A Numerical Unitarity Formalism for Evaluating One-Loop Amplitudes

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Constructing Loop Amplitudes from Tree Amplitudes

R. K. Ellis, W. Giele, Z.K, arXiv.0708.2398

LHC physics is coming

- It is expected that least the Higgs boson will be found
- New heavy particles give complex final states in terms of leptons and jets
- Standard Model physics gives significant background
- Precise understanding of the background is beneficial both in searching for the signal and clarifying the nature of new physics
- New technical challenge: calculate differential cross-sections of 5,6,… leg processes in NLO accuracy in QCD perturbation theory

At LO general purpose software packages
MADGRAPH, ALPGEN, HELAC, CompHEP, …
UPGRADE THEM by including NLO QCD and perhaps EWK corrections.

Unitarity cut methods for one-loop calculations

One loop amplitudes in terms of tree amplitudes of physical states

BASIC SETUP

- Four dimensional unitarity cut method + structure of the collinear limit

  Bern Dixon Kosower: $pp \to W, Z + 2$ jets (1998)

  i) Only physical states, tree amplitudes, (no 1-loop Feynman diagrams)
  ii) It does not catch the rational parts -> cut constructible part

- $D=4-2\varepsilon$ dimensional unitarity cut method

  van Nerveen; Bern, Morgan; Bern, Dixon, Dunbar, Kosower

  i) Rational parts: integrals are always convergent in $D=4-2\varepsilon$
  ii) One has to use D-dimensional states, "tree level input" is more complicated, full ε-dependence. Too complicated?

- RECENT DEVELOPMENTS

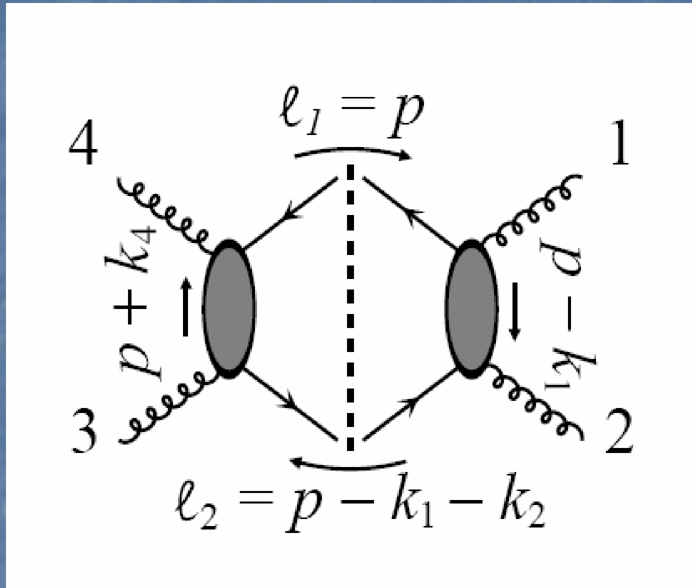
Unitarity as on-shell method of calculation

Bern, Dixon, Dunbar, Kosower

$$T^\dagger - T = -2iT^\dagger T$$

Sewing versus cutting

Im $T^{1\text{-loop}} = \sum c_j$ Im I_j



$$-i\,\mathrm{Disc}\; A_4(1,2,3,4)\Big|_{s-\mathrm{cut}} = \int \frac{d^4p}{(2\pi)^4}\, 2\pi\delta^{(+)}(\ell_1^2 - m^2)\, 2\pi\delta^{(+)}(\ell_2^2 - m^2) \times A_4^{\mathrm{tree}}(-\ell_1, 1, 2, \ell_2)\, A_4^{\mathrm{tree}}(-\ell_2, 3, 4, \ell_1)\,,$$

Unitarity cut method: recent developments

i) Twistors and use of complex kinematics

Witten; Caczazo, Witten (talk by C. Schwinn)

ii) On-shell recursion relations for tree amplitudes

Britto, Feng, Caczaco; Britto, Feng, Cachazo, Witten

iii) Generalized unitarity (talk by P. Mastrolia)

more than two internal particles are on-shell

Britto, Cachazo, Feng; Brandhuber, Spence, Travaglini

iv) Spinorial integration

Cachazo, Witten; Britto, Feng, Mastrolia, Svreck (D=4)

v) On shell recursion relation for loop amplitudes

Bern, Dixon, Kosower

vi) Algebraic tensor reduction (talks by Papadopoulos and Forde)

Ossala, Pittau, Papadopoulos; Forde;

vii) Unitarity in D-dimension and spinorial integrals (talk by P. Mastrolia)

Anastasiou, Britto, Feng, Kunszt, Mastrolia (D=4-2ε)

One-loop amplitudes from tree amplitudes

Main ingredients

- Decomposing one-loop N-point amplitude in terms master integrals
- Use of generalized unitarity and complex kinematics
- Algebraic reduction of the amplitude at the integrand level
- Rational parts: separate tree-type algorithm

Technical issues

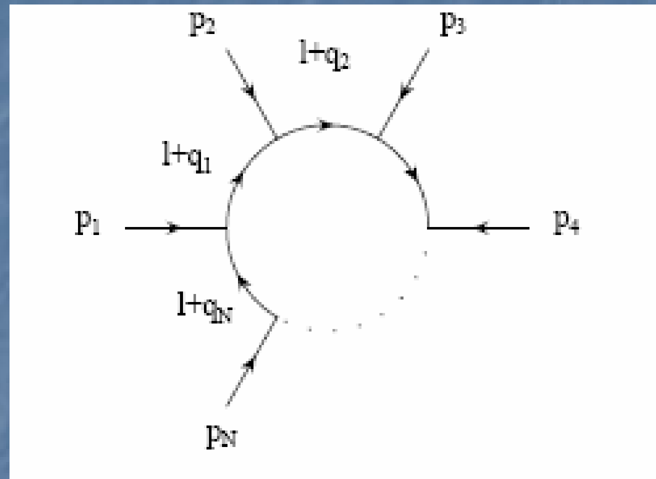
- Reduction: algebraic (OPP), spinorial (BFCM,BTS)
- Implementation: analytic (BFCM,BBDFK), numerical (OPP,EGK)

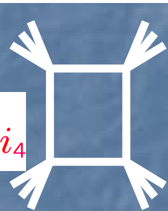
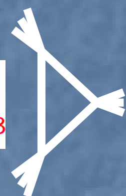
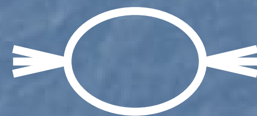
1) Decomposing one-loop N-point amplitudes in terms of master integrals

$$\mathcal{A}_N(p_1, p_2, \ldots, p_N) = \int [d\,l]\; \mathcal{A}(p_1, p_2, \ldots, p_N; l)$$

$$\mathcal{A}_N(p_1, p_2, \ldots, p_N; l) = \frac{\mathcal{N}(p_1, p_2, \ldots, p_N; l)}{d_1 d_2 \cdots d_N}$$

$$d_i = (l + q_i)^2 - m_i^2 = (l - q_0 + \sum_{j=1}^{i} p_i)^2 - m_i^2$$



$\mathcal{A}_N(\{p_i\}) = \sum d_{i_1 i_2 i_3 i_4}$  $+ \sum c_{i_1 i_2 i_3}$  $+ \sum b_{i_1 i_2}$ 

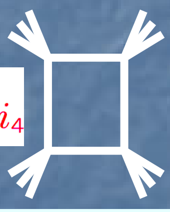
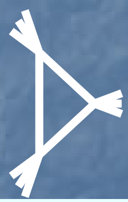
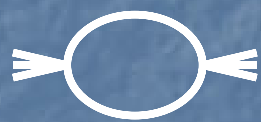
+ Tadpoles

+ Rational part

Decomposing one-loop N-point amplitudes in terms of master integrals (cont.)

$$\mathcal{A}_N(p_1,p_2,\dots,p_N) = \sum_{1\le i_1<i_2<i_3<i_4\le N} d_{i_1i_2i_3i_4}(p_1,p_2,\dots,p_N) I_{i_1i_2i_3i_4}$$
$$+ \sum_{1\le i_1<i_2<i_3\le N} c_{i_1i_2i_3}(p_1,p_2,\dots,p_N) I_{i_1i_2i_3}$$
$$+ \sum_{1\le i_1<i_2\le N} b_{i_1i_2}(p_1,p_2,\dots,p_N) I_{i_1i_2}$$
$$+ \sum_{1\le i_1\le N} a_{i_1}(p_1,p_2,\dots,p_N) I_{i_1}$$

$$I_{i_1\cdots i_M} = \int [d\,l]\,\frac{1}{d_{i_1}\cdots d_{i_M}}$$

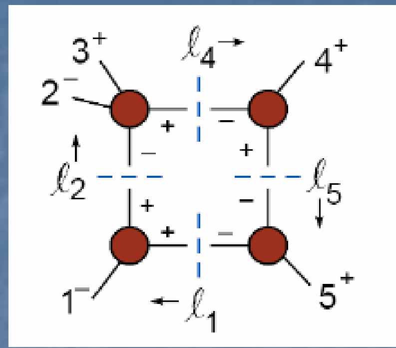
$\mathcal{A}_N(\{p_i\}) = \sum d_{i_1i_2i_3i_4}$  $+\sum c_{i_1i_2i_3}$  $+\sum b_{i_1i_2}$ 

"Cut constructible" part: coefficients are calculated in 4 dimensions

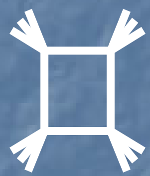
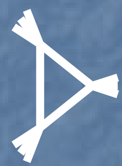
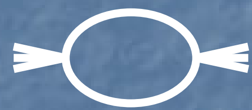
Rational part is generated by the order ε part of b_{ij}

2) Generalized unitarity to read out coefficients

$$T^\dagger - T = -2iT^\dagger T$$



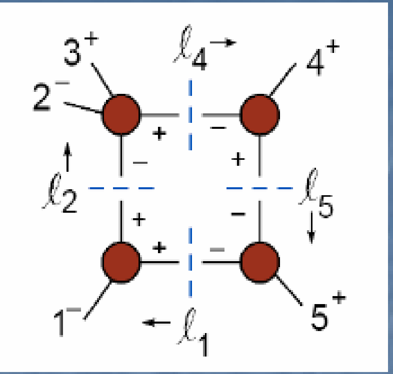
Britto, Cachazo, Feng

$$\mathcal{A}_N(\{p_i\}) = \sum d_{i_1i_2i_3i_4}$$

$$+\sum c_{i_1i_2i_3}$$

$$+\sum b_{i_1i_2}$$


$$\frac{i}{(l+P_i)^2} \to (2\pi)\delta((l+P_i)^2)\,, \quad d_0 = d_i = d_j = d_k = 0$$

- Unitarity constraints can only be solved if we allow for complex momenta
- Tree-level helicity amplitudes can be analytically continued to complex momenta

Generalized unitarity to read out coefficients (cont.)



Factorized expression for the cut diagrams

The box coefficient can be extracted both analytically and numerically.

$$d_{ijk} = \frac{1}{2} \sum_{a=1}^{2} A_1(l_{ijk;a}) A_2(l_{ijk;a}) A_3(l_{ijk;a}) A_4(l_{ijk;a})$$

c_{ij}, (b_i) can be calculated after the box (triangle) contributions are subtracted

How to extract the triangle and bubble coefficients?
a) Spinorial integrals. b) Algebraic reduction.

3) Algebraic reduction, subtraction terms

Ossola, Papadopoulos, Pittau: there is a systematic way of calculating the subtraction terms at the integrand level. The numerator can be decomposed as linear combination of 4-,3-,2,-1 denominator factors

We follow OPP but use the van Neerven Vermaseren basis and multi-pole expansion of rational functions

$$\mathcal{A}_N(p_1, p_2, \ldots, p_N; l) = \frac{\mathcal{N}(p_1, p_2, \ldots, p_N; l)}{d_1 d_2 \cdots d_N} =$$

$$\sum_{1 \le i_1 < i_2 < i_3 < i_4 \le N} \frac{\overline{d}_{i_1 i_2 i_3 i_4}(l)}{d_{i_1} d_{i_2} d_{i_3} d_{i_4}} + \sum_{1 \le i_1 < i_2 < i_3 \le N} \frac{\overline{c}_{i_1 i_2 i_3}(l)}{d_{i_1} d_{i_2} d_{i_3}} + \sum_{1 \le i_1 < i_2 \le N} \frac{\overline{b}_{i_1 i_2}(l)}{d_{i_1} d_{i_2}} + \sum_{1 \le i_1 \le N} \frac{\overline{a}_{i_1}(l)}{d_{i_1}}$$

- We can re-express the rational function in an expansion over 4-,3-,2.- and 1-propagator terms with partial fractioning
- The residues of these pole terms contain the l-independent master integral coefficients plus finite number of "spurious terms".

The residues of the poles and unitarity cuts

The residue is taken at special loop momentum defined by the unitarity conditions. It is the same then to calculate the generalized cuts of the amplitude.

Cutting operation :

$$\text{Cut}_{ij\cdots k}\left[F(l)\right] \equiv \left[d_i(l)d_j(l)\cdots d_k(l)F(l)\right]\Big|_{l=l_{ij\cdots k}}.$$

$$\bar{d}_{ijkl}(l) = \text{Cut}_{ijkl}\Big(\mathcal{A}_N(l)\Big)$$

$d_i=d_j=d_k=d_l=0$

$$\bar{c}_{ijk}(l) = \text{Cut}_{ijk}\left(\mathcal{A}_N(l) - \sum_{l\neq i,j,k}\frac{\bar{d}_{ijkl}(l)}{d_i d_j d_k d_l}\right)$$

$d_i=d_j=d_k=0$

$$\bar{b}_{ij}(l) = \text{Cut}_{ij}\left(\mathcal{A}_N(l) - \sum_{k\neq i,j}\frac{\bar{c}_{ijk}(l)}{d_i d_j d_k} - \frac{1}{2!}\sum_{k,l\neq i,j}\frac{\bar{d}_{ijkl}(l)}{d_i d_j d_k d_l}\right)$$

$d_i=d_j=0$

Unitarity conditions : use of the van Neerven Vermaseren (NV) basis for parameterizing the loop-momentum

Van Neerven Vermaseren basis for tensor reduction

For box, triangle and bubble integrals split the 4 dimensional space-time to physical space and trivial space; the physical space is spanned by the inflow momenta, $k_1,\ldots,k_R$ $R \leq 4$; the trivial space is the orthogonal completions

$4 = D_P + D_T$

Box: $D_P=3$ $D_T=1$ use dual momenta v_i $p_i v_j = \delta_{ij}$

$$v_1^\mu(k_1,k_2,k_3) = \frac{\delta^{\mu k_2 k_3}_{k_1 k_2 k_3}}{\Delta(k_1,k_2,k_3)};\ v_2^\mu(k_1,k_2,k_3) = \frac{\delta^{k_1 \mu k_3}_{k_1 k_2 k_3}}{\Delta(k_1,k_2,k_3)};\ v_3^\mu(k_1,k_2,k_3) = \frac{\delta^{k_1 k_2 \mu}_{k_1 k_2 k_3}}{\Delta(k_1,k_2,k_3)}$$

Unit vector in trivial space n_1, and projection operator $w_{\mu\nu}$ with $p_i^\mu w_{mu\nu}=0$

$$w_\mu{}^\nu(k_1,k_2,k_3) = \frac{\delta^{k_1 k_2 k_3 \nu}_{k_1 k_2 k_3 \mu}}{\Delta(k_1,k_2,k_3)} = n_{1\mu} n_1{}^\nu = \frac{\varepsilon_{k_1 k_2 k_3 \mu}\varepsilon^{k_1 k_2 k_3 \nu}}{\Delta(k_1,k_2,k_3)}$$

Decomposition of the loop-momentum

$$l^\mu = V_R^\mu + \sum_{i=1}^{D_P} \frac{1}{2}(d_i - d_{i-1})\, v_i^\mu + \sum_{i=1}^{D_T} \alpha_i\, n_i^\mu\ ,\quad V_R^\mu = -\frac{1}{2}\sum_{i=1}^{D_P}\left((q_i^2 - m_i^2) - (q_{i-1}^2 - m_{i-1}^2)\right) v_i^\mu$$

Triangle: $D_P=2$ $D_T=2$. Bubble: $D_P=1$ $D_T=3$. Similar expressions.

Van Neerven Vermaseren basis for tensor reduction (cont.)

Box, two solutions

$$l^\mu = V_4^\mu + \alpha_1\, n_1^\mu$$

$$l_\pm^\mu = V_4^\mu \pm i\,\sqrt{V_4^2 - m_l^2} \times n_1^\mu$$

Triangle, infinite # of solutions (on a circle)

$$l^\mu = V_3^\mu + \alpha_1 n_1^\mu + \alpha_2 n_2^\mu$$

$$l_{\alpha_1\alpha_2}^\mu = V_3^\mu + \alpha_1\, n_1^\mu + \alpha_2\, n_2^\mu;\;\; \alpha_1^2 + \alpha_2^2 = -(V_3^2 - m_k^2)$$

$n_i.\, n_j = \delta_{ij};\; n_i\, .k_j = 0$

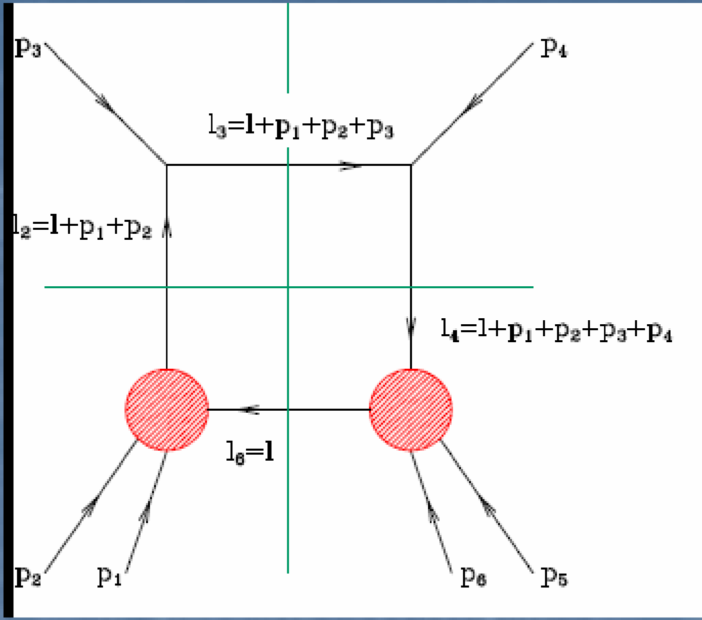
Bubble, infinite # of solutions (on a "sphere")

$$l^\mu = V_2^\mu + \alpha_1 n_1^\mu + \alpha_2 n_2^\mu + \alpha_3 n_3^\mu$$

$$l_{\alpha_1\alpha_2\alpha_3}^\mu = V_2^\mu + \alpha_1\, n_1^\mu + \alpha_2\, n_2^\mu + \alpha_3\, n_3^\mu;\;\; \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = -(V_2^2 - m_j^2)\;.$$

Forde: special choice for α_1 and α_2 :

Calculating the box residue



$$d_{ijkl} = \frac{\text{Cut}_{ijkl}\left(\mathcal{A}_N(l^+)\right) + \text{Cut}_{ijkl}\left(\mathcal{A}_N(l^-)\right)}{2}$$

$$\tilde{d}_{ijkl} = \frac{\text{Cut}_{ijkl}\left(\mathcal{A}_N(l^+)\right) - \text{Cut}_{ijkl}\left(\mathcal{A}_N(l^-)\right)}{2i\sqrt{V_4^2 - m_l^2}}$$

l-dependence of the box residue:
only one spurious term

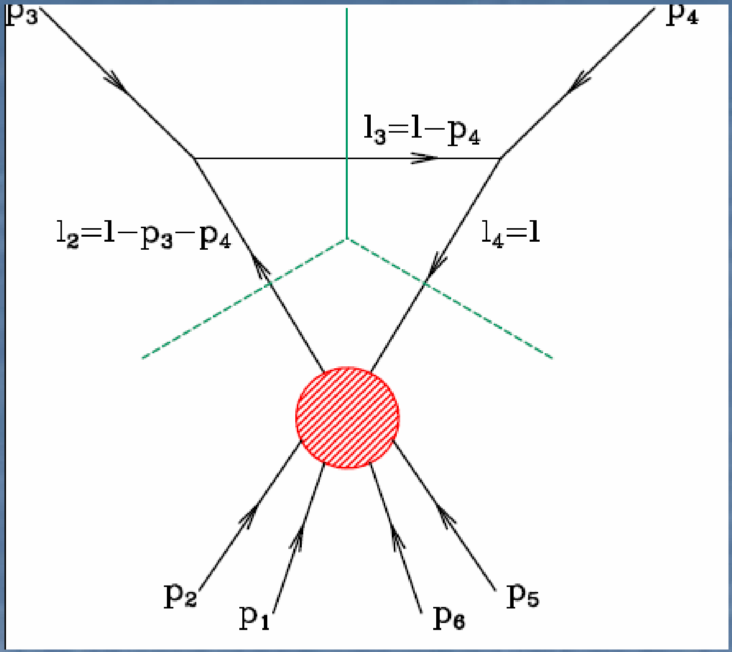
$$\bar{d}_{ijkl}(l) \equiv \bar{d}_{ijkl}(n_1 \cdot l)\,, \quad (n_1 \cdot l)^2 \sim n_1^2 = 1\,, \quad \bar{d}_{ijkl}(l) = d_{ijkl} + \tilde{d}_{ijkl}\, l \cdot n_1$$

the residue of the amplitude factorizes to the product of tree amplitudes

$$\text{Cut}_{2346}\left(\mathcal{A}_6(l^\pm)\right) = \mathcal{M}_4^{(0)}(l_6^\pm; p_1, p_2; -l_2^\pm) \times \mathcal{M}_3^{(0)}(l_2^\pm; p_3; -l_3^\pm)\mathcal{M}_3^{(0)}(l_3^\pm; p_4; -l_4^\pm)$$

$$\times \mathcal{M}_4^{(0)}(l_4^\pm; p_5, p_6; -l_6^\pm)$$

Calculating the triangle residue



l-dependence of the triangle residue

$$\bar{c}_{ijk}(l) \equiv \bar{c}_{ijk}(s_1, s_2); \;\; s_1 = n_1 \cdot l, \; s_2 = n_2 \cdot l$$

Maximum rank: 3 -> 10 possible terms

$$\{1, s_1, s_2, s_1^2, s_1 s_2, s_2^2, s_1^3, s_1^2 s_2, s_1 s_2^2, s_2^3\}$$

One constraint : $s_1^2 + s_2^2 \sim n_1^2 + n_2^2 = 2$

$C^{(0)}$ + six spurious terms

$$\bar{c}_{ijk}(l) = c_{ijk}^{(0)} + c_{ijk}^{(1)} s_1 + c_{ijk}^{(2)} s_2 + c_{ijk}^{(3)} (s_1^2 - s_2^2) + s_1 s_2 (c_{ijk}^{(4)} + c_{ijk}^{(5)} s_1 + c_{ijk}^{(6)} s_2)$$

the residue of the amplitude factorizes to the product of tree amplitudes

$$\mathrm{Cut}_{234}\left(\mathcal{A}_6(l^{\alpha_1\alpha_2})\right) = \mathcal{M}_3^{(0)}(l_2^{\alpha_1\alpha_2}; p_3; -l_3^{\alpha_1\alpha_2}) \times \mathcal{M}_3^{(0)}(l_3^{\alpha_1\alpha_2}; p_4; -l_4^{\alpha_1\alpha_2})$$

$$\times \mathcal{M}_6^{(0)}(l_4^{\alpha_1\alpha_2}; p_5, p_6, p_1, p_2; -l_2^{\alpha_1\alpha_2})$$

Bubble residue: $b^{(0)}$ + 8 spurious terms

Numerical Implementation

First application: calculation of 4, 5, 6 gluon amplitudes

Given set of external momenta -> v_i and n_i -> special loop momenta of generalized unitarity cut -> tree gluon amplitudes -> all coefficients d^{(0)}_{ijkl} …c^{(6)}_{ijk}..b^{(0)}_{ij} -> loop amplitudes

Check the singular parts:

$$m^{(1)}(1,2,\ldots,n) \sim -\frac{n}{\epsilon^2} \times m^{(0)}(1,2,\ldots,n) + \mathcal{O}\left(\epsilon^{-1}\right) .$$

I_3^{1m} and I_4 contributions

$$m^{(1)}(1,2,\ldots,n) \sim \left(-\frac{n}{\epsilon^2} + \frac{1}{\epsilon}\left(-\frac{11}{3} + \sum_{i=1}^{n} \log\left(\frac{s_{i,i+1}}{\mu^2}\right)\right)\right) \times m^{(0)}(1,2,\ldots,n) + \mathcal{O}(1)$$

all except I_3^{3m} contribute

Numerical Implementation

Comparison with previous results

i) known analytic results (Bern,Kosower, Britto; Feng, Mastrolia)
ii) known semi-numerical results (IBP) (Ellis, Giele, Zanderighi)

100000 points are generated away from soft and collinear region.
Cuts on transverse momenta, rapidity and separation of the outgoing gluons

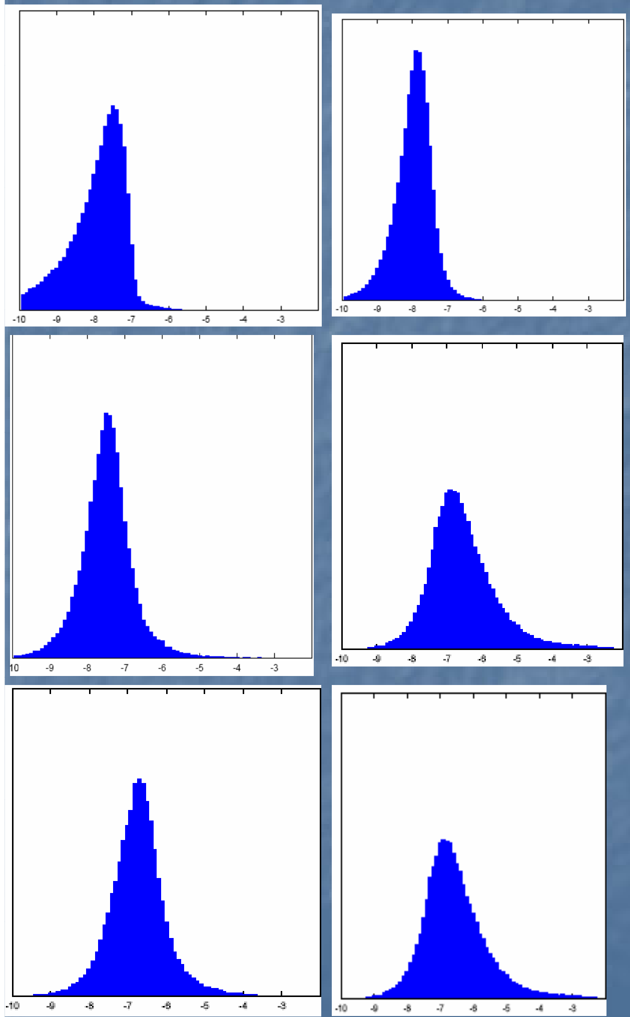
EGZ: 9s per ordered amplitude on 2.8GHz Pentium processor
EGK: 0.01s per ordered amplitude on 2.8GHz Pentium processor

	ev.time	# of cuts
4 gluon:	0.0009s	6
5 gluon:	0.0035s	20
6 gluon :	0.0107s	44

Computer time: scales with $\approx$ n^4 (# of cuts) not as n!

Relative errors for 100000 ordered amplitudes

4-gluon 6-gluon



Horizontal axis: $$S = \log_{10}\left(\left|\frac{m^{(1)}_{\text{unitarity}} - m^{(1)}_{\text{analytic}}}{m^{(1)}_{\text{analytic}}}\right|\right).$$

Range of S: (-10, -2)

Vertical axis: number of events

Majority of events agree with rel. precision 10^{-6} or better

Numerical instabilities

1) Matrix inversion needed for the calculation of triangle and bubble spurious contributions has numerical instabilities.

Possible improvements: χ^2 fit to a large number of points in the space of solutions of the unitarity constraints
Forde's method?

2) Presence of Gram-determinants in the box, triangle and bubble coefficients.
In the solution of the unitarity constraints we have maximum power: -2

3) Decomposition to the given integrals basis may become degenerate
Change the integration basis?

Concluding Remarks

- We have developed a numerical 4D unitarity cut method using the van Neerven-Vermaseren basis for calculating the cut-constructible part of one-loop amplitudes
  - it appears to be competitively fast
  - applicable also for amplitudes with massive internal and external lines
- Master integral and "spurious" coefficients are calculable in terms of factorized product of tree-level amplitudes. Existing tree-level numerical programs for 5, 6,… legs amplitudes can be upgraded to one-loop level programs.
- Numerical instabilities have to be better understood.
- Improvements needed:

  rational part and integration over the external phase space