Perturbation theory of computing QCD jet cross sections beyond NLO accuracy

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Outline

1. **Introduction**
   - Standard motivation
   - Less standard motivation
2. **pQCD computation of jet cross sections**
   - Perturbative expansion
   - NLO correction
3. **Extension to NNLO**
   - Structure of NNLO subtraction
   - Naive generalization of NLO subtraction fails
4. **NLO subtraction revisited**
   - Separation of collinear and purely-soft subtractions
   - NLO subtraction with new phase-space mappings
   - NLO subtraction with fixed helicities
5. **Summary**
6. **Extra slides**

Standard motivation

Precision QCD sometimes requires computations beyond NLO

- to reduce the dependence on unphysical renormalization and factorization scales
- if the NLO corrections are large (can be more than 100 %), such as Higgs production in hadron collisions
- the main source of uncertainty in experimental results is due to theory, such as α_s measurements
- the NLO computation is effectively LO, such as energy distribution inside jet cones
- reliable error estimate is needed, such as, precise measurement of parton luminosity, but rather always
- ...

Less standard motivation

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- fast development in computations of loop amplitudes raises the hope of accessing NLO corrections for multileg processes

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  . . . can we compute real radiation fast enough?
- current approaches to fixed-order and parton shower computations have mutually exclusive elements, which may hamper their combination
  . . . understanding NNLO helps further development of parton showers

The perturbative expansion at NNLO accuracy

$$\sigma = \sigma^{\mathrm{LO}} + \sigma^{\mathrm{NLO}} + \sigma^{\mathrm{NNLO}} + \ldots$$

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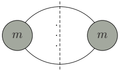
- Consider $e^+e^- \to m$ jet production
  - LO



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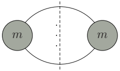

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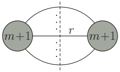

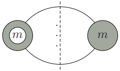

$$\sigma^{\mathrm{NLO}} = \int_{m+1} \mathrm{d}\sigma^{\mathrm{R}}_{m+1} + \int_m \mathrm{d}\sigma^{\mathrm{V}}_m$$
$$= \int \mathrm{d}\phi_{m+1} |\mathcal{M}^{(0)}_{m+1}|^2 J_{m+1} + \int \mathrm{d}\phi_m 2\mathrm{Re}\langle \mathcal{M}^{(1)}_m | \mathcal{M}^{(0)}_m \rangle J_m$$

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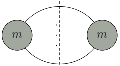

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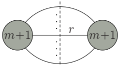

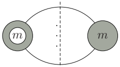

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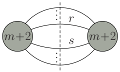

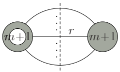

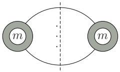

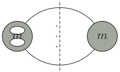

$$\sigma^{\mathrm{NNLO}} = \int_{m+2} \mathrm{d}\sigma_{m+2}^{\mathrm{RR}} + \int_{m+1} \mathrm{d}\sigma_{m+1}^{\mathrm{RV}} + \int_m \mathrm{d}\sigma_m^{\mathrm{VV}} =$$
$$= \int \mathrm{d}\phi_{m+2} |\mathcal{M}_{m+2}^{(0)}|^2 J_{m+2} + \int \mathrm{d}\phi_{m+1} 2\mathrm{Re}\langle \mathcal{M}_{m+1}^{(1)} | \mathcal{M}_{m+1}^{(0)} \rangle J_{m+1} +$$
$$+ \int \mathrm{d}\phi_m \left[|\mathcal{M}_m^{(1)}|^2 + 2\mathrm{Re}\langle \mathcal{M}_m^{(2)} | \mathcal{M}_m^{(0)} \rangle \right] J_m$$

The perturbative expansion at NNLO accuracy

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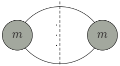
- Consider $e^+e^- \to m$ jet production
  - LO



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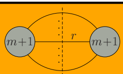

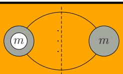

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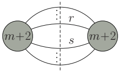
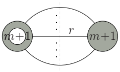
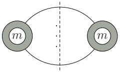
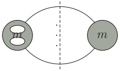
  - NNLO









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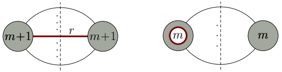


$$\underbrace{\int_1 \mathrm{d}\phi(k) \qquad\qquad \int \mathrm{d}^d k}_{\text{divergent in } d = 4!}$$

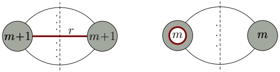
Process independent methods (phase space slicing, residuum, dipole or antennae subtraction) use

- regularized integrals in $d = 4 - 2\epsilon$ dimensions
- universal soft- and collinear factorization of QCD (squared) matrix elements
  - $\mathbf{C}_{ir}$ is a symbolic operator that takes the collinear limit

$$\mathbf{C}_{ir}|\mathcal{M}^{(0)}_{m+1}(p_i, p_r, \ldots)|^2 \propto \frac{1}{s_{ir}} \langle \mathcal{M}^{(0)}_m(p_{ir}, \ldots)|\hat{P}^{(0)}_{ir}|\mathcal{M}^{(0)}_m(p_{ir}, \ldots)\rangle$$

  - $\mathbf{S}_r$ is a symbolic operator that takes the soft limit

$$\mathbf{S}_r|\mathcal{M}^{(0)}_{m+1}(p_r, \ldots)|^2 \propto \sum_{\substack{i,k \\ i \neq k}} \frac{s_{ik}}{s_{ir} s_{kr}} \langle \mathcal{M}^{(0)}_m(\ldots)|\boldsymbol{T}_i \boldsymbol{T}_k|\mathcal{M}^{(0)}_m(\ldots)\rangle$$



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divergent in $d = 4$!

Process independent methods (phase space slicing, residuum, dipole or antennae subtraction) use

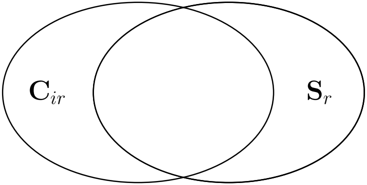
- regularized integrals in $d = 4 - 2\epsilon$ dimensions
- universal soft- and collinear factorization of QCD (squared) matrix elements
- to construct approximate cross section to regularize real emissions:

$$\sigma^{\mathrm{NLO}} = \int_{m+1} \left[\left(\mathrm{d}\sigma^{\mathrm{R}}\right)_{\varepsilon=0} - \left(\mathrm{d}\sigma^{\mathrm{A}}\right)_{\varepsilon=0} \right] + \int_m \left[\mathrm{d}\sigma^{\mathrm{V}} + \int_1 \mathrm{d}\sigma^{\mathrm{A}} \right]_{\varepsilon=0}$$

$$\equiv \int_{m+1} \mathrm{d}\sigma^{\mathrm{NLO}}_{m+1} + \int_m \mathrm{d}\sigma^{\mathrm{NLO}}_m$$

Construction of the subtraction terms at NLO

- The collinear and soft regions overlap:



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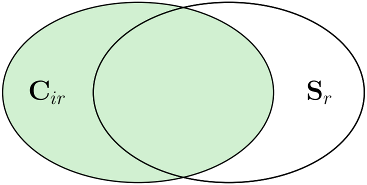

- The candidate subtraction term...

$$\mathbf{A}_1 |\mathcal{M}_{m+1}^{(0)}|^2 \stackrel{?}{=} \sum_r \left[\sum_{i \neq r} \frac{1}{2} \mathbf{C}_{ir} \qquad\qquad \right] |\mathcal{M}_{m+1}^{(0)}|^2$$

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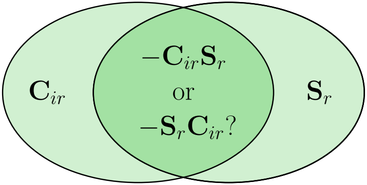

- The candidate subtraction term…

$$\mathbf{A}_1 |\mathcal{M}^{(0)}_{m+1}|^2 \stackrel{?}{=} \sum_r \left[\sum_{i \neq r} \frac{1}{2} \mathbf{C}_{ir} + \mathbf{S}_r \qquad\qquad \right] |\mathcal{M}^{(0)}_{m+1}|^2$$

  - …has the correct singularity structure but performs double subtraction in the regions of phase space where the limits overlap

Construction of the subtraction terms at NLO

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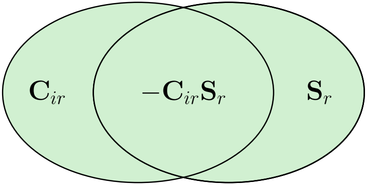

- The candidate subtraction term...

$$\mathbf{A}_1|\mathcal{M}_{m+1}^{(0)}|^2 = \sum_r \left[\sum_{i\neq r}\frac{1}{2}\mathbf{C}_{ir} + \left(\mathbf{S}_r - \sum_{i\neq r}\mathbf{C}_{ir}\mathbf{S}_r\right)\right]|\mathcal{M}_{m+1}^{(0)}|^2$$

  - ...is now free of double subtractions
  - ...but only defined in the strict collinear and/or soft limits

Construction of the subtraction terms at NLO

- Extension over the full phase space requires momentum mappings,

$$\{p\}_{m+1} \to \{\tilde{p}\}_m$$

leading to phase space factorization, such that

$$\int_1 \mathrm{d}\sigma^{\mathrm{A}} = \mathrm{d}\sigma^{\mathrm{B}}_m \otimes \boldsymbol{I}(\epsilon)$$

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- The pole part of the insertion operator is universal

$$\boldsymbol{I}(\epsilon) = \frac{\alpha_{\mathrm{s}}}{2\pi} \sum_i \left[\frac{1}{\epsilon}\gamma_i - \frac{1}{\epsilon^2} \sum_{k \neq i} \boldsymbol{T}_i \cdot \boldsymbol{T}_k \left(\frac{4\pi\mu^2}{s_{ik}} \right)^{\epsilon} \right] + \mathrm{O}(\varepsilon^0)$$

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- and equals that of the virtual correction (up to a sign)

$$\mathrm{d}\sigma_m^{\mathrm{NLO}} = \left[\mathrm{d}\sigma_m^{\mathrm{V}} + \int_1 \mathrm{d}\sigma^{\mathrm{A}} \right]_{\varepsilon=0} = \mathrm{O}(\varepsilon^0)$$

Three terms contribute at NNLO accuracy

- $\sigma^{\mathrm{NNLO}} = \sigma^{\mathrm{RR}}_{m+2} + \sigma^{\mathrm{RV}}_{m+1} + \sigma^{\mathrm{VV}}_{m} =$

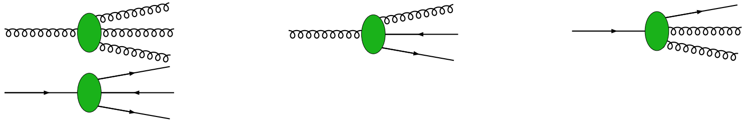
$$= \int_{m+2} \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} + \int_{m+1} \mathrm{d}\sigma^{\mathrm{RV}}_{m+1} J_{m+1} + \int_{m} \mathrm{d}\sigma^{\mathrm{VV}}_{m} J_{m}$$

$$\underbrace{\int_2 \mathrm{d}\phi(k_1, k_2) \qquad \int \mathrm{d}^d k_1 \int_1 \mathrm{d}\phi(k_2) \qquad \int \mathrm{d}^d k_1 \mathrm{d}^d k_2}_{\text{separately divergent in } d=4!}$$

- Attempt to apply the same strategy as at NLO: regularize in $d = 4 - 2\epsilon$ dimensions and use universal IR structure to subtract divergences

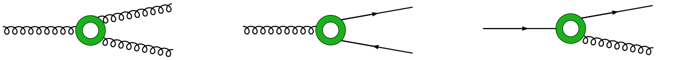
Thanks to many people, the universal IR structure at NNLO is well-known:

- Tree level 3-parton splitting functions and double soft gg and $q\bar{q}$ currents



J. M. Campbell, E. W. N. Glover 1997, S. Catani, M. Grazzini 1998
V. Del Duca, A. Frizzo, F. Maltoni, 1999, D. Kosower, 2002

- One-loop 2-parton splitting fucntions and soft gluon current



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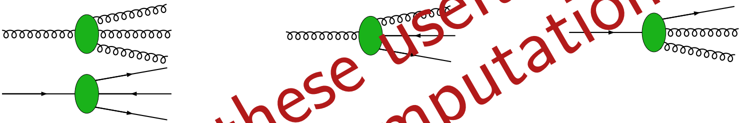

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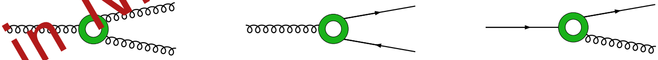

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Are these useful
in NNLO computations?

Successful computations use different strategy

- Antennae subtraction uses complete squared matrix elements instead of IR structure

See talk by T. Gehrmann

- For processes involving massive particles and/or simple kinematics, direct numerical evaluation of the coefficients in the Laurent expansion of the three contributions (based on sector decomposition) has been more successful

C. Anastasiou, K. Melnikov, F. Petriello 2004–2007

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C. Anastasiou, K. Melnikov, F. Petriello 2004–2007

- Existing NLO subtraction schemes cannot naively be extended to NNLO

G. Somogyi, V. Del Duca, Z.T. 2006–2007

Structure of subtraction is governed by the jet function

$$\sigma^{\mathrm{NNLO}} = \sigma^{\mathrm{NNLO}}_{m+2} + \sigma^{\mathrm{NNLO}}_{m+1} + \sigma^{\mathrm{NNLO}}_{m} =$$

$$= \int_{m+2} \left\{ \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} J_m - \left(\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} J_m \right) \right\}$$

$$+ \int_{m+1} \left\{ \left(\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right) J_{m+1} - \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] J_m \right\}$$

$$+ \int_{m} \left\{ \mathrm{d}\sigma^{\mathrm{VV}}_{m} + \int_2 \left(\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} \right) + \int_1 \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] \right\} J_m$$

Structure of subtraction is governed by the jet function

$$\sigma^{\rm NNLO} = \sigma^{\rm NNLO}_{m+2} + \sigma^{\rm NNLO}_{m+1} + \sigma^{\rm NNLO}_{m} =$$

$$= \int_{m+2} \left\{ {\rm d}\sigma^{\rm RR}_{m+2} J_{m+2} \boxed{- {\rm d}\sigma^{\rm RR,A_2}_{m+2} J_m} - \left({\rm d}\sigma^{\rm RR,A_1}_{m+2} J_{m+1} - {\rm d}\sigma^{\rm RR,A_{12}}_{m+2} J_m \right) \right\}$$

$$+ \int_{m+1} \left\{ \left({\rm d}\sigma^{\rm RV}_{m+1} + \int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right) J_{m+1} - \left[{\rm d}\sigma^{\rm RV,A_1}_{m+1} + \left(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right)^{\rm A_1} \right] J_m \right\}$$

$$+ \int_m \left\{ {\rm d}\sigma^{\rm VV}_m \boxed{+ \int_2 \left({\rm d}\sigma^{\rm RR,A_2}_{m+2}\right.} - {\rm d}\sigma^{\rm RR,A_{12}}_{m+2} \left.\right) + \int_1 \left[{\rm d}\sigma^{\rm RV,A_1}_{m+1} + \left(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right)^{\rm A_1} \right] \right\} \boxed{J_m}$$

- The approximate cross section ${\rm d}\sigma^{\rm RR,A_2}_{m+2}$ regularizes the doubly-unresolved limits of ${\rm d}\sigma^{\rm RR}_{m+2}$

Structure of subtraction is governed by the jet function

$$\sigma^{\mathrm{NNLO}} = \sigma^{\mathrm{NNLO}}_{m+2} + \sigma^{\mathrm{NNLO}}_{m+1} + \sigma^{\mathrm{NNLO}}_{m} =$$

$$= \int_{m+2} \left\{ \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A}_2}_{m+2} J_m \boxed{- \left(\mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} J_{m+1} \right.} - \left. \mathrm{d}\sigma^{\mathrm{RR,A}_{12}}_{m+2} J_m \right) \right\}$$

$$+ \int_{m+1} \left\{ \left(\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} \boxed{+ \int_1 \mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} \right) J_{m+1}} - \left[\mathrm{d}\sigma^{\mathrm{RV,A}_1}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} \right)^{\mathrm{A}_1} \right] J_m \right\}$$

$$+ \int_m \left\{ \mathrm{d}\sigma^{\mathrm{VV}}_m + \int_2 \left(\mathrm{d}\sigma^{\mathrm{RR,A}_2}_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A}_{12}}_{m+2} \right) + \int_1 \left[\mathrm{d}\sigma^{\mathrm{RV,A}_1}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} \right)^{\mathrm{A}_1} \right] \right\} J_m$$

- The approximate cross section $\mathrm{d}\sigma^{\mathrm{RR,A}_2}_{m+2}$ regularizes the doubly-unresolved limits of $\mathrm{d}\sigma^{\mathrm{RR}}_{m+2}$
- The approximate cross section $\mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2}$ regularizes the singly-unresolved limits of $\mathrm{d}\sigma^{\mathrm{RR}}_{m+2}$

Structure of subtraction is governed by the jet function

$$\sigma^{\mathrm{NNLO}} = \sigma^{\mathrm{NNLO}}_{m+2} + \sigma^{\mathrm{NNLO}}_{m+1} + \sigma^{\mathrm{NNLO}}_{m} =$$

$$= \int_{m+2} \left\{ \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} J_m \boxed{-} \left(\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} J_{m+1} \boxed{- \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} J_m} \right) \right\}$$

$$+ \int_{m+1} \left\{ \left(\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right) J_{m+1} - \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] J_m \right\}$$

$$+ \int_m \left\{ \mathrm{d}\sigma^{\mathrm{VV}}_m + \int_2 \left(\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} \boxed{- \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2}} \right) + \int_1 \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] \right\} \boxed{J_m}$$

- The approximate cross section $\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2}$ regularizes the doubly-unresolved limits of $\mathrm{d}\sigma^{\mathrm{RR}}_{m+2}$
- The approximate cross section $\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2}$ regularizes the singly-unresolved limits of $\mathrm{d}\sigma^{\mathrm{RR}}_{m+2}$
- The approximate cross section $\mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2}$ regularizes the singly-unresolved limits of $\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2}$ **and** the doubly-unresolved limits of $\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2}$

Structure of subtraction is governed by the jet function

$$\sigma^{\mathrm{NNLO}} = \sigma^{\mathrm{NNLO}}_{m+2} + \sigma^{\mathrm{NNLO}}_{m+1} + \sigma^{\mathrm{NNLO}}_{m} =$$

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$$+ \int_{m+1} \left\{ \left(\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} \right) J_{m+1} \boxed{- \left[\mathrm{d}\sigma^{\mathrm{RV,A}_1}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} \right)^{\mathrm{A}_1} \right] J_m} \right\}$$

$$+ \int_m \left\{ \mathrm{d}\sigma^{\mathrm{VV}}_m + \int_2 \left(\mathrm{d}\sigma^{\mathrm{RR,A}_2}_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A}_{12}}_{m+2} \right) \boxed{+ \int_1 \left[\mathrm{d}\sigma^{\mathrm{RV,A}_1}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} \right)^{\mathrm{A}_1} \right] \right\} J_m}$$

- The approximate cross section $\mathrm{d}\sigma^{\mathrm{RR,A}_2}_{m+2}$ regularizes the doubly-unresolved limits of $\mathrm{d}\sigma^{\mathrm{RR}}_{m+2}$
- The approximate cross section $\mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2}$ regularizes the singly-unresolved limits of $\mathrm{d}\sigma^{\mathrm{RR}}_{m+2}$
- The approximate cross section $\mathrm{d}\sigma^{\mathrm{RR,A}_{12}}_{m+2}$ regularizes the singly-unresolved limits of $\mathrm{d}\sigma^{\mathrm{RR,A}_2}_{m+2}$ **and** the doubly-unresolved limits of $\mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2}$
- The approximate cross sections $\mathrm{d}\sigma^{\mathrm{RV,A}_1}_{m+1}$ and $\left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} \right)^{\mathrm{A}_1}$ regularize the singly-unresolved limits of $\mathrm{d}\sigma^{\mathrm{RV}}_{m+1}$ and $\int_1 \mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2}$

Structure of subtraction is governed by the jet function

$$\sigma^{\rm NNLO} = \sigma^{\rm NNLO}_{m+2} + \sigma^{\rm NNLO}_{m+1} + \sigma^{\rm NNLO}_{m} =$$

$$= \int_{m+2}\left\{{\rm d}\sigma^{\rm RR}_{m+2}J_{m+2} - {\rm d}\sigma^{\rm RR,A_2}_{m+2}J_m - \left({\rm d}\sigma^{\rm RR,A_1}_{m+2}J_{m+1} - {\rm d}\sigma^{\rm RR,A_{12}}_{m+2}J_m\right)\right\}$$

$$+\boxed{\int_{m+1}\left\{\left({\rm d}\sigma^{\rm RV}_{m+1} + \int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2}\right)J_{m+1} - \left[{\rm d}\sigma^{\rm RV,A_1}_{m+1} + \left(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2}\right)^{\rm A_1}\right]J_m\right\}}$$

$$+\int_m\left\{{\rm d}\sigma^{\rm VV}_m + \int_2\left({\rm d}\sigma^{\rm RR,A_2}_{m+2} - {\rm d}\sigma^{\rm RR,A_{12}}_{m+2}\right) + \int_1\left[{\rm d}\sigma^{\rm RV,A_1}_{m+1} + \left(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2}\right)^{\rm A_1}\right]\right\}J_m$$

- The approximate cross section ${\rm d}\sigma^{\rm RR,A_2}_{m+2}$ regularizes the doubly-unresolved limits of ${\rm d}\sigma^{\rm RR}_{m+2}$
- The approximate cross section ${\rm d}\sigma^{\rm RR,A_1}_{m+2}$ regularizes the singly-unresolved limits of ${\rm d}\sigma^{\rm RR}_{m+2}$
- The approximate cross section ${\rm d}\sigma^{\rm RR,A_{12}}_{m+2}$ regularizes the singly-unresolved limits of ${\rm d}\sigma^{\rm RR,A_2}_{m+2}$ **and** the doubly-unresolved limits of ${\rm d}\sigma^{\rm RR,A_1}_{m+2}$
- The approximate cross sections ${\rm d}\sigma^{\rm RV,A_1}_{m+1}$ and $\left(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2}\right)^{\rm A_1}$ regularize the singly-unresolved limits of ${\rm d}\sigma^{\rm RV}_{m+1}$ and $\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2}$

The $m+1$-parton contribution:

$$\int_{m+1} \left\{ \left(\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} \right) J_{m+1} - \left[\mathrm{d}\sigma^{\mathrm{RV,A}_1}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} \right)^{\mathrm{A}_1} \right] J_m \right\}$$

- Construction of approximate cross section $\mathrm{d}\sigma^{\mathrm{RV,A}_1}_{m+1}$ that regularizes the kinematical singularities of $\mathrm{d}\sigma^{\mathrm{RV}}_{m+1}$ in the singly-unresolved regions is straightforward

The $m+1$-parton contribution:

$$\int_{m+1}\left\{\left(\mathrm{d}\sigma^{\mathrm{RV}}_{m+1}+\int_1\mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2}\right)J_{m+1}-\left[\mathrm{d}\sigma^{\mathrm{RV},\mathrm{A}_1}_{m+1}+\left(\int_1\mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2}\right)^{\mathrm{A}_1}\right]J_m\right\}$$

- Construction of approximate cross section $\mathrm{d}\sigma^{\mathrm{RV},\mathrm{A}_1}_{m+1}$ that regularizes the kinematical singularities of $\mathrm{d}\sigma^{\mathrm{RV}}_{m+1}$ in the singly-unresolved regions is straightforward
- ... but

$$\left[\mathrm{d}\sigma^{\mathrm{RV}}_{m+1}-\mathrm{d}\sigma^{\mathrm{RV},\mathrm{A}_1}_{m+1}\right]_{\epsilon=0}=\text{infinite}$$

The $m+1$-parton contribution:

$$\int_{m+1}\left\{\left(\mathrm{d}\sigma^{\mathrm{RV}}_{m+1}+\int_1\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2}\right)J_{m+1}-\left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1}+\left(\int_1\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2}\right)^{\mathrm{A_1}}\right]J_m\right\}$$

- Construction of approximate cross section $\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1}$ that regularizes the kinematical singularities of $\mathrm{d}\sigma^{\mathrm{RV}}_{m+1}$ in the singly-unresolved regions is straightforward
- …but need to subtract the universal pole part too

$$\int_{m+1}\left[\mathrm{d}\sigma^{\mathrm{RV}}_{m+1}-\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1}+\left(\mathrm{d}\sigma^{\mathrm{B}}_{m+1}\otimes\boldsymbol{I}(\epsilon)-?\right)\right]_{\epsilon=0}=\text{finite}$$

- and recall

$$\boldsymbol{I}(\epsilon)\propto\frac{\alpha_s}{2\pi}\sum_i\left[\frac{1}{\epsilon}\gamma_i-\frac{1}{\epsilon^2}\sum_{k\neq i}\boldsymbol{T}_i\cdot\boldsymbol{T}_k\left(\frac{4\pi\mu^2}{s_{ik}}\right)^{\epsilon}\right]+\mathrm{O}(\varepsilon^0)$$

The standard integrated approximate cross sections do not obey universal IR collinear factorization

- Due to coherent soft-gluon emission from unresolved partons only the sum $\langle \mathcal{M}_{m+1}^{(0)} | (\boldsymbol{T}_j \cdot \boldsymbol{T}_k + \boldsymbol{T}_r \cdot \boldsymbol{T}_k) | \mathcal{M}_{m+1}^{(0)} \rangle$ factorizes in the collinear limit ($\boldsymbol{T}_{jr} = \boldsymbol{T}_j + \boldsymbol{T}_r$)

$$\mathbf{C}_{jr} \langle \mathcal{M}_{m+1}^{(0)} | (\boldsymbol{T}_j \cdot \boldsymbol{T}_k + \boldsymbol{T}_r \cdot \boldsymbol{T}_k) | \mathcal{M}_{m+1}^{(0)} \rangle \propto \frac{1}{s_{jr}} \langle \mathcal{M}_m^{(0)} | \boldsymbol{T}_{jr} \cdot \boldsymbol{T}_k \, \hat{P}_{jr}^{(0)} | \mathcal{M}_m^{(0)} \rangle$$

The standard integrated approximate cross sections do not obey universal IR collinear factorization

- Due to coherent soft-gluon emission from unresolved partons only the sum $\langle\mathcal{M}^{(0)}_{m+1}|(\boldsymbol{T}_j\cdot\boldsymbol{T}_k+\boldsymbol{T}_r\cdot\boldsymbol{T}_k)|\mathcal{M}^{(0)}_{m+1}\rangle$ factorizes in the collinear limit ($\boldsymbol{T}_{jr}=\boldsymbol{T}_j+\boldsymbol{T}_r$)

$$\mathbf{C}_{jr}\langle\mathcal{M}^{(0)}_{m+1}|(\boldsymbol{T}_j\cdot\boldsymbol{T}_k+\boldsymbol{T}_r\cdot\boldsymbol{T}_k)|\mathcal{M}^{(0)}_{m+1}\rangle \propto \frac{1}{s_{jr}}\langle\mathcal{M}^{(0)}_{m}|\boldsymbol{T}_{jr}\cdot\boldsymbol{T}_k\,\hat{P}^{(0)}_{jr}|\mathcal{M}^{(0)}_{m}\rangle$$

- This factorization is violated by the factors $s_{ik}^{-\epsilon}/\epsilon^2$

$$\mathbf{C}_{jr}\,\frac{1}{\epsilon^2}\langle\mathcal{M}^{(0)}_{m+1}|(\boldsymbol{T}_j\cdot\boldsymbol{T}_k\,s_{jk}^{-\epsilon}+\boldsymbol{T}_r\cdot\boldsymbol{T}_k\,s_{rk}^{-\epsilon})|\mathcal{M}^{(0)}_{m+1}\rangle \propto$$
$$\times\frac{1}{s_{jr}}\Bigg[\langle\mathcal{M}^{(0)}_{m}|\boldsymbol{T}_{jr}\cdot\boldsymbol{T}_k\,\hat{P}^{(0)}_{jr}\left(\frac{1}{\epsilon^2}-\frac{1}{\epsilon}\ln s_{(jr)k}\right)|\mathcal{M}^{(0)}_{m}\rangle$$
$$-\frac{1}{\epsilon}\langle\mathcal{M}^{(0)}_{m}|\boldsymbol{T}_j\cdot\boldsymbol{T}_k\ln z_j+\boldsymbol{T}_r\cdot\boldsymbol{T}_k\ln z_r\,|\mathcal{M}^{(0)}_{m}\rangle\Bigg]$$

The standard integrated approximate cross sections do not obey universal IR collinear factorization

- Due to coherent soft-gluon emission from unresolved partons only the sum $\langle\mathcal{M}_{m+1}^{(0)}|(\boldsymbol{T}_j\cdot\boldsymbol{T}_k+\boldsymbol{T}_r\cdot\boldsymbol{T}_k)|\mathcal{M}_{m+1}^{(0)}\rangle$ factorizes in the collinear limit ($\boldsymbol{T}_{jr}=\boldsymbol{T}_j+\boldsymbol{T}_r$)

$$\mathbf{C}_{jr}\langle\mathcal{M}_{m+1}^{(0)}|(\boldsymbol{T}_j\cdot\boldsymbol{T}_k+\boldsymbol{T}_r\cdot\boldsymbol{T}_k)|\mathcal{M}_{m+1}^{(0)}\rangle\propto\frac{1}{s_{jr}}\langle\mathcal{M}_m^{(0)}|\boldsymbol{T}_{jr}\cdot\boldsymbol{T}_k\,\hat{P}_{jr}^{(0)}|\mathcal{M}_m^{(0)}\rangle$$

- This factorization is violated by the factors $s_{ik}^{-\epsilon}/\epsilon^2$
- $\Rightarrow$ need to use
  - either colour stripped amplitudes (as in antennae subtraction)
  - or properly defined new approximate cross sections $\longrightarrow$ remark about $\mathrm{d}\sigma_{m+2}^{\mathrm{RR,A}_1}$

Pure soft limit of the squared matrix element

- Using the soft insertion rules one obtains

$$\mathbf{S}_r|\mathcal{M}^{(0)}_{m+1}(p_r,\dots)|^2 \propto$$
$$\sum_{i=1}^{m}\sum_{k=1}^{m}\sum_{\text{hel.}}\varepsilon_\mu(p_r)\varepsilon^*_\nu(p_r)\frac{2p_i^\mu p_k^\nu}{s_{ir}s_{kr}}\langle\mathcal{M}^{(0)}_m(\dots)|\boldsymbol{T}_i\cdot\boldsymbol{T}_k|\mathcal{M}^{(0)}_m(\dots)\rangle$$

$$d^{\mu\nu}(p_r,n) = \sum_{\text{hel.}}\varepsilon_\mu(p_r)\varepsilon^*_\nu(p_r) = -g^{\mu\nu} + \frac{p_r^\mu n^\nu + p_r^\nu n^\mu}{p_r\cdot n}\,.$$

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$$\mathbf{S}_r|\mathcal{M}^{(0)}_{m+1}(p_r,\dots)|^2 \propto \sum_{i=1}^{m}\sum_{k=1}^{m}\sum_{\text{hel.}}\varepsilon_\mu(p_r)\varepsilon^*_\nu(p_r)\frac{2p_i^\mu p_k^\nu}{s_{ir}s_{kr}}\langle\mathcal{M}^{(0)}_m(\dots)|\boldsymbol{T}_i\cdot\boldsymbol{T}_k|\mathcal{M}^{(0)}_m(\dots)\rangle$$

$$d^{\mu\nu}(p_r,n) = \sum_{\text{hel.}}\varepsilon_\mu(p_r)\varepsilon^*_\nu(p_r) = -g^{\mu\nu} + \frac{p_r^\mu n^\nu + p_r^\nu n^\mu}{p_r\cdot n}\,.$$

- Soft-collinear contributions are given by the colour-diagonal terms,

$$\mathbf{S}_r|\mathcal{M}^{(0)}_{m+1}(p_r,\dots)|^2 \propto \sum_{i=1}^{m}\Bigg[\frac{1}{2}\sum_{k\neq i}^{m}\left(\frac{s_{ik}}{s_{ir}s_{rk}} - \frac{2s_{in}}{s_{rn}s_{ir}} - \frac{2s_{kn}}{s_{rn}s_{kr}}\right)\langle\mathcal{M}^{(0)}_m(\dots)|\boldsymbol{T}_i\cdot\boldsymbol{T}_k|\mathcal{M}^{(0)}_m(\dots)\rangle - \boldsymbol{T}_i^2\frac{2}{s_{ir}}\frac{s_{in}}{s_{rn}}|\mathcal{M}^{(0)}_m(\dots)|^2\Bigg] \qquad s_{in} = 2p_i\cdot n$$

Pure soft limit of the squared matrix element

- Using the soft insertion rules one obtains

$$\mathbf{S}_r |\mathcal{M}_{m+1}^{(0)}(p_r,\dots)|^2 \propto \sum_{i=1}^{m}\sum_{k=1}^{m}\sum_{\text{hel.}} \varepsilon_\mu(p_r)\varepsilon_\nu^*(p_r)\frac{2p_i^\mu p_k^\nu}{s_{ir}s_{kr}}\langle\mathcal{M}_m^{(0)}(\dots)|\boldsymbol{T}_i\cdot\boldsymbol{T}_k|\mathcal{M}_m^{(0)}(\dots)\rangle$$

$$d^{\mu\nu}(p_r,n) = \sum_{\text{hel.}} \varepsilon_\mu(p_r)\varepsilon_\nu^*(p_r) = -g^{\mu\nu} + \frac{p_r^\mu n^\nu + p_r^\nu n^\mu}{p_r\cdot n}\,.$$

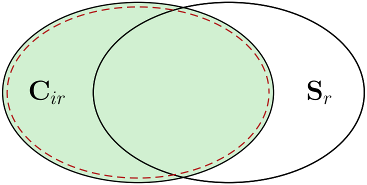
- choose Coulomb gauge and keep the pure soft only

$$\mathbf{S}_r |\mathcal{M}_{m+1}^{(0)}(p_r,\dots)|^2 \longrightarrow \sum_{i=1}^{m}\left[\frac{1}{2}\sum_{k\neq i}^{m}\left(\frac{s_{ik}}{s_{ir}s_{rk}} - \frac{2s_{iQ}}{s_{rQ}s_{ir}} - \frac{2s_{kQ}}{s_{rQ}s_{kr}}\right)\langle\mathcal{M}_m^{(0)}(\dots)|\boldsymbol{T}_i\cdot\boldsymbol{T}_k|\mathcal{M}_m^{(0)}(\dots)\rangle\right]$$

$$\text{with } n^\mu = Q^\mu - p_r^\mu\, Q^2/s_{rQ} \text{ and colour} - \text{conservation}$$

Construction of the subtraction terms at NLO

- Collinear and soft limits are automatically disjunct



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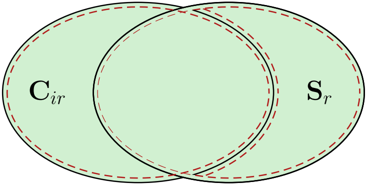

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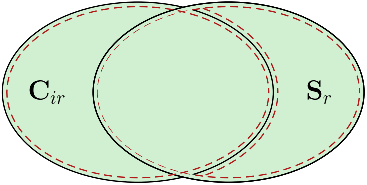
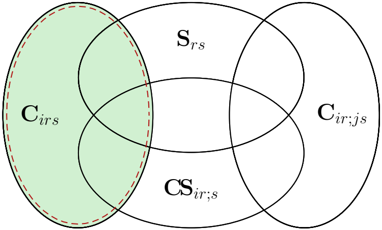


- works at any order in PT, e.g. at NNLO



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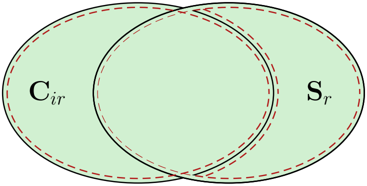

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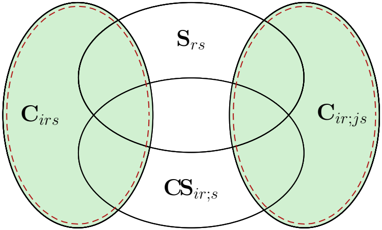

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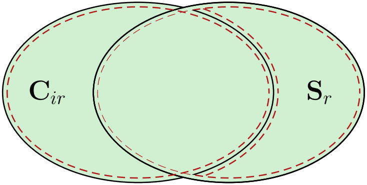

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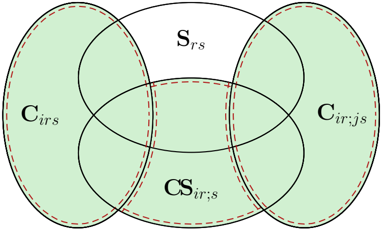

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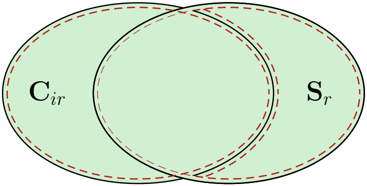

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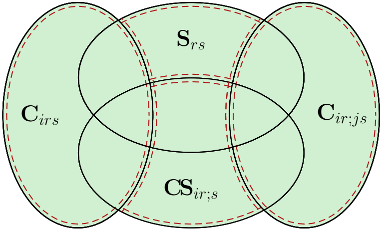

Construction of the subtraction terms at NLO

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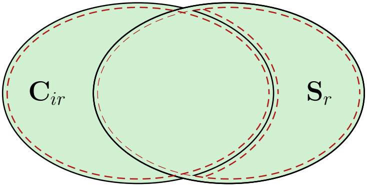

- Extension over the full phase space requires momentum mappings, $\{p\}_{m+1} \to \{\tilde{p}\}_m$ that
  - implement exact momentum conservation
  - lead to exact phase-space factorization
  - can be generalized to any number of unresolved partons

Construction of the subtraction terms at NLO

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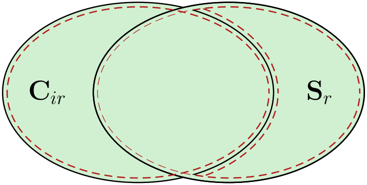
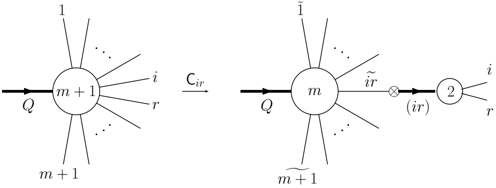


- Extension over the full phase space requires momentum mappings, $\{p\}_{m+1} \to \{\tilde{p}\}_m$ that
  - implement exact momentum conservation
  - lead to exact phase-space factorization
  - can be generalized to any number of unresolved partons
- We use separate phase space mappings for the collinear and soft limits

Collinear mapping

$$\tilde{p}_{ir}^{\mu} = \frac{1}{1-\alpha_{ir}}(p_i^{\mu} + p_r^{\mu} - \alpha_{ir} Q^{\mu}) \,, \qquad \tilde{p}_{n}^{\mu} = \frac{1}{1-\alpha_{ir}} p_n^{\mu} \,, \quad n \neq i, r$$

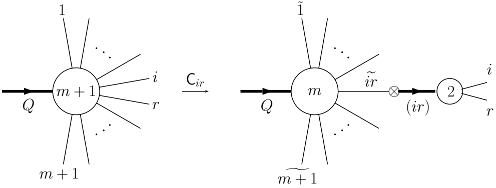
$$\alpha_{ir} = \frac{1}{2}\left[y_{(ir)Q} - \sqrt{y_{(ir)Q}^2 - 4y_{ir}} \right]$$



Collinear mapping

$$\tilde{p}_{ir}^{\mu} = \frac{1}{1-\alpha_{ir}}(p_i^{\mu}+p_r^{\mu}-\alpha_{ir}Q^{\mu})\,, \qquad \tilde{p}_n^{\mu} = \frac{1}{1-\alpha_{ir}}p_n^{\mu}\,, \quad n \neq i, r$$

$$\alpha_{ir} = \frac{1}{2}\left[y_{(ir)Q} - \sqrt{y_{(ir)Q}^2 - 4y_{ir}}\right]$$

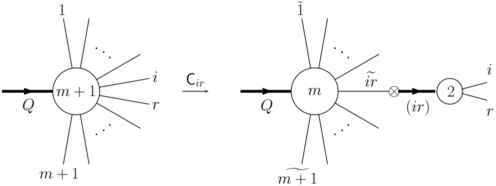


- momentum is conserved $\tilde{p}_{ir}^{\mu} + \sum_n \tilde{p}_n^{\mu} = p_i^{\mu} + p_r^{\mu} + \sum_n p_n^{\mu}$
- phase-space factorization is exact
  $\mathrm{d}\phi_{m+1}(p_1,\ldots;Q) = \mathrm{d}\phi_m(\widetilde{p}_1,\ldots;Q) \otimes \mathrm{d}\phi_2(p_i,p_r;p_{(ir)})$

Collinear mapping

$$\tilde{p}^{\mu}_{ir} = \frac{1}{1-\alpha_{ir}}(p^{\mu}_i + p^{\mu}_r - \alpha_{ir} Q^{\mu}) \,, \qquad \tilde{p}^{\mu}_n = \frac{1}{1-\alpha_{ir}} p^{\mu}_n \,, \quad n \neq i, r$$

$$\alpha_{ir} = \frac{1}{2}\left[y_{(ir)Q} - \sqrt{y^2_{(ir)Q} - 4 y_{ir}} \right]$$



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- integral over convolution can be constrained to improve numerical efficiency

Soft mapping

$$\tilde{p}_n^\mu = \Lambda^\mu_\nu[Q,(Q-p_r)/\lambda_r](p_n^\nu/\lambda_r)\,, \quad n \neq r\,, \qquad \lambda_r = \sqrt{1-y_{rQ}}\,,$$

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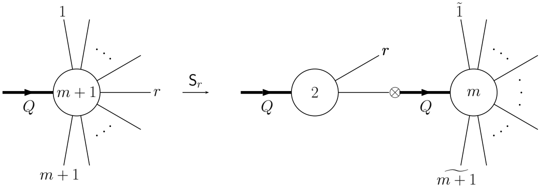

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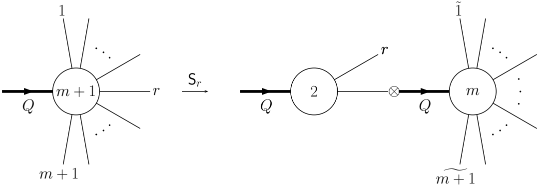

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The approximate cross section at NLO

The collinear and soft momentum mappings define extensions of the limit formulae over the full phase space

$$\mathbf{C}_{ir}|\mathcal{M}^{(0)}_{m+2}|^2 \longrightarrow \mathcal{C}^{(0,0)}_{ir}$$
$$\mathbf{S}_{r}|\mathcal{M}^{(0)}_{m+2}|^2 \longrightarrow \mathcal{S}^{(0,0)}_{r}$$

- The true subtraction term is

$$\boldsymbol{\mathcal{A}}_1|\mathcal{M}^{(0)}_{m+2}|^2 = \sum_r \left[\sum_{i\neq r} \frac{1}{2}\mathcal{C}^{(0,0)}_{ir} + \mathcal{S}^{(0,0)}_{r}\right]$$

  - $\mathcal{C}^{(0,0)}_{ir}$, $\mathcal{S}^{(0,0)}_{r}$ are functions of the original momenta that inherit the notation of the operators, but have nothing to do with taking limits
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- The approximate cross section is written formally as

$$\mathrm{d}\sigma_{m+2}^{\mathrm{RR,A}_1} = \mathrm{d}\phi_{m+1}[\mathrm{d}p_1]\boldsymbol{\mathcal{A}}_1|\mathcal{M}_{m+2}^{(0)}|^2$$

Similarly and completely systematically one can define

- the fully differencial $(m+2)$-parton contriubtion

$$\mathrm{d}\sigma^{\mathrm{NNLO}}_{m+2} = \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} J_m - \left(\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} J_m\right)$$

- $\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} = \mathrm{d}\phi_{m+1}[\mathrm{d}p_1]\boldsymbol{\mathcal{A}}_1|\mathcal{M}^{(0)}_{m+2}|^2$
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- and the fully differential $(m+1)$-parton contribution

$$\mathrm{d}\sigma_{m+1}^{\mathrm{NNLO}} = \left(\mathrm{d}\sigma_{m+2}^{\mathrm{RV}} + \int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1} \right) J_{m+1} - \left[\mathrm{d}\sigma_{m+1}^{\mathrm{RV},\mathrm{A}_1} + \left(\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1} \right)^{\mathrm{A}_1} \right] J_m$$

  - $\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1} = \mathrm{d}\phi_{m+1}|\mathcal{M}_{m+1}^{(0)}|^2 \otimes \boldsymbol{I}(m,\varepsilon)$
  - $\mathrm{d}\sigma_{m+1}^{\mathrm{RV},\mathrm{A}_1} = \mathrm{d}\phi_{m}[\mathrm{d}p_1]\boldsymbol{\mathcal{A}}_1 2\mathrm{Re}\langle\mathcal{M}_{m+1}^{(0)}|\mathcal{M}_{m+1}^{(1)}\rangle$
  - $\left(\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1} \right)^{\mathrm{A}_1} = \mathrm{d}\phi_{m}[\mathrm{d}p_1]\boldsymbol{\mathcal{A}}_1 \left(|\mathcal{M}_{m+1}^{(0)}|^2 \otimes \boldsymbol{I}(m,\varepsilon) \right)$

- both are integrable in $d=4$ dimensions

Monte Carlo summation over helicity in NLO computations

- proved to be useful to gain speed in multileg computations at Born level

P. Draggiotis, R. H. P. Kleiss, C. G. Papadopoulos 1998, 2002

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- The pure-soft factorization is independent of the soft-gluon helicity

$$\mathbf{S}_r|\mathcal{M}_{m+1}^{(0)}(p_r^{\lambda},\dots)|^2 \propto \frac{1}{2}\sum_{i=1}^{m}\Bigg[\frac{1}{2}\sum_{k\neq i}^{m}\left(\frac{s_{ik}}{s_{ir}s_{rk}}-\frac{2s_{iQ}}{s_{rQ}s_{ir}}-\frac{2s_{kQ}}{s_{rQ}s_{kr}}\right)\langle\mathcal{M}_m^{(0)}(\dots)|\boldsymbol{T}_i\cdot\boldsymbol{T}_k|\mathcal{M}_m^{(0)}(\dots)\rangle -\boldsymbol{T}_i^2\frac{2}{s_{ir}}\frac{s_{in}}{s_{rn}}|\mathcal{M}_m^{(0)}(\dots)|^2\Bigg]$$

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- The pure-soft factorization is independent of the soft-gluon helicity
- Can define collinear and pure-soft subtractions for fixed helicities

$$\mathbf{S}_r|\mathcal{M}^{(0)}_{m+1}(p_r^{\lambda},\dots)|^2 \longrightarrow$$
$$\frac{1}{2}\sum_{i=1}^{m}\left[\frac{1}{2}\sum_{k\neq i}^{m}\left(\frac{s_{ik}}{s_{ir}s_{rk}}-\frac{2s_{iQ}}{s_{rQ}s_{ir}}-\frac{2s_{kQ}}{s_{rQ}s_{kr}}\right)\langle\mathcal{M}^{(0)}_m(\dots)|\boldsymbol{T}_i\cdot\boldsymbol{T}_k|\mathcal{M}^{(0)}_m(\dots)\rangle\right]$$

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Thanks for your attention!

Towards event shapes

- Constructed $\mathrm{d}\sigma_5^{\mathrm{NNLO}}$ (RR) and $\mathrm{d}\sigma_4^{\mathrm{NNLO}}$ (RV) for $e^+e^- \to 3$ jets ($e^+e^- \to q\bar{q}ggg$ and $e^+e^- \to q\bar{q}gg$ subprocesses respectively)

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- Checked numerically that ($J = C$ or $1 - T$)
  - In all singly- and doubly-unresolved limits

$$\frac{\mathrm{d}\sigma_5^{\mathrm{RR,A}_2} J_3 + \mathrm{d}\sigma_5^{\mathrm{RR,A}_1} J_4 - \mathrm{d}\sigma_5^{\mathrm{RR,A}_{12}} J_3}{\mathrm{d}\sigma_5^{\mathrm{RR}}} \to 1$$

  - In all singly-unresolved limits

$$\frac{\mathrm{d}\sigma_4^{\mathrm{RV,A}_1} J_3 - \int_1 \mathrm{d}\sigma_5^{\mathrm{RR,A}_1} J_4 - \left(\int_1 \mathrm{d}\sigma_5^{\mathrm{RR,A}_1}\right)^{\mathrm{A}_1} J_3}{\mathrm{d}\sigma_4^{\mathrm{RV}}} \to 1$$

$\Rightarrow$ The counterterms are fully local, azimuthal and color correlations fully included

Towards event shapes

- Computed the RR and RV contributions to first three moments of 3-jet event shape variables thurst (T) and C-parameter

$$\langle O^n \rangle \equiv \int O^n \frac{\mathrm{d}\sigma}{\sigma_0} = \left(\frac{\alpha_s(Q)}{2\pi}\right) A_O^{(n)} + \left(\frac{\alpha_s(Q)}{2\pi}\right)^2 B_O^{(n)} + \left(\frac{\alpha_s(Q)}{2\pi}\right)^3 C_O^{(n)}$$

where the NNLO contribution $C_O^{(n)}$ is a sum of the RR, RV and VV pieces

$$C_O^{(n)} = C_{O;5}^{(n)} + C_{O;4}^{(n)} + C_{O;3}^{(n)}$$

- The quantities $C_{O;5}^{(n)}$ and $C_{O;4}^{(n)}$ are found to be finite ($O = C$ or $O = \tau \equiv 1 - T$; $n = 1, 2, 3$)
- Up to NLO accuracy perfect agreement with known results for $e^+e^- \to 3$ jets

Towards event shapes – RR contribution

- Prediction for moments of event shapes – RR contribution

n	$C^{(n)}_{\tau;5}$	$C^{(n)}_{C;5}$
1	$-(9.27 \pm 0.34) \cdot 10^1$	$-(3.44 \pm 0.14) \cdot 10^2$
2	-3.07 ± 0.43	$-(1.42 \pm 0.03) \cdot 10^2$
3	2.01 ± 0.12	6.29 ± 1.87

- Technical details
  - No. of MC points used: $n = 40 \times 2.5 \cdot 10^5$ (VEGAS)
  - χ^2/d.o.f. as reported by VEGAS: $\chi^2/\text{d.o.f.} = 0.79$
  - No. of subtractions: 535 at 139 different PS points for each event [compare with 12 subtractions at 12 different PS points for $e^+e^- \to$ 4 jets at NLO needed in this scheme ($q\bar{q}ggg$ subprocess)]
  - Speed of code on an AMD Athlon 1.3 GHz machine with 256 MB RAM: $2.5 \cdot 10^5$ pts. $\approx$ 2.5 h

Towards event shapes – RV contribution

- Prediction for moments of event shapes – RV contribution

n	$C^{(n)}_{\tau;4}$	$C^{(n)}_{C;4}$
1	$(1.23 \pm 0.01) \cdot 10^3$	$(4.33 \pm 0.05) \cdot 10^3$
2	$(2.55 \pm 0.02) \cdot 10^2$	$(3.25 \pm 0.02) \cdot 10^3$
3	$(4.79 \pm 0.03) \cdot 10^1$	$(1.80 \pm 0.01) \cdot 10^3$

- Technical details
  - No. of MC points used: $n = 20 \times 2.5 \cdot 10^5$ (VEGAS)
  - χ^2/d.o.f. as reported by VEGAS: χ^2/d.o.f. $= 1.24$
  - No. of subtractions: 15 at 7 different PS points for each event
  - Speed of code on an AMD Athlon 1.3 GHz machine with 256 MB RAM: $2.5 \cdot 10^5$ pts. $\approx$ 7 h

Collinear limit of color-connected SME

- Due to color coherence only the sum

$$|\mathcal{M}^{(0)}_{m+1;(i,l)}|^2 + |\mathcal{M}^{(0)}_{m+1;(r,l)}|^2$$

has a universal collinear limit as $p_i || p_r$, not $|\mathcal{M}^{(0)}_{m+1;(i,l)}|^2$ or $|\mathcal{M}^{(0)}_{m+1;(r,l)}|^2$ separately

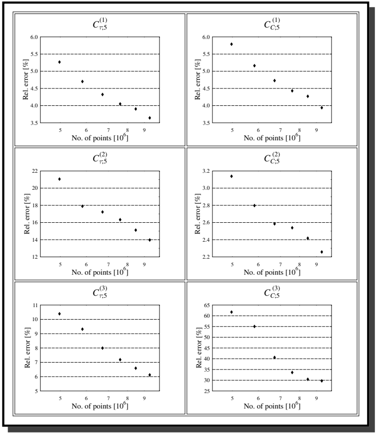
- Generally we have

$$\boldsymbol{I} = \mathcal{V}_{il}|\mathcal{M}^{(0)}_{m+1;(i,l)}(\{\tilde{p}\}^{(il)}_{m+1})|^2 + \mathcal{V}_{rl}|\mathcal{M}^{(0)}_{m+1;(r,l)}(\{\tilde{p}\}^{(rl)}_{m+1})|^2 + \dots$$

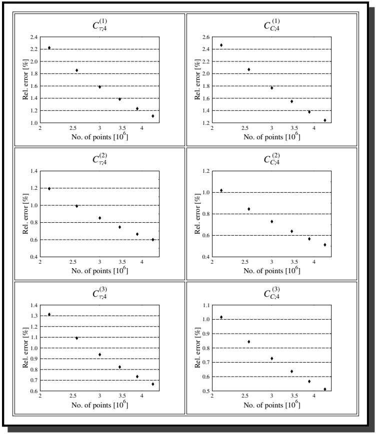
- $\mathbf{C}_{ir}\boldsymbol{I}$ exists iff
  - $\mathbf{C}_{ir}\mathcal{V}_{il} = \mathbf{C}_{ir}\mathcal{V}_{rl} \equiv \mathcal{V}_{(ir)l}$
  - $\{\tilde{p}\}^{(il)}_{m+1} \xrightarrow{i||r} \{\tilde{p}\}^{[(ir)l]}_{m}, \quad \{\tilde{p}\}^{(rl)}_{m+1} \xrightarrow{i||r} \{\tilde{p}\}^{[(ir)l]}_{m}$
- Then
$\mathbf{C}_{ir}\boldsymbol{I} \propto \frac{1}{s_{ir}}\mathcal{V}_{(ir)l}\langle\mathcal{M}^{(0)}_{m}(\{\tilde{p}\}^{[(ir)l]}_{m})|\boldsymbol{T}_{ir}\boldsymbol{T}_{l}\hat{P}_{ir}|\mathcal{M}^{(0)}_{m}(\{\tilde{p}\}^{[(ir)l]}_{m})\rangle$

Rate of convergence

RR part

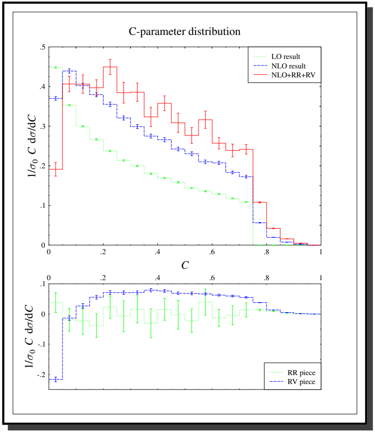


RV part



Event shape distributions (still unphysical at NNLO)

C-parameter



Thrust