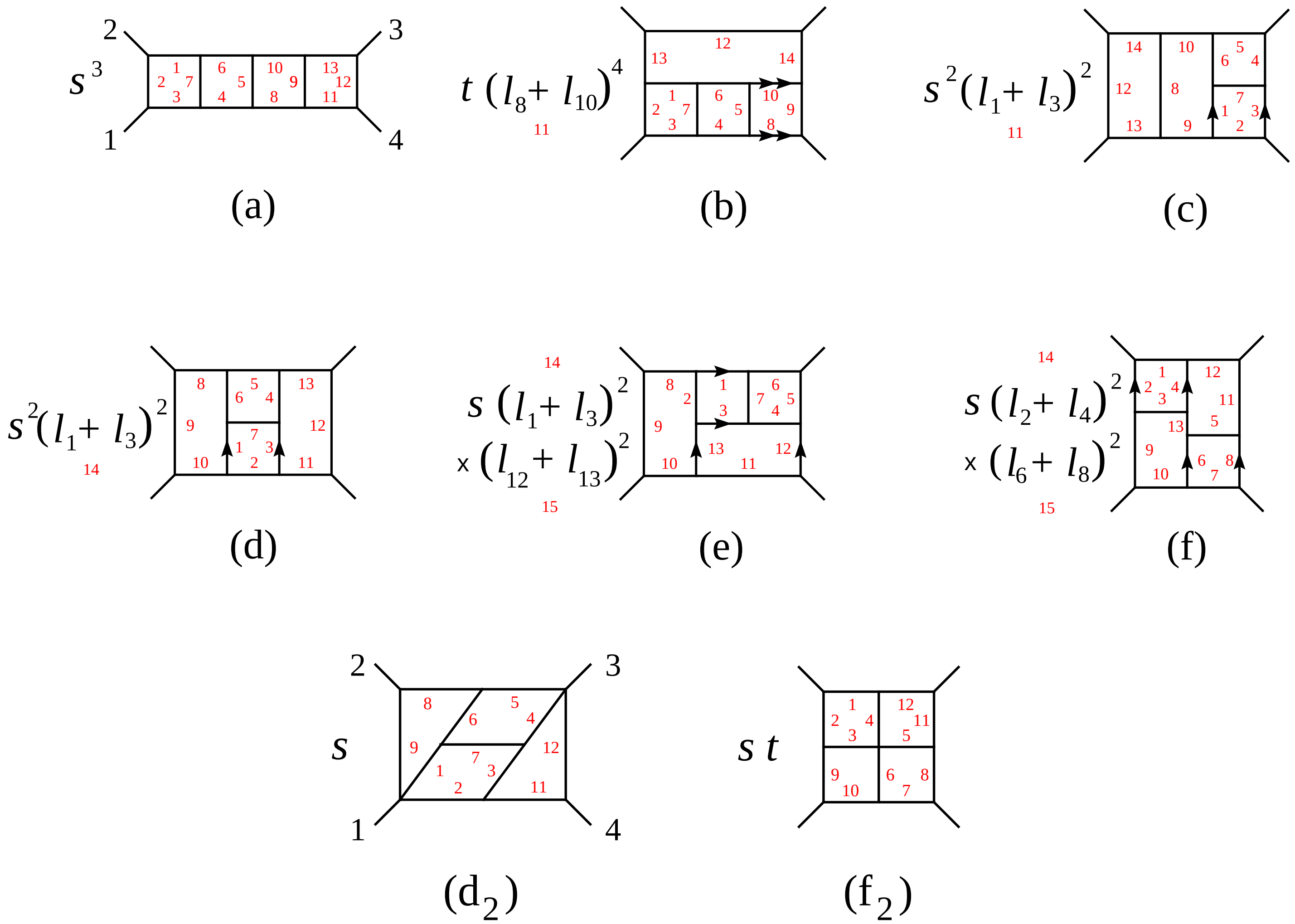
Two loops

[C. Anastasiou, Z. Bern, L.J. Dixon & D.A. Kosower'03; Z. Bern, L.J. Dixon & D.A. Kosower'04]

Three loops

[Z. Bern, L.J. Dixon & V.A. Smirnov'05]

Four loops [Z. Bern, M. Czakon, L. Dixon, D.A. Kosower, & V.A. Smirnov'06]



The poles from $1/\epsilon^8$ to $1/\epsilon^4$ were evaluated analytically, the $1/\epsilon^3$ and $1/\epsilon^2$ numerically.
The $1/\epsilon^2$ part $\to$ the cusp anomalous dimension

$$f_0(\hat{a}) \;=\; \hat{a} - \frac{\pi^2}{6}\,\hat{a}^2 + \frac{11}{180}\pi^4\,\hat{a}^3 - \left(\frac{73}{2520}\,\pi^6 - (1+r)\zeta_3^2\right)\hat{a}^4 + \;\cdots$$

where $\hat{a} \equiv \frac{g^2 N_c}{8\pi^2} = \frac{N_c \alpha_s}{2\pi}$ is the expansion parameter

$r = -2.03 \to r = -2$

$r = -2.00002$ [F. Cachazo, M. Spradlin& A. Volovich'06]

Integrals in five loops

[Z. Bern, J.J.M. Carrasco, H. Johansson & D.A. Kosower'07]

Updating `MB`.
Sometimes it does not work, for unknown reasons, especially when auxiliary analytic parameters are introduced.

Further automation and optimization. For example, `MBmerge`.
Automatic integration, looking for appropriate linear changes of variables.

`EvalMB` [D. Kosower]

The static colour-singlet potential

$$V(|\boldsymbol{q}|) = -\frac{4\pi C_F \alpha_s(|\boldsymbol{q}|)}{\boldsymbol{q}^2}\left[1 + \frac{\alpha_s(|\boldsymbol{q}|)}{4\pi} a_1 + \left(\frac{\alpha_s(|\boldsymbol{q}|)}{4\pi}\right)^2 a_2 \right.$$
$$\left. + \left(\frac{\alpha_s(|\boldsymbol{q}|)}{4\pi}\right)^3 \left(a_3 + 8\pi^2 C_A^3 \ln \frac{\mu^2}{\boldsymbol{q}^2}\right) + \cdots \right]$$

One loop

[W. Fischler '77; A. Billoire '80]

$$a_1 = \frac{31}{9} C_A - \frac{20}{9} T_F n_l$$

Two loops

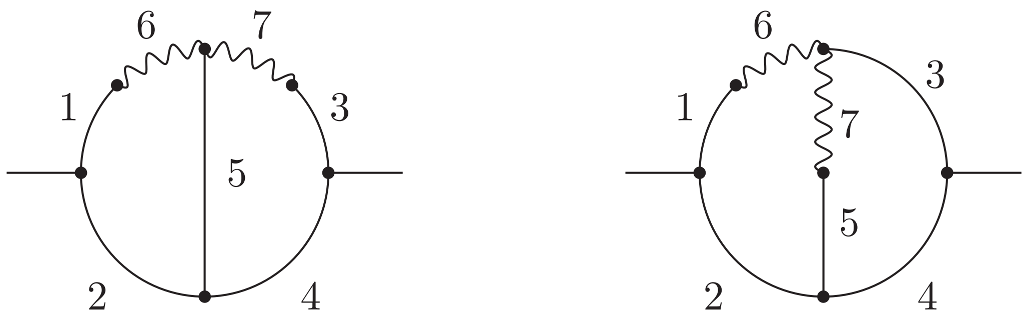
[M. Peter'97; Y. Schröder'99]

$$
\begin{aligned}
a_2 \;\; = & \;\; \left[\frac{4343}{162} + 4\pi^2 - \frac{\pi^4}{4} + \frac{22}{3}\zeta(3)\right] C_A^2 - \left[\frac{1798}{81} + \frac{56}{3}\zeta(3)\right] C_A T_F n_l \\
& - \left[\frac{55}{3} - 16\zeta(3)\right] C_F T_F n_l + \left(\frac{20}{9} T_F n_l\right)^2
\end{aligned}
$$

Three loops: $a_3 = a_3^{(3)} n_l^3 + a_3^{(2)} n_l^2 + a_3^{(1)} n_l + a_3^{(0)} =?$

A.V. Smirnov, V.A. Smirnov, and M. Steinhauser
(work in progress)
$a_3^{(3)}, a_3^{(2)}$ done, $a_3^{(1)}$ pending
Generation of diagrams by `QGRAPH`

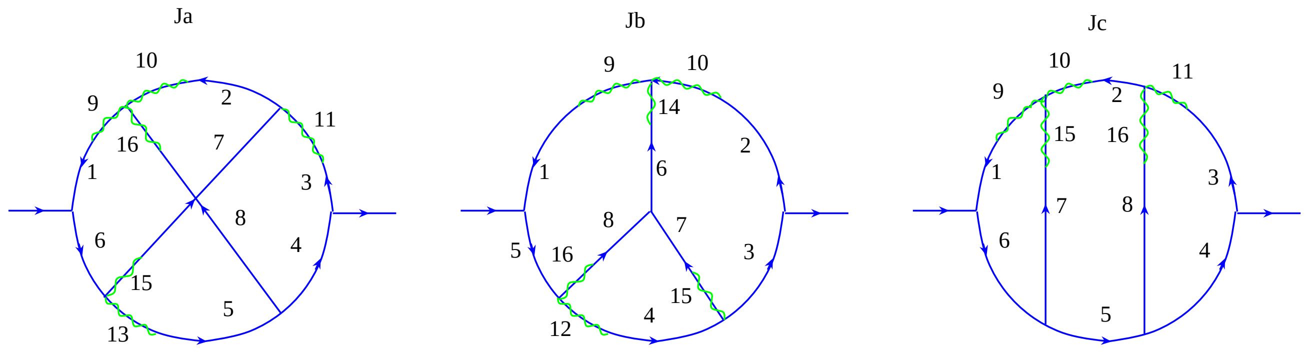
Two loop correction $\rightarrow$



$(\mathbf{A})$ $(\mathbf{B})$

Propagators: $\quad \frac{1}{(-k^2-i0)^{a_i}}\;, \qquad \frac{1}{(-v\cdot k-i0)^{a_i}}$

with $q \cdot v = 0, \quad v = (1, \vec{0})$



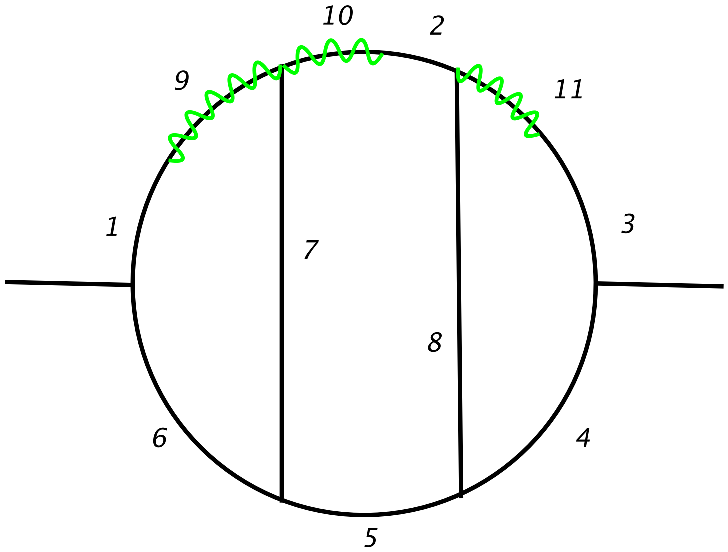
$$J_a(a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8;a_0;a_9,a_{10},a_{11},a_{13},a_{15},a_{16};s_9,s_{10},s_{11},s_{13},s_{15},s_{16})$$
$$= \int\int\int \frac{(-p_0^2)^{-a_0}\mathrm{d}k\,\mathrm{d}l\,\mathrm{d}r}{(-p_1^2)^{a_1}(-p_2^2)^{a_2}(-p_3^2)^{a_3}(-p_4^2)^{a_4}(-p_5^2)^{a_5}(-p_6^2)^{a_6}(-p_7^2)^{a_7}(-p_8^2)^{a_8}}$$
$$\times\frac{1}{(-v\cdot p_1-s_9 i0)^{a_9}(-v\cdot p_2-s_{10}i0)^{a_{10}}(-v\cdot p_3-s_{11}i0)^{a_{11}}(-v\cdot p_5-s_{13}i0)^{a_{13}}(-v\cdot p_7-s_{15}i0)^{a_{15}}(-v\cdot p_8-s_{16}i0)^{a_{16}}}$$
$$p_0=k-r,p_1=k,p_2=l,p_3=r,p_4=q+r,p_5=q+k-l+r,p_6=q+k,p_7=l-r,p_8=k-l$$

$$J_b(a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8;a_0;a_9,a_{10},a_{12},a_{14},a_{15},a_{16};s_9,s_{10},s_{12},s_{14},s_{15},s_{16})$$
$$= \int\int\int \frac{(-p_0^2)^{-a_0}\mathrm{d}k\,\mathrm{d}l\,\mathrm{d}r}{(-p_1^2)^{a_1}(-p_2^2)^{a_2}(-p_3^2)^{a_3}(-p_4^2)^{a_4}(-p_5^2)^{a_5}(-p_6^2)^{a_6}(-p_7^2)^{a_7}(-p_8^2)^{a_8}}$$
$$\times\frac{1}{(-v\cdot p_1-s_9 i0)^{a_9}(-v\cdot p_2-s_{10}i0)^{a_{10}}(-v\cdot p_4-s_{12}i0)^{a_{12}}(-v\cdot p_6-s_{14}i0)^{a_{14}}(-v\cdot p_7-s_{15}i0)^{a_{15}}(-v\cdot p_8-s_{16}i0)^{a_{16}}}$$
$$p_0=l-q,p_1=k-q,p_2=r-q,p_3=r,p_4=l,p_5=k,p_6=k-r,p_7=l-r,p_8=k-l$$

$$J_c(a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8;a_0;a_9,a_{10},a_{11},a_{15},a_{16};s_9,s_{10},s_{11},s_{15},s_{16})$$
$$= \int\int\int \frac{(-p_0^2)^{-a_0}\mathrm{d}k\,\mathrm{d}l\,\mathrm{d}r}{(-p_1^2)^{a_1}(-p_2^2)^{a_2}(-p_3^2)^{a_3}(-p_4^2)^{a_4}(-p_5^2)^{a_5}(-p_6^2)^{a_6}(-p_7^2)^{a_7}(-p_8^2)^{a_8}}$$
$$\times\frac{1}{(-v\cdot p_1-s_9 i0)^{a_9}(-v\cdot p_2-s_{10}i0)^{a_{10}}(-v\cdot p_3-s_{11}i0)^{a_{11}}(-v\cdot p_7-s_{15}i0)^{a_{15}}(-v\cdot p_8-s_{16}i0)^{a_{16}}}$$
$$p_0=k-r,p_1=k,p_2=l,p_3=r,p_4=q+r,p_5=q+l,p_6=q+k,p_7=k-l,p_8=l-r$$

Here $s_i=1$ or -1. The casual -$i0$ is omitted in all the propagators with quadratic momentum dependence, i.e. we have $-p_1^2-i0$, etc.

Reduction to integrals with three static propagators $\rightarrow$ ~ 70000 integrals in the general ξ-gauge with different distributions of static lines.
Ten families of resulting integrals.
For example, $\#350$:



with the numerator chosen as $(-(k-r)^2)^{-a_{12}}$

Evaluating master integrals by MB.
For example,

$$G(1,1,1,1,1,1,1,1,1,0,1,0) = \frac{(i\pi^{d/2})^3}{(q^2)^3 v^2}$$
$$\times \left[\frac{56\pi^4}{135\epsilon} + \frac{112\pi^4}{135} + \frac{16\pi^2\zeta(3)}{9} + \frac{8\zeta(5)}{3} + O(\epsilon)\right]$$

< 100 master integrals in the whole problem

Solving reduction problems algorithmically:

- ‘Laporta’s algorithm’

  [S. Laporta and E. Remiddi’96; S. Laporta’00; T. Gehrmann and E. Remiddi’01]

  One public version `AIR` [C. Anastasiou and A. Lazopoulos’04]

  Private versions

  [T. Gehrmann and E. Remiddi, M. Czakon, Y. Schröder, C. Sturm, P. Marquard and D. Seidel, A. Onishchenko, O. Veretin, . . .]

- Baikov’s method

- Gröbner bases. Suggested by O.V. Tarasov [O.V. Tarasov’98]

  [A.V. Smirnov & V.A. Smirnov’05–07;

  A.G. Grozin, A.V. Smirnov and V.A. Smirnov’06]

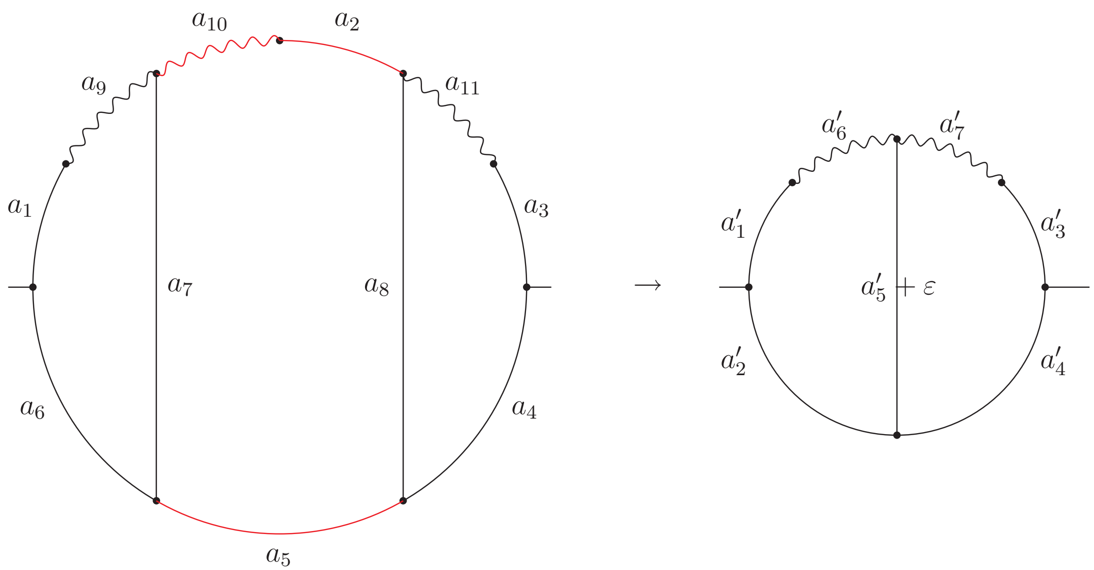
FIRE = Feynman Integrals REduction [A.V. Smirnov]
(implemented in `Mathematica`)
Sectors
2^n regions labelled by subsets $\nu \subseteq \{1, \ldots, n\}$:
σ_ν = $\{(a_1, \ldots, a_n) : a_i > 0$ if $i \in \nu\,, \;\; a_i \leq 0$ if $i \notin \nu\}$
Natural ordering.

The goal of reduction: to make more non-positive indices.
Three different strategies in FIRE.

1. In sectors with a small number of non-positive indices, apply s-bases (generalizations of Gröbner bases).
Constructing them automatically by a kind of Buchberger algorithm.

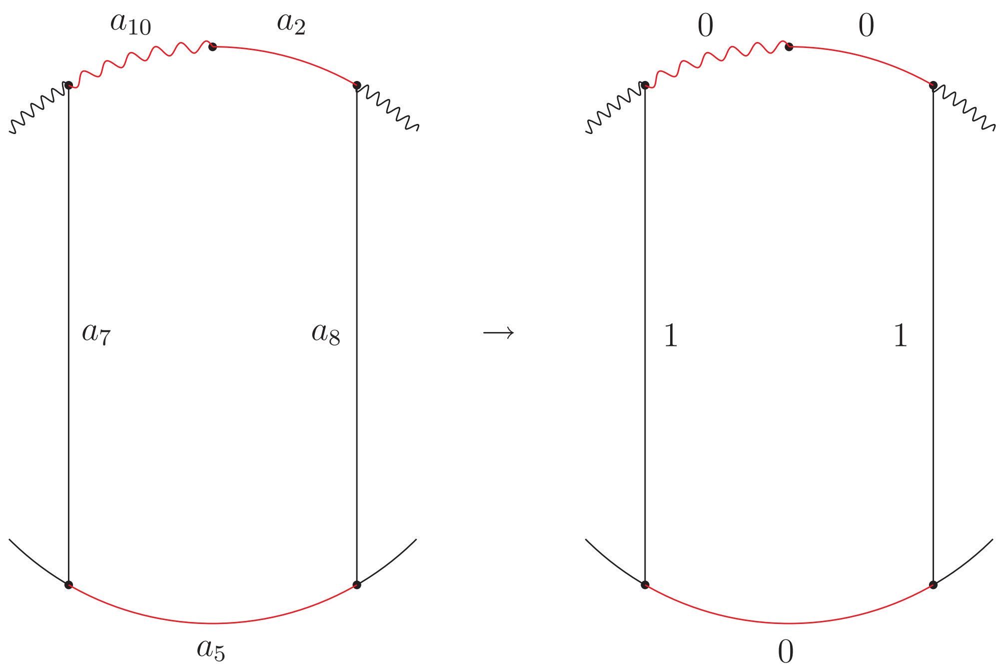
2. In sectors with a large number of non-positive indices, integrate over a loop momentum explicitly and reduce the problem to a family of two-loop integrals where the index of one propagator is, possibly, shifted by ϵ or 2ϵ.

Consider the region $a_2, a_5, a_{10} \le 0\ , \quad a_7, a_8 > 0$



Apply s-bases (within FIRE) to such 2loop reduction problems with 7 indices.

Reduce indices $a_2, a_5, a_{10}, a_7, a_8$ to their boundary values, i.e. $a_2, a_5, a_{10} = 0, \quad a_7, a_8 = 1$



At these values, the transition to the 2loop problem because very simple (without multiple summations).

3. In ‘intermediate sectors’, the Laporta’s algorithm (implemented within FIRE) is applied.

```
In[37]:= F[350, {1, 1, 1, 1, 1, 1, 1, 1, 1, -4, 1, 0}]

        Total time: 0.×10^-8 seconds

Out[37]= -((-13 + 4 d) (717 - 1380 d + 812 d^2 - 192 d^3 + 16 d^4)
             vv^2 G[350, {1, -1, 1, 0, 1, 0, 1, 1, 1, 0, 1, 0}]) /
           (4 (-4 + d) (-2 + d) (-1 + d) (-9 + 2 d) (-7 + 2 d) QQ^2) -
         ((-7 + 2 d) (6561 - 8329 d + 3832 d^2 - 764 d^3 + 56 d^4) vv^2
            G[350, {1, 0, 1, 0, 1, 0, 1, 1, 1, 0, 1, 0}]) /
          (4 (-4 + d) (-3 + d) (-2 + d) (-1 + d) (-9 + 2 d) QQ) -
         ((-5 + d) (-37782 + 92949 d - 101781 d^2 + 62834 d^3 - 23292 d^4 + 5144 d^5 - 624 d^6 + 32 d^7)
            vv G[350, {1, 1, 0, 1, 0, 1, 1, 1, -1, 0, 1, 0}]) /
          (3 (-3 + d) (-2 + d) (-1 + d)^2 (-9 + 2 d) (-7 + 2 d)^2 QQ) +
         ((-9 + 2 d) (-7 + 2 d) QQ^2 vv^2 G[350, {1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 1, 0}]) /
          (16 (-3 + d) (-1 + d)) -
         ((-76 + 104 d - 41 d^2 + 5 d^3) Gfunction2 vv^2 G[420, {0, 0, 1, 1, 1, 2, 0}]) /
          ((-4 + d) (-3 + d) (-1 + d)^2 QQ) +
         (6 (-11 + 3 d) (-10 + 3 d) (-53 + 57 d - 19 d^2 + 2 d^3) Gfunction1 vv^2
            G[420, {0, 1, 1, 0, 1, 0, 1}]) / ((-4 + d) (-2 + d) (-1 + d) (-9 + 2 d) (-7 + 2 d) QQ) -
         (4 (-2 + d) (-5 - 2 d + d^2) Gfunction2 vv G[420, {1, 1, 1, 1, 0, 0, 0}]) /
          ((-4 + d) (-1 + d)^3 QQ) -
         ((5 - 5 d + d^2) Gfunction1 vv^2 G[420, {1, 1, 1, 1, 0, 0, 1}]) / ((-2 + d) (-1 + d)) -
         ((-15 + 4 d) (-44670 + 67908 d - 40361 d^2 + 11776 d^3 - 1692 d^4 + 96 d^5) Gfunction2 vv^2
            G[440, {0, 1, 1, 0, 1, 1, 1}]) / (2 (-4 + d) (-2 + d) (-1 + d) (-9 + 2 d)^2 (-7 + 2 d) QQ) -
         ((155520 - 225318 d + 241812 d^2 - 232553 d^3 + 148996 d^4 - 56923 d^5 + 12542 d^6 - 1476 d^7 + 72 d^8)
            Gfunction2 vv G[440, {1, 1, 1, 1, 1, 0, 0}]) /
          (4 (-5 + d) (-2 + d) (-1 + d)^2 d (-9 + 2 d) (-7 + 2 d)) -
         ((-49766400 + 104930880 d - 117068888 d^2 + 112909728 d^3 - 98678530 d^4 + 65472598 d^5 -
               30090049 d^6 + 9350518 d^7 - 1930379 d^8 + 254086 d^9 - 19332 d^10 + 648 d^11)
            Gfunction2 QQ vv^2 G[440, {1, 1, 1, 1, 1, 0, 2}]) /
          (16 (-5 + d) (-4 + d) (-3 + d) (-2 + d) (-1 + d)^2 d (-9 + 2 d) (-7 + 2 d) (-10 + 3 d) (-8 + 3 d)) +
         ((-11 + 3 d) (19 - 13 d + 2 d^2) Gfunction2 QQ vv^2 G[440, {1, 1, 1, 1, 1, 1, 1}]) /
          (4 (-4 + d) (-2 + d) (-1 + d)) -
         ((7505867980800 - 22016263812480 d + 25326867194640 d^2 - 11825087159232 d^3 -
               1470559502076 d^4 + 3199467138972 d^5 + 985029320158 d^6 - 3233584816247 d^7 +
               2566794162679 d^8 - 1208872787569 d^9 + 388923007523 d^10 - 89318966836 d^11 +
               14749375348 d^12 - 1719639440 d^13 + 134826736 d^14 - 6393792 d^15 + 138816 d^16)
            Gfunction2 vv G[460, {1, 0, 0, 1, 1, 0, 0}]) / (9 (-5 + d) (-4 + d)^3 (-3 + d)
            (-2 + d) (-1 + d)^2 d (-9 + 2 d)^2 (-7 + 2 d)^2 (-14 + 3 d) (-8 + 3 d) QQ^2) +
         (2 (606311136 - 2762926020 d + 5492115846 d^2 - 6345136554 d^3 + 4764530284 d^4 - 2453286651 d^5 +
               887384218 d^6 - 226325153 d^7 + 40064194 d^8 - 4721708 d^9 + 339192 d^10 - 12080 d^11 + 96 d^12)
            Gfunction2 vv G[460, {1, 1, 1, 0, 1, 0, 0}]) /
          (9 (-4 + d)^2 (-3 + d) (-2 + d) (-1 + d)^2 (-9 + 2 d) (-7 + 2 d)^2 (-10 + 3 d) (-8 + 3 d) QQ) +
         ((-29357280 + 151369812 d - 354726930 d^2 + 485963836 d^3 - 428920128 d^4 + 255982293 d^5 -
               105658449 d^6 + 30255753 d^7 - 5908509 d^8 + 751490 d^9 - 56160 d^10 + 1872 d^11)
            Gfunction2 vv G[460, {1, 1, 1, 1, 0, 0, 0}]) /
          (3 (-4 + d)^2 (-3 + d) (-2 + d) (-1 + d)^3 (-9 + 2 d) (-7 + 2 d)^2 (-8 + 3 d) QQ) +
         ((-2690072985600 + 7630467347520 d - 9683092802232 d^2 + 8021206341564 d^3 -
               6158979260610 d^4 + 5202781287593 d^5 - 3996006291042 d^6 + 2359320979898 d^7 -
               1024980166906 d^8 + 326929794923 d^9 - 76568837728 d^10 + 13044141222 d^11 - 1575877234 d^12 +
               128169024 d^13 - 6302088 d^14 + 141696 d^15) Gfunction3 vv^3 G[480, {0, 0, 1, 1, 1, 2, 0}]) /
          (36 (-4 + d)^2 (-3 + d) (-2 + d) (-1 + d)^2 d (-9 + 2 d)^2 (-7 + 2 d)^2
```

```
    (-14 + 3 d) (-10 + 3 d) (-8 + 3 d) QQ) +
((-3 + d) (781 - 1127 d + 545 d^2 - 110 d^3 + 8 d^4) Gfunction3 vv^3 G[480, {0, 1, 1, 0, 1, 1, 1}]) /
 (4 (-4 + d) (-2 + d) (-1 + d) (-9 + 2 d) (-7 + 2 d) QQ) +
((-5 + d) (-1829452608 + 6886763352 d - 11293400484 d^2 + 10701728298 d^3 -
      6523409351 d^4 + 2683727318 d^5 - 757436875 d^6 + 145418540 d^7 - 18294934 d^8 +
      1383296 d^9 - 50936 d^10 + 384 d^11) Gfunction3 vv^3 G[480, {0, 1, 1, 0, 1, 2, 0}]) /
 (36 (-4 + d)^2 (-2 + d) (-1 + d)^2 (-9 + 2 d) (-7 + 2 d)^2 (-10 + 3 d) (-8 + 3 d) (-18 + 5 d) QQ) +
((-3 + d)^2 (-11 + 3 d) Gfunction3 vv^3 G[480, {1, 1, 0, 0, 1, 1, 1}]) /
 (2 (-1 + d) (-9 + 2 d) (-7 + 2 d) QQ) +
((10448 - 10608 d + 3704 d^2 - 524 d^3 + 25 d^4) Gfunction1 Gfunction3 vv^3
   G[480, {1, 1, 1, 1, 0, 1, 0}]) / (32 (-6 + d)^2 (-2 + d) (-1 + d) Gfunction2) +
((296 - 1572 d + 1098 d^2 - 269 d^3 + 22 d^4) Gfunction3 vv^3 G[480, {1, 1, 1, 1, 0, 1, 1}]) /
 (32 (-6 + d)^2 (-1 + d) (-7 + 2 d)) -
((-5 + d) (-3 + d)^2 Gfunction3 vv^3 G[480, {1, 1, 1, 1, 0, 2, 0}]) / (4 (-1 + d) (-7 + 2 d)^2) -
((-4 + d) (-19 + 6 d) Gfunction3 QQ vv^3 G[480, {1, 1, 1, 1, 1, 1, 1}]) / (8 (-1 + d) (-7 + 2 d))
```

The status: almost all integrals contributing to n_l part were reduced.
Most master integrals were calculated.

Time needed for the reduction of most complicated integrals by FIRE $\sim$ 1-2 days.
Operative memory: several Gb.
s-bases $\sim$ 1-10 Mb each

Creating tables of results (which are not big)

Priorities: to complete the n_l part, then n_l^0 part.
Various checks, e.g. ξ-independence.
Numerical checks by Czakon's MB.m. and by sector decompositions

[T. Binoth & G. Heinrich'00,04; C. Bogner & S. Weinzierl'07]