$B_{(s)} \longrightarrow D_{(s)}\ell\nu$ form factors through the Step Scaling Method

Nazario Tantalo

INFN sez. "Tor Vergata"

Centro Ricerche e Studi "E. Fermi"

08-02-2007

outline

- introduction
- the step scaling method
- parametrization of the form factors
- lattice calculation of the form factors
- numerical results
- conclusions
- outlooks

the differential decay rate for the process $B_{(s)} \longrightarrow D_{(s)}\ell\nu$ is given by

$$\frac{d\Gamma(B \longrightarrow D\ell\nu)}{d\omega} = (\text{known factors})|V_{cb}|^2(\omega^2 - 1)^{\frac{3}{2}} F_D^2(\omega)$$

$$\omega = \frac{p_B \cdot p_D}{M_B M_D} = v_B \cdot v_D$$

- an accurate knowledge of the hadronic form factor $F_D^2(\omega)$ is required in order to extract V_{cb} from exclusive decays
- $F_D(\omega)$ approaches the Isgur-Wise limit as $m_b, m_c \longrightarrow \infty$

Q: is there still room for a quenched calculation?

- heavy–light systems are challenging; on currently affordable lattice sizes (at least in unquenched simulations) one has

$$am_b > 1 \quad Lm_d > 1 \qquad \text{or} \qquad am_b < 1 \quad Lm_d < 1$$

- the Fermilab group has already carried out quenched,

  S. Hashimoto et al Phys. Rev. D **66** (2002) 014503

  S. Hashimoto et al Phys. Rev. D **61** (2000) 014502

- and preliminary unquenched calculations of the form factors

  M. Okamoto et al Nucl. Phys. Proc. Suppl. **140** (2005) 461

Fermilab the Fermilab approach consists in simulating the following action with $am_0 > 1$

$$S = \sum_n \bar{\psi}_n \left[m_0 + \gamma_0 D_0 + \zeta \vec{\gamma} \cdot \vec{D} - r_t \frac{aD_0^2}{2} - r_s \frac{a\vec{D}^2}{2} + c_B \frac{i\sigma_{ij} F_{ij}}{4} + c_E \frac{i\sigma_{0i} F_{0i}}{2} \right] \psi_n$$

i.e. the Symanzik effective action for quarks with $|a\vec{p}| \ll 1$ with *mass dependent* coefficients usually computed perturbatively

A X El-Khadra et al Phys. Rev. D **55** (1997) 3933

S Aoki et al Prog. Theor. Phys. **109** (2003) 383

N H Christ et al hep-lat/0608006

- the unquenched results have been carried out by using "Rooted Staggered fermions"
- staggered fermions are introduced on the lattice by simulating the following quark action (actually in its improved version):

$$\bar{\chi} D_{\text{stag}} \chi = \sum_n \bar{\chi}_n \left[\sum_\mu \frac{\eta_{n,\mu}}{2} \left(U_{n,\mu} \chi_{n+\mu} - U^\dagger_{n-\mu,\mu} \chi_{n-\mu} \right) + m_0 \chi_n \right]$$

affected by doubling, i.e. it has $2^4 = 16$ one-component fermions

rooting means that gauge configurations are generated according to the following partition function:

$$Z^{\text{root}}_{N_f=3} = \int DU e^{-S_g} \left\{ \det[D_{\text{stag}}(m_u)] \det[D_{\text{stag}}(m_d)] \det[D_{\text{stag}}(m_s)] \right\}^{1/4}$$

S. R. Sharpe@LATTICE 2006 [hep-lat/0610094]:

Q: **"Rooted staggered fermions: Good, bad or ugly?"**

A: **ugly**! in the sense that are affected by unphysical contributions at regulated stage that need a complicate analysis to be removed

SSM the Step Scaling Method has been introduced in order to deal with two–scale problems on the lattice

M Guagnelli et al Phys. Lett. B **546** (2002) 237

on a very general ground, it is based on a simple identity

$$\mathcal{O}(E_h, E_l, \infty) = \mathcal{O}(E_h, E_l, L_0) \underbrace{\frac{\mathcal{O}(E_h, E_l, 2L_0)}{\mathcal{O}(E_h, E_l, L_0)}}_{\sigma(E_h, E_l, L_0)} \underbrace{\frac{\mathcal{O}(E_h, E_l, 4L_0)}{\mathcal{O}(E_h, E_l, 2L_0)}}_{\sigma(E_h, E_l, 2L_0)} \dots$$

- and on a reasonable "phenomenological assumption", i.e finite volume effects are due to the low energy scale

$$\sigma(E_h, E_l, L) \simeq \sigma(E_l, L) \qquad \frac{\partial}{\partial(\frac{1}{E_h})}\sigma(E_h, E_l, L) \simeq 0 \quad E_h \gg E_l$$

- so, provided that $E_h \gg 4E_l$, one has

$$\mathcal{O}(E_h, E_l, \infty) \simeq \mathcal{O}(E_h, E_l, L_0) \quad \sigma(E_h/2, E_l, L_0) \quad \sigma(E_h/4, E_l, 2L_0) \quad \dots$$

- in the case of heavy-light systems the argument can be made rigorous by using HQET predictions
- let us take f_B as an example

G M de Divitiis et al Nucl. Phys. B **672** (2003) 372

D Guazzini et al PoS **LAT2006** (2006) 084

$$\sigma(m_h, m_d, L) = \frac{f_B^0(m_d, 2L)\left(1 + \frac{f_B^1(m_d,2L)}{m_h} + \dots\right)}{f_B^0(m_d, L)\left(1 + \frac{f_B^1(m_d,L)}{m_h} + \dots\right)} = \sigma^{\mathrm{stat}}(m_d, L)\left(1 + \frac{f_B^1(m_d, 2L) - f_B^1(m_d, L)}{m_h}\right)$$

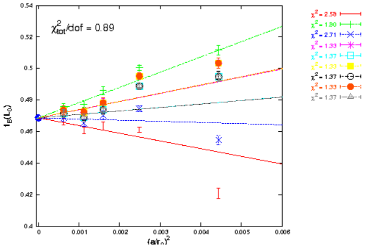
$$= \sigma^{\mathrm{stat}}(m_d, L)\left(1 + \frac{f_B^{1,1}(m_d)}{m_h L}\right)$$

- even better in the case of the meson masses (b–quark mass calculation)

M Guagnelli et al Nucl. Phys. B **675** (2003) 309

$$\sigma(m_h, m_d, L) = \frac{M(m_h, m_d, 2L)}{M(m_h, m_d, L)} = \frac{m_h + \bar{\Lambda}(m_d, 2L) + \dots}{m_h + \bar{\Lambda}(m_d, L) + \dots} = 1 + \frac{\bar{\Lambda}(m_d, 2L) - \bar{\Lambda}(m_d, L)}{m_h} + \dots$$

the step scaling method, does it works in practice?



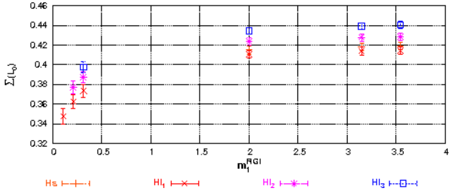
UNC: The calculation is quenched.

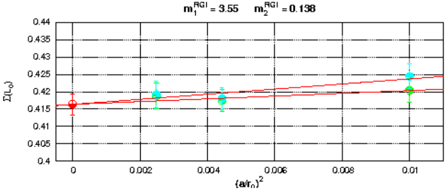
EFT: Fully non perturbative through SSM.

χE: The strange quark is under control.

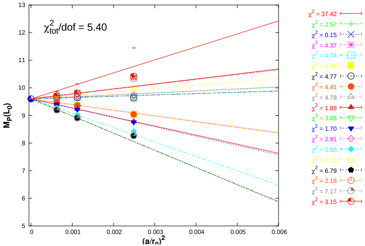
***a*E:** 4 lattice spacings.

***L*E:** Naturally estimated.





the step scaling method, does it works in practice?



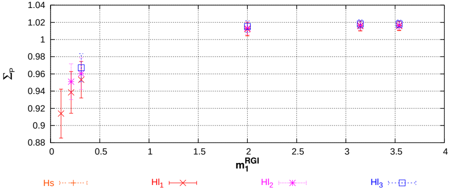
UNC: The calculation is quenched.

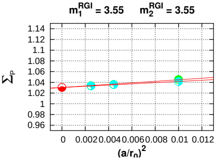
EFT: Fully non perturbative through SSM.

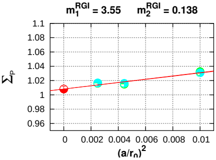
χE: The strange quark is under control.

***a*E:** 3 lattice spacings.

***L*E:** Naturally estimated.







$B_{(s)} \to D_{(s)}\ell\nu$, parametrization of the form factors

on the lattice one has to calculate the matrix element of the heavy–heavy vector current between the parent and daughter hadronic particles

$$\langle \mathcal{M}_2(p_2) \| \underbrace{\bar{h}_2(x)\gamma^\mu h_1(x)}_{V^\mu} \| \mathcal{M}_1(p_1) \rangle$$

these matrix elements can be parametrized in terms of two independent form factors

$$\omega = \frac{p_1 \cdot p_2}{M_1 M_2} = v_1 \cdot v_2$$

$$\langle \mathcal{M}_2(p_2) \| V^\mu \| \mathcal{M}_1(p_1) \rangle = f_+(\omega)(p_1^\mu + p_2^\mu) + f_-(\omega)(p_1^\mu - p_2^\mu)$$

$$\langle \mathcal{M}_2(p_2) \| V^\mu \| \mathcal{M}_1(p_1) \rangle = \sqrt{M_1 M_2}\,\{h_+(\omega)(v_1^\mu + v_2^\mu) + h_-(\omega)(v_1^\mu - v_2^\mu)\}$$

obviously the two parametrization are simply related each other

$$h_\pm(\omega) = \frac{(M_1 + M_2)f_\pm(\omega) + (M_1 - M_2)f_\mp(\omega)}{2\sqrt{M_1 M_2}} \qquad f_\pm(\omega) = \frac{(M_1 + M_2)h_\pm(\omega) - (M_1 - M_2)h_\mp(\omega)}{2\sqrt{M_1 M_2}}$$

$B_{(s)} \to D_{(s)}\ell\nu$, static limit of the form factors

HQET interactions at leading order (static theory) are blind with respect to the spin and flavour of the heavy quarks

as a consequence the semileptonic form factors reduce to a single universal function in this limit, the Isgur–Wise function $\xi(\omega)$:

$$\begin{cases} h_+(\omega) & \longrightarrow & \xi(\omega) \\ h_-(\omega) & \longrightarrow & 0 \end{cases} \qquad \begin{cases} f_+(\omega) & \longrightarrow & \frac{r+1}{2\sqrt{r}}\xi(\omega) \\ f_-(\omega) & \longrightarrow & \frac{r-1}{2\sqrt{r}}\xi(\omega) \end{cases}$$

where the limit $m_{h_1}, m_{h_2} \to \infty$ has been taken fixing the ratio $r = m_{h_2}/m_{h_1}$

the form factor appearing in the differential decay rate is

$$G(\omega) = F_D(\omega) \quad = \quad \frac{2\sqrt{r}}{r+1} f_+(\omega) \quad = \quad h_+(\omega) - \left(\frac{1-r}{r+1}\right) h_-(\omega) \quad \longrightarrow \quad \xi(\omega)$$

lattice computation of the matrix elements, continuous momenta

G. M. de Divitiis et al Phys. Lett. B **595** (2004) 408
ZeRo Collaboration Nucl. Phys. B **664** (2003) 276
P. F. Bedaque Phys. Lett. B **593**, 82 (2004)
C. T. Sachrajda et al Phys. Lett. B **609**, 73 (2005)
many others

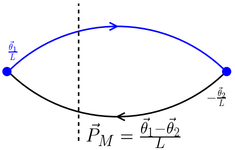
having in mind this computation we experimented flavour–twisted boundary conditions in order to have a continuous momentum transfer in between one–particle states

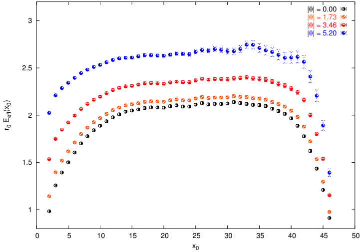
$$\psi(x + \mathbf{e}_i L) = e^{i\theta_i}\psi(x) \qquad \theta_0 = 0$$

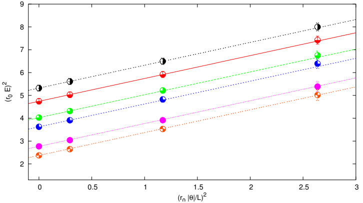
$$\int d\mathbf{p}\ e^{i\mathbf{p}\cdot(\mathbf{x}+\mathbf{e}_i L)}\psi(t;\mathbf{p}) = \int d\mathbf{p}\ e^{i(\mathbf{p}\cdot\mathbf{x}+\theta_i)}\psi(t;\mathbf{p})$$

$$e^{ip_i L} = e^{i\theta_i}$$

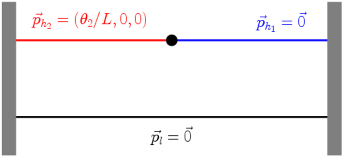
$$p_i = \frac{\theta_i}{L} + \frac{2\pi n}{L}, \quad n \in Z^3$$







- let us introduce the following correlation functions in the Schrödinger Functional regularization



$$\mathcal{O}_1 = \frac{a^6}{L^3} \sum_{\mathbf{y},\mathbf{z}} \bar{\zeta}_{h_1}(\mathbf{y};\theta_1)\gamma^5 \zeta_l(\mathbf{z})$$

$$\mathcal{O}_2 = \frac{a^6}{L^3} \sum_{\mathbf{y}',\mathbf{z}'} \bar{\zeta}_l(\mathbf{y}')\gamma^5 \zeta_{h_2}(\mathbf{z}';\theta_2)$$

$$V^\mu(x) = \bar{\psi}_{h_2}(x;\theta_2)\gamma^\mu \psi_{h_1}(x;\theta_1)$$

$$f_{\mathcal{M}_2 V^\mu \mathcal{M}_1}(x_0;\mathbf{p}_1,\mathbf{p}_2) = \frac{a^3}{2} \sum_{\mathbf{x}} \langle\, \mathcal{O}_2\ V^\mu(x)\ \mathcal{O}_1 \,\rangle$$

- by assuming single state dominance one gets

$$f_{\mathcal{M}_2 V^\mu \mathcal{M}_1}(x_0;\mathbf{p}_1,\mathbf{p}_2) \simeq \frac{\rho_1(\mathbf{p}_1)\rho_2(\mathbf{p}_2)}{4\sqrt{E_1 E_2}} \ \langle \mathcal{M}_2(\mathbf{p}_2) \| \, V^\mu \, \| \mathcal{M}_1(\mathbf{p}_1) \rangle \ e^{-x_0 E_1} e^{-(T-x_0)E_2}$$

"single ratios": the crucial observation is that, by the conservation of the vector current one gets:

$$f_{\mathcal{M}V^0\mathcal{M}}(x_0;\mathbf{p},\mathbf{p}) \simeq \frac{\rho(\mathbf{p})^2}{4E} \underbrace{2E}_{\langle\mathcal{M}(\mathbf{p})\|V^0\|\mathcal{M}(\mathbf{p})\rangle} e^{-TE}$$

so that the matrix elements are given by (renormalization factors cancel in the ratio)

$$\langle\mathcal{M}_2(\mathbf{p}_2)\|\, V^\mu \,\|\mathcal{M}_1(\mathbf{p}_1)\rangle = 2\sqrt{E_1E_2}\,\frac{f_{\mathcal{M}_2V^\mu\mathcal{M}_1}(x_0;\mathbf{p}_1,\mathbf{p}_2)}{\sqrt{f_{\mathcal{M}_2V^0\mathcal{M}_2}(x_0;\mathbf{p}_2,\mathbf{p}_2)f_{\mathcal{M}_1V^0\mathcal{M}_1}(x_0;\mathbf{p}_1,\mathbf{p}_1)}}$$

furthermore, in the mass diagonal case the form factors reduce to a single one

$$\mathbf{p}_1 = \mathbf{0} \qquad \mathbf{p}_2 = (\theta_2/L,0,0)$$

$$\omega = \frac{p_2\cdot p_1}{M_2M_2} = \frac{E_2}{M_2}$$

$$\left|\frac{f_{\mathcal{M}_2V^1\mathcal{M}_2}(x_0;\mathbf{p}_2,\mathbf{0})}{f_{\mathcal{M}_2V^0\mathcal{M}_2}(x_0;\mathbf{p}_2,\mathbf{0})}\right| = \frac{\sqrt{\omega^2-1}}{\omega+1}$$

"double ratios": our technique gives the same level of accuracy of the "double ratios" technique previously introduced by the Fermilab group:

$$R_0(x_0) = \frac{f_{\mathcal{M}_2 V^0 \mathcal{M}_1}(x_0; \mathbf{0}, \mathbf{0}) f_{\mathcal{M}_1 V^0 \mathcal{M}_2}(x_0; \mathbf{0}, \mathbf{0})}{f_{\mathcal{M}_2 V^0 \mathcal{M}_2}(x_0; \mathbf{0}, \mathbf{0}) f_{\mathcal{M}_1 V^0 \mathcal{M}_1}(x_0; \mathbf{0}, \mathbf{0})} \simeq |h_+(\omega = 1)|$$

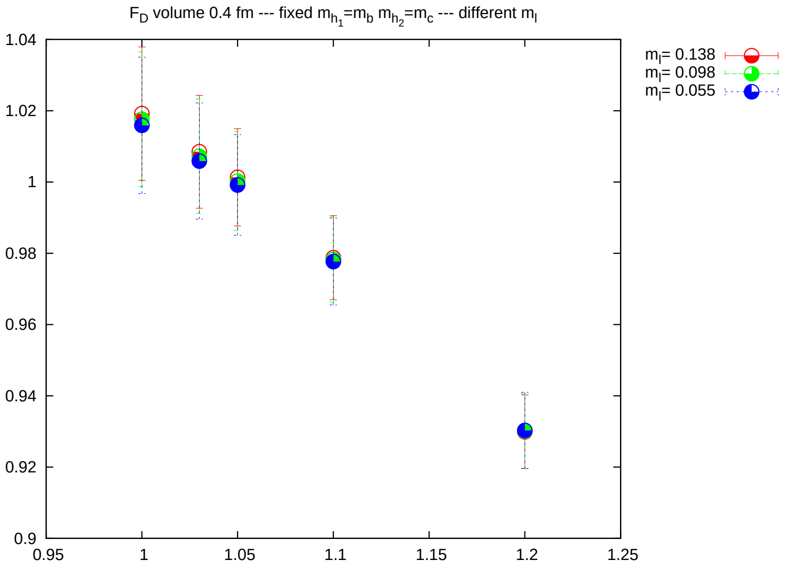
$$R_k(x_0, \mathbf{p}_2) = \frac{f_{\mathcal{M}_2 V^k \mathcal{M}_1}(x_0; \mathbf{p}_2, \mathbf{0}) f_{\mathcal{M}_2 V^0 \mathcal{M}_2}(x_0; \mathbf{p}_2, \mathbf{0})}{f_{\mathcal{M}_2 V^0 \mathcal{M}_1}(x_0; \mathbf{p}_2, \mathbf{0}) f_{\mathcal{M}_2 V^k \mathcal{M}_2}(x_0; \mathbf{p}_2, \mathbf{0})} \simeq$$

$$\simeq \left[1 - \frac{h_-(\omega)}{h_+(\omega)}\right] \left[1 + \frac{h_-(\omega)}{2h_+(\omega)}(\omega - 1)\right] \qquad \omega \simeq 1$$

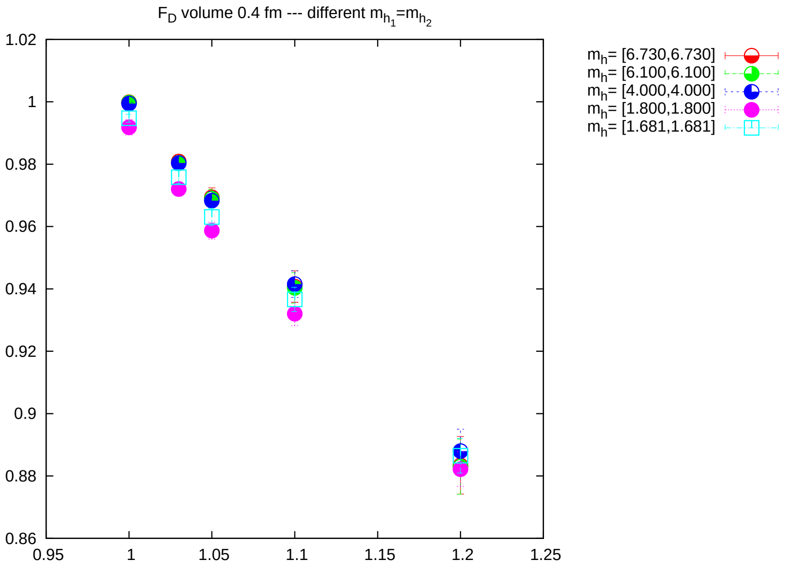
but "single ratios" work well also at $\omega \neq 1$

numerical results, small volume

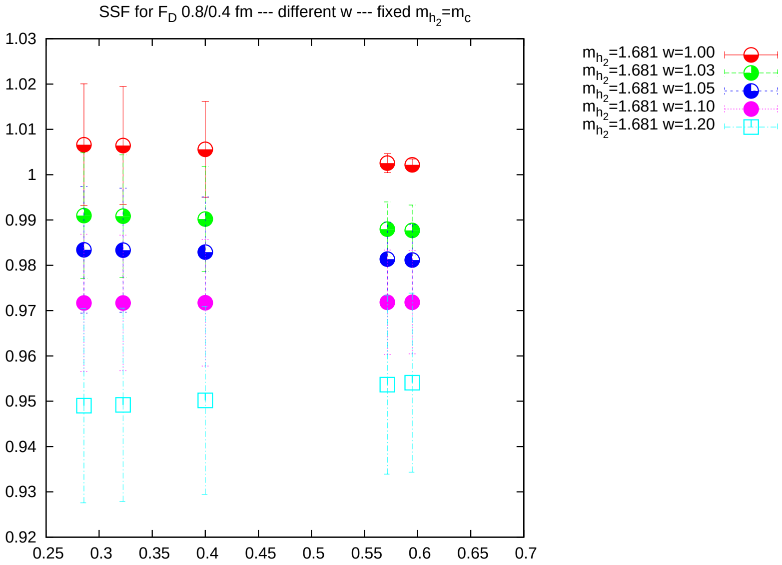
on the small volume, $L_0 = 0.4$ fm, we have $m_b = m_b^{phys}$:

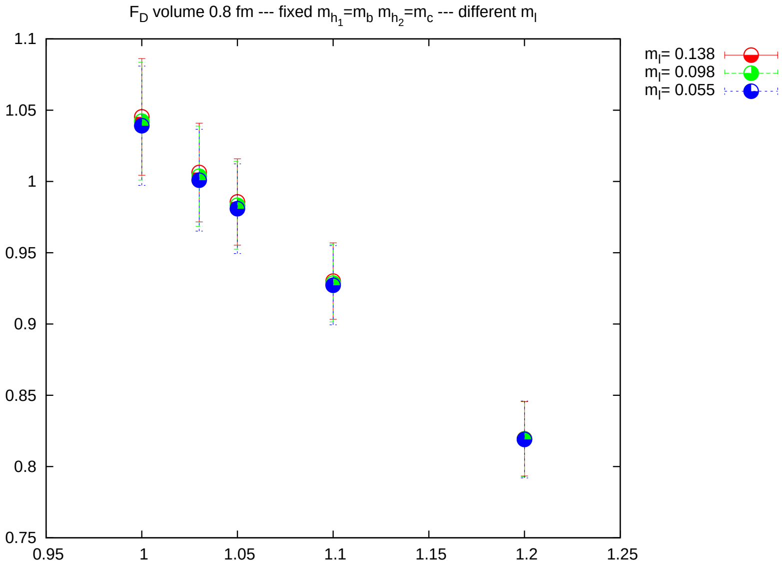


on the small volume, $L_0 = 0.4$ fm, we have $m_b = m_b^{phys}$:



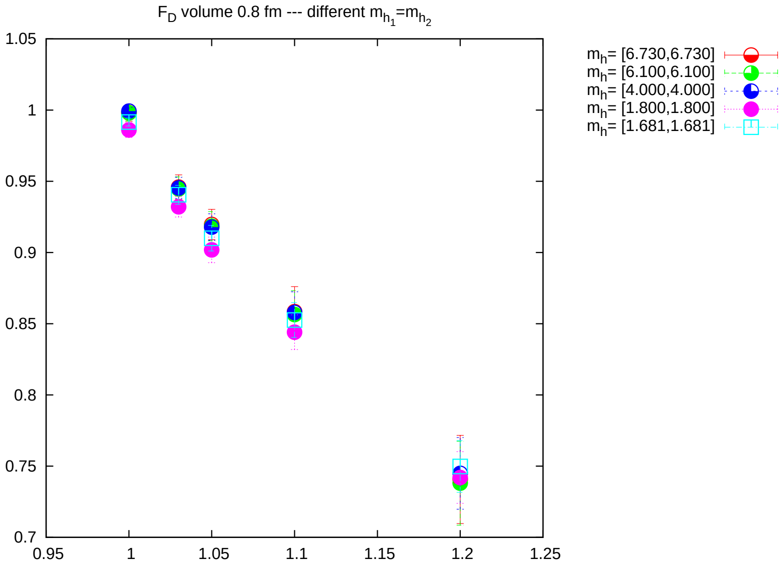
the step scaling functions are extremely flat



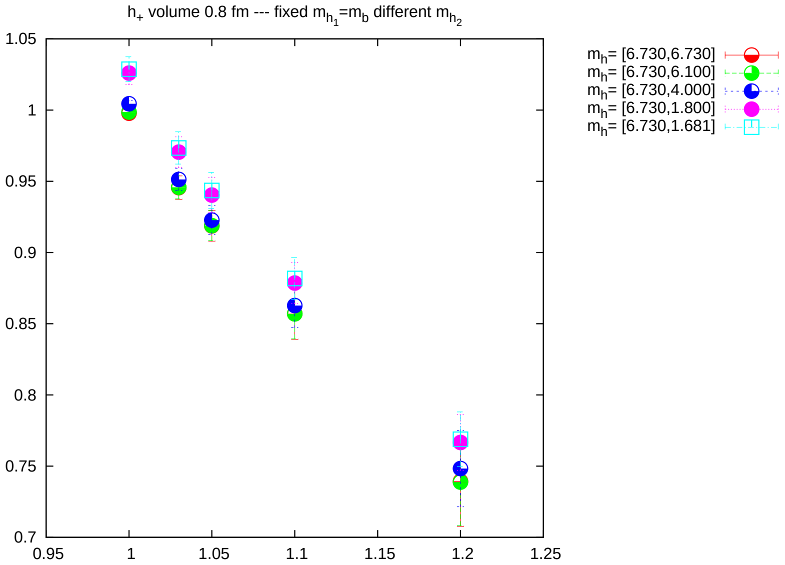


F_D volume 0.8 fm --- fixed m_h1=m_b m_h2=m_c --- different m_l
1.1
1.05
1
0.95
0.9
0.85
0.8
0.75
0.95
1
1.05
1.1
1.15
1.2
1.25
m_l= 0.138
m_l= 0.098
m_l= 0.055

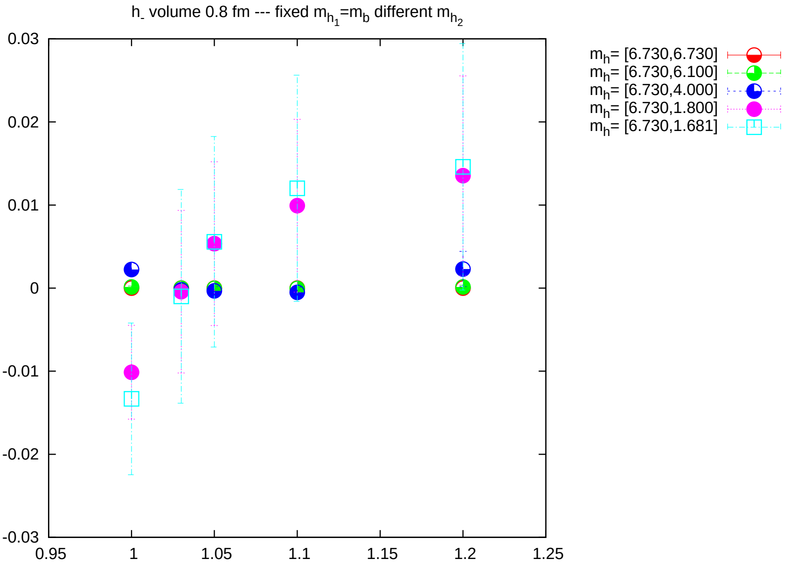
numerical results, volume 0.8 fm



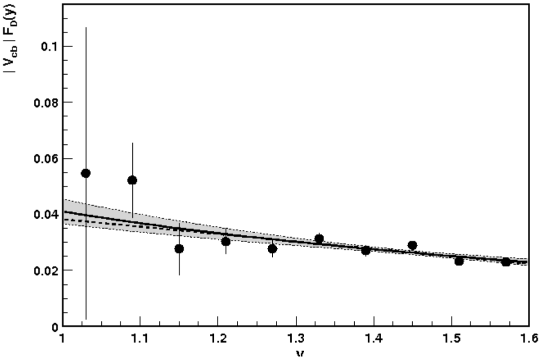
numerical results, volume 0.8 fm

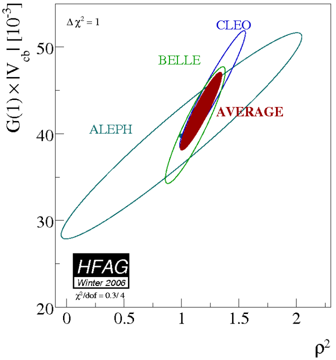


numerical results, volume 0.8 fm

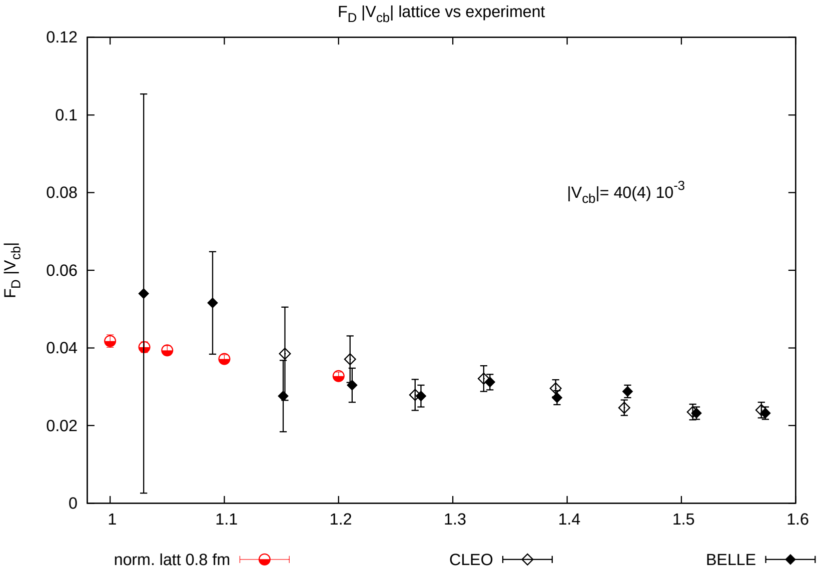


experimental situation

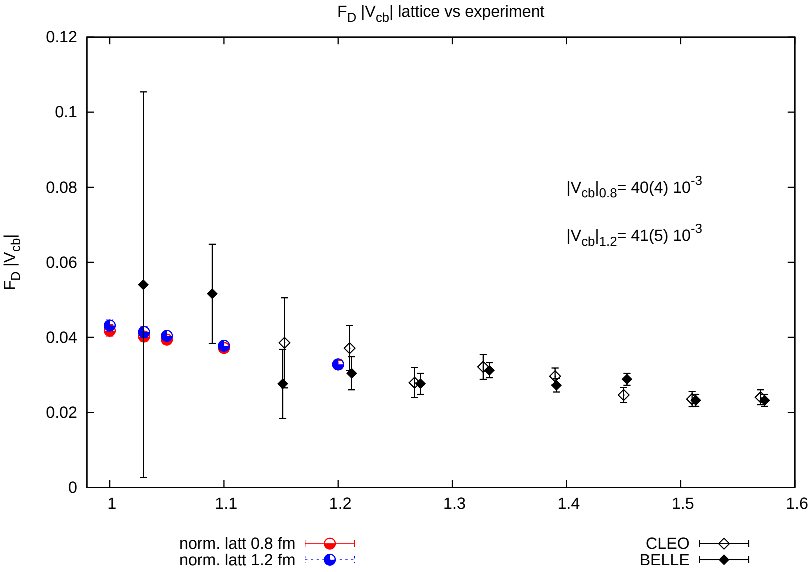




experimental situation vs lattice at $\omega > 1$



experimental situation vs lattice at $\omega > 1$ another small step?



conclusions

- the step scaling method has been shown to work also in the case of matrix elements between one–particle states
- we have extracted the relativistic heavy–heavy form factors at $\omega > 1$ with a numerical precision that is comparable with previous lattice calculations at $\omega = 1$
- after having verified that residual finite volume effects are negligible we plan to extend the technique to $B_{(s)} \longrightarrow D^{\star}_{(s)}\ell\nu$ (matrix elements already calculated. . .)
- bag parameters?
- unquench all. . .

we have generated a big set of $N_f = 2$ gauge configurations on big volumes and in the chiral regime:
L. Del Debbio, L. Giusti, M. Lüscher, R. Petronzio, N. T.

Table 1. Lattice parameters and simulation statistics

Run	Lattice	β	c_{sw}	κ_{sea}	N_{trj}	N_{sep}	N_{cfg}
A_{1a}	32×24^3	5.6	0	0.15750	6300	100	64
A_{1b}				0.15750	5070	30	169
A_2				0.15800	10800	100	109
A_{3a}				0.15825	6100	100	62
A_{3b}				0.15825	3800	100	38
A_4				0.15835	4950	50	100
B_1	64×32^3	5.8	0	0.15410	5050	50	100
B_2				0.15440	5200	50	101
B_3				0.15455	5150	50	104
B_4				0.15462	5050	50	102
C_1	64×24^3	5.6	0	0.15800	3450	30	116
D_1	48×24^3	5.3	1.90952	0.13550	5150	50	104
D_2				0.13590	5130	30	171
D_3				0.13610	5040	30	168
D_4				0.13620	5010	30	168
D_5				0.13625	5040	30	169
E_1	64×32^3	5.3	1.90952	0.13550	5344	32	168
E_2				0.13590	5024	32	158
E_3				0.13605	5024	32	158

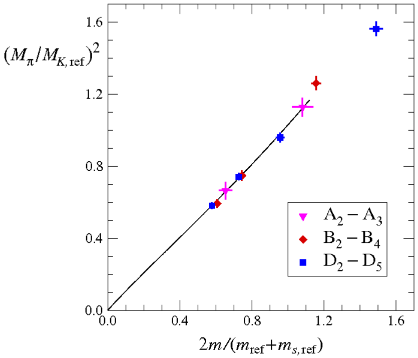
A: $a = 0.0717(15)$ fm $L = 1.721(36)$ fm

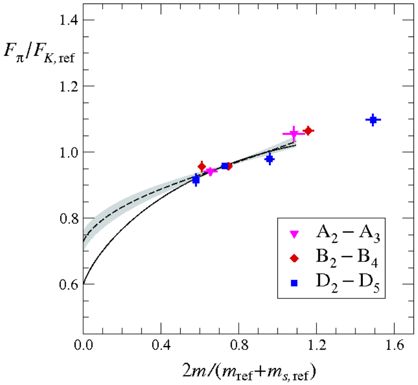
B: $a = 0.0521(07)$ fm $L = 1.667(22)$ fm

D: $a = 0.0784(10)$ fm $L = 1.882(24)$ fm

E: $a \simeq 0.078$ fm $L \simeq 2.5$ fm these configurations have been used in order to study the dependence of the mass and decay constant of the "pion" as a function of the sea quark mass and make contact with chiral perturbation theory

L. Del Debbio et al hep-lat/0610059 (accepted JHEP)
L. Del Debbio et al hep-lat/0701009 (accepted JHEP)





outlooks

- in the forthcoming months we plan to apply the step scaling method in the unquenched case ($N_f = 2$) in order to compute m_b, f_B, V_{cb}, V_{ub}, B_B and renormalization factors (structure functions, etc.).

- the idea is to use the set of gauge configurations already generated in collaboration with Del Debbio et al. as the big volume

- and generate the small volumes after performing the appropriate matching of the parameters (a, m_{sea}, etc.)

outlooks

- in the forthcoming months we plan to apply the step scaling method in the unquenched case ($N_f = 2$) in order to compute m_b, f_B, V_{cb}, V_{ub}, B_B and renormalization factors (structure functions, etc.).
- the idea is to use the set of gauge configurations already generated in collaboration with Del Debbio et al. as the big volume
- and generate the small volumes after performing the appropriate matching of the parameters (a, m_{sea}, etc.)

Q: how much it will cost?

A1: the calculation of the observables on the "big volumes" (64×32^3) takes (CGNE even/odd preconditioned)

$$\text{single propagator} \simeq 2\ \text{h/crate}$$

$$\text{observables} \simeq 10 \times \text{single propagator} \times N_{cnfg} \simeq 3\ \text{y/crate}$$

A2: the generation of the small volume gauge configurations is cheep w.r.t. than in the big volume case (Schrödinger Functional cutoff $1/L$). The cost can be reasonably estimated to be the same as the calculation of the observables on the big volumes...