CUSP ANOMALY, INTEGRABILITY AND ADS/CFT

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ANOMALOUS DIMENSIONS AND SPIN CHAINS

HEISENBERG MODEL FOR THE SU(2) SECTOR

BETHE ANSATZ

PSU(2,2|4) SYMMETRY IN THE FULL N=4 SYM

BES/BHL "DRESSING" PHASE

INTEGRAL EQUATION

NUMERICAL STUDY

AdS/CFT, ANOMALOUS DIMENSIONS AND STRINGS

BASIC EXAMPLE:

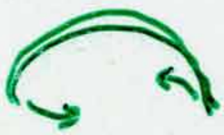
$\mathcal{N}=4$ SYM $\Longleftrightarrow$ $AdS_5 \times S^5$ TYPE IIB

SCALING DIMENSION OF GAUGE INVARIANT OPERATORS $\Longleftrightarrow$ ENERGIES OF STRING STATES

$\Delta(\lambda)$ $\qquad\qquad$ $E(\lambda)$

PERTURBATIVE IF $\lambda \ll 1$ $\qquad\qquad$ PERTURBATIVE IF $\lambda \gg 1$

$\mathrm{tr}(\cdots X Y D^2 Y X Z \cdots)$



SPINNING STRINGS

(WE WILL ALWAYS BE IN THE LARGE N LIMIT)

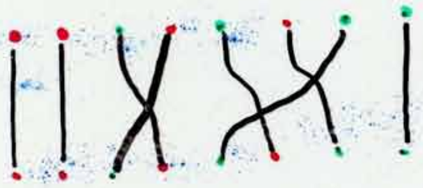
$\lambda = g_{YM}^2 N$

FROM CFT'S TO SPIN CHAINS

WE WANT TO COMPUTE
THE ANOMALOUS DIMENSION
OF OPERATORS LIKE

$$\mathrm{tr}(xxyxyxyy)$$

FEYNMAN
DIAGRAMS



WE CONSIDER THE QUANTUM MECHANICS OF J SPINS

$$\uparrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\downarrow\cdots$$

SU(2) SPIN CHAIN OF LENGTH J

HAMILTONIAN:

$$\mathcal{H} = \lambda \sum_{i=1}^{J} \left(\mathbb{1} - P_{i,i+1}\right) + \quad \leftarrow \text{1-LOOP: HEISENBERG MODEL}$$

$$+ \lambda^2 \sum \left(-4\,\mathbb{1} + 6 P_{i,i+1} - P_{i,i+1} P_{i+1,i+2} - P_{i+1,i+2} P_{i,i+1}\right)$$

$$+ \lambda^3 \ldots$$

AT ORDER λ^k THE INTERACTION LENGTH IS K

SPECTRUM OF SPIN CHAINS : BETHE ANSATZ

GROUND STATE: $\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow$

1^{ST} EXCITATIONS : $\uparrow\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow$ IMPURITY TRAVELS WITH MOMENTUM p
(SINGLE MAGNON)

$$E(p) = 4\sin^2(p/2) \qquad 0 \leq p < 2\pi$$

2-MAGNON STATES:

$$|\psi\rangle = \sum_{x_1, x_2} \psi(x_1, x_2)\, |\uparrow\uparrow\underset{x_1}{\downarrow}\uparrow\uparrow\uparrow\underset{x_2}{\downarrow}\downarrow\downarrow\rangle$$

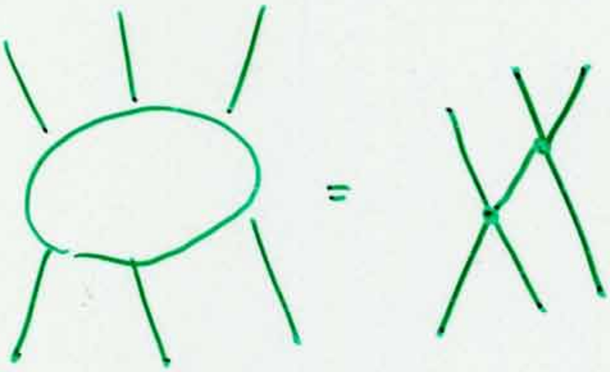
$$\psi(x_1, x_2) = e^{ip_1x_1 + ip_2x_2} + \underset{\substack{\uparrow \\ \text{S-MATRIX}}}{S(p_1,p_2)}\, e^{ip_1x_2 + ip_2x_1}$$

$$S(p_1,p_2) = \frac{2e^{ip_1} - e^{ip_1+ip_2} - 1}{2e^{ip_2} - e^{ip_1+ip_2} - 1}$$

THIS DESCRIBES 2→2 SCATTERING OF MAGNONS

MULTI-MAGNON STATES

IF THE HAMILTONIAN IS INTEGRABLE, THE SCATTERING OF MANY MAGNONS <u>FACTORIZES</u> IN $2 \to 2$ SCATTERING



THE ALLOWED MOMENTA ARE OBTAINED BY THE BETHE EQUATIONS

$$e^{ip_k L} = \prod_{\substack{j=1 \\ j \neq k}}^{M} S(p_k, p_j) \qquad k = 1 \dots M$$

FULL $N=4$ SYM

BEISERT 05

PSU(2,2|4) SYMMETRY

$16 = 8_B + 8_F$ EXCITATIONS

- DISPERSION RELATION:

$$E(p) = -1 + \sqrt{1 + 8\lambda \sin^2(p/2)}$$

VALID AT LARGE J
FOR ANY λ

- S-MATRIX $(16^2 \times 16^2)$

FIXED BY SYMMETRIES
MODULO A SCALAR FACTOR

$$\sigma(p_1, p_2; \lambda) = e^{i\theta}$$

"DRESSING" PHASE

MAIN REQUIREMENT: CROSSING SYMMETRY

JANIK 06

THE DRESSING PHASE

- UP TO 3 LOOPS: $\Theta = 0$
- IN CLASSICAL STRINGS: A.F.S.

ARUTYUNOV
FROLOV
STAUDACHER
06

$$\Theta(p_1,p_2) = 2\sum_{r=2}^{\infty} q_r(p_1)\, q_{r+1}(p_2) - q_r(p_2)\, q_{r+1}(p_1)$$

$$q_r(p) = \frac{2\sin((r-1)p/2)}{r-1}\left[\frac{-1+\sqrt{1+g^2\sin^2 p/2}}{2g\sin(p/2)}\right]^{r-1}$$

- STRING 1 LOOP HERNANDEZ-LOPEZ 06
- CONJECTURE FOR LARGE-λ ASYMPTOTIC EXPANSION BEISERT-HERNANDEZ-LOPEZ 06

$$\Theta = \sum_{r,s} c_{r,s}(g)\left[q_r(p_1)\, q_s(p_2) - q_r(p_2)\, q_s(p_1)\right]$$

- EXACT PROPOSAL BEISERT-EDEN-STAUDACHER 06

$$c_{r,s}(g) = 2\cos\left[\frac{\pi}{2}(s-r-1)\right](r-1)(s-1)\int_0^{\infty} \frac{J_{r-1}(2gt)\, J_{s-1}(2gt)}{t\,(e^t-1)}\, dt$$

THE CUSP ANOMALY

IN GAUGE THEORY WE CONSIDER
THE OPERATORS:

$$tr\left(\phi D^s \phi\right)$$

THEIR ANOMALOUS DIMENSION AT LARGE S IS

$$\Delta - S = f(g)\ln(S) + \dots$$

AT SMALL g

CUSP ANOMALY

$$f(g) = 8g^2 - \frac{8}{3}\pi^2 g^4 + \frac{88}{45}\pi^4 g^6 \dots$$

IN STRING THEORY WE CONSIDER
A FOLDED STRING ROTATING IN AdS_5

$$ds^2 = d\rho^2 - \cosh^2\rho\, dt^2 + \sinh^2\rho\, d\theta^2 \qquad (AdS_3)$$

$$\begin{cases} t = \kappa\tau \\ \theta \simeq \kappa\tau \\ \rho \simeq \sigma \end{cases} \qquad \kappa = \frac{1}{\pi}\ln\left(\frac{S}{\sqrt{\lambda}}\right)$$

GUBSER
KLEBANOV
POLYAKOV 02

FROLOV
TSEYTLIN 02

1-LOOP

2-LOOP

$$f(g) = 4g - \frac{3\ln(2)}{\pi} + O\left(\frac{1}{g}\right)$$

BES EQUATION

BASED ON THE BEHLS S-MATRIX,

B.E.S. STUDIED THE BETHE ANSATZ

FOR THE CUSP ANOMALY.

INTEGRAL EQUATION FOR $f(g)$:

$$f(g) = 16\, g^2\, \hat{\sigma}(0)$$

$$\hat{\sigma}(t;g) = \frac{t}{e^t - 1}\left(K(2gt, 0) - 4g^2 \int_0^\infty dt'\, K(2gt, 2gt')\, \hat{\sigma}(t') \right)$$

$$K(t,t') = K^m(t,t') + 2K^c(t,t')$$

$$K^m(t,t') = \frac{J_1(t)\, J_0(t') - J_0(t)\, J_1(t')}{t - t'}$$

$$K^c(t,t') = 4g^2 \int_0^\infty dt''\, K_1(t, 2gt'') \frac{t''}{e^{t''} - 1}\, K_0(2gt'', t')$$

$$K_0(t,t') = \frac{t\, J_1(t)\, J_0(t') - t'\, J_0(t)\, J_1(t')}{t^2 - t'^2}$$

$$K_1(t,t') = \frac{t'\, J_1(t)\, J_0(t') - t\, J_0(t)\, J_1(t')}{t^2 - t'^2}$$

PERTURBATIVE EXPANSION OF $f(g)$ FROM BES EQUATION

4-LOOP:
- BERN, CZAKON, DIXON, KOSOWER, SMIRNOV 06
- CACHAZO SPRADLIN VOLOVICH 06

$$f(g) = 8g^2 - \frac{8}{3}\pi^2 g^4 + \frac{88}{45}\pi^4 g^6$$
$$- 16\left(\frac{73}{630}\pi^6 + 4\,\zeta(3)^2\right) g^8$$
$$+ 32\left(\frac{887}{14175}\pi^8 + \frac{4}{3}\pi^2\,\zeta(3)^2 + 40\,\zeta(3)\,\zeta(5)\right) g^{10}$$
$$- \cdots\cdots$$

TRANSCENDENTALITY PRINCIPLE

(KOTIKOV LIPATOV)

THE TERM g^{2k} HAS TRANSCENDENTALITY $2k-2$

$$t(\pi^n) = n$$
$$t(\zeta(n)) = n$$

CONJECTURE FOR QCD:

HIGHER TRANSCENDENTALITY TERMS IN QCD COINCIDE WITH THE ABOVE EXPANSION.

QUANTITATIVE STUDY OF $f(g)$ AT FINITE g.

THE BES INTEGRAL EQUATION ALLOWS, FOR THE FIRST TIME, TO COMPUTE ANOMALOUS DIMENSIONS AT FINITE VALUES OF THE COUPLING

BENNA, S.B., KLEBANOV SCARDICCHIO 06

EXPANDING

$$\hat{\sigma}(t) = \frac{t}{e^t-1} \sum_{n=1}^{\infty} s_n(g) \frac{J_n(2gt)}{2gt}$$

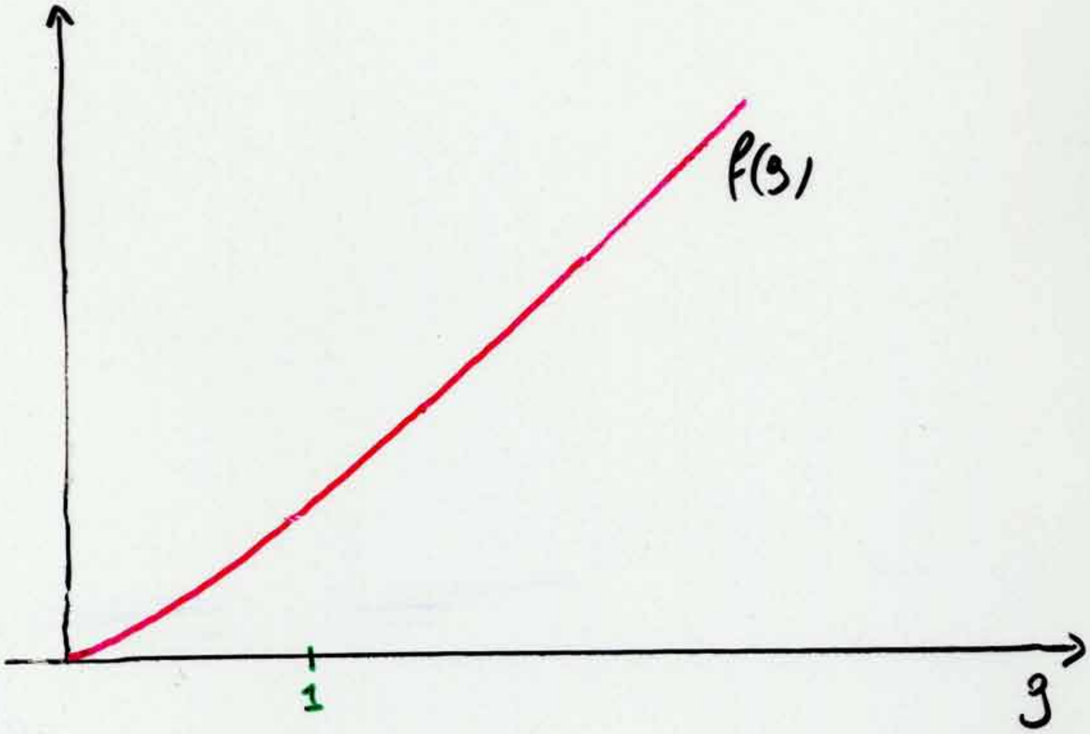
ONE OBTAINS A MATRIX EQUATION FOR $s_n(g)$

TO COMPUTE $s_n(g)$ at FINITE g IT IS ENOUGH TO CONSIDER FINITE MATRICES $K \times K$ WITH $K \sim g$

WITH THIS METHOD IT'S POSSIBLE TO COMPUTE

$$f(g) = 16\, g^2\, s_1(g)$$

NUMERICALLY



AT LARGE g:

$$f(g) = (4 \pm 10^{-6})\, g + $$ ← GKP

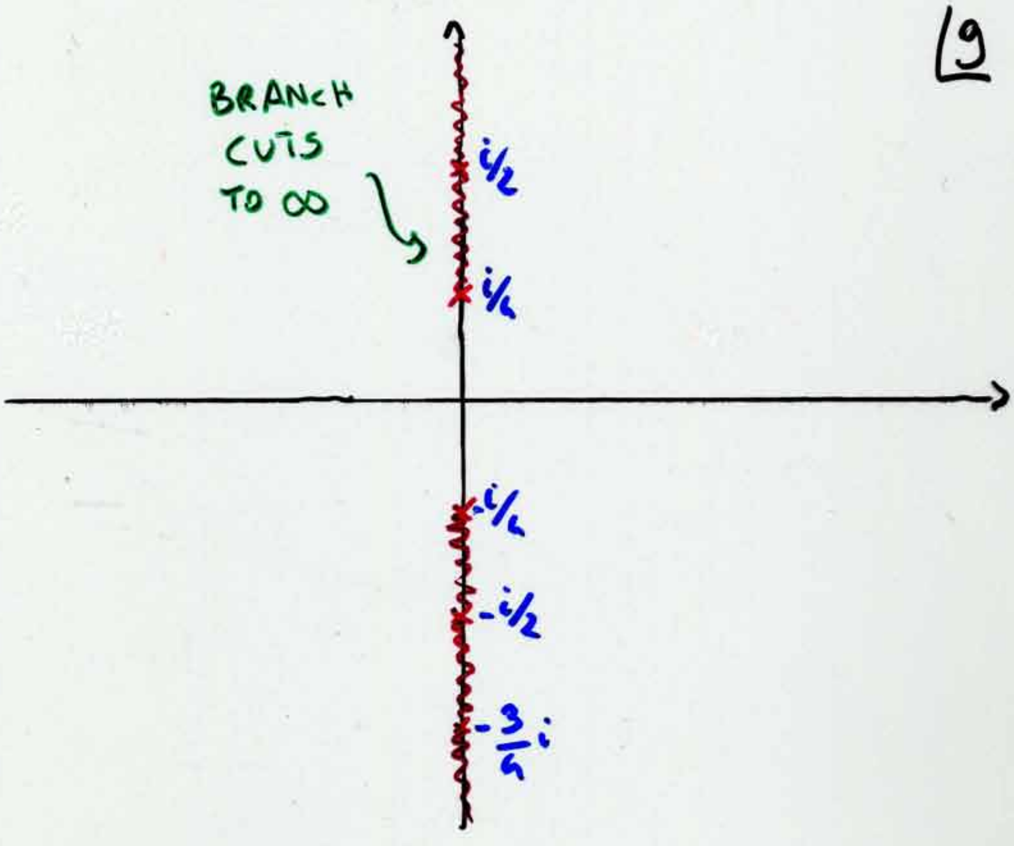
$$-\left(0.661907(2)\right) + $$ ← FROLOV-TSEYTLIN $3\frac{\ln(2)}{\pi}$

$$-\left(0.0232(1)\right)\frac{1}{g}$$ ← 2 LOOP PREDICTION

$4g$ TERM OBTAINED ANALYTICALLY — ALDAY, ARUTYUNOV, BENNA, EDEN, KLEBANOV 07

ANALYTIC STRUCTURE

- AT SMALL g, $f(g)$ HAS A FINITE RADIUS OF CONVERGENCE, $|g| < \frac{1}{4}$
- AT LARGE g THE EXPANSION IS ONLY ASYMPTOTIC. ESSENTIAL SINGULARITY.



SUMMARY

- SPECTRUM OF GAUGE THEORIES IN TERMS OF SPIN CHAINS
- BETHE ANSATZ FOR $\mathcal{N}=4$ SYM AND $AdS_5 \times S^5$
- CUSP ANOMALY: NON BPS OBSERVABLE

OPEN QUESTIONS

- FIND FURTHER EVIDENCE FOR THE BEHLS S-MATRIX
- STUDY ANALYTICAL STRUCTURE OF ANOMALOUS DIMENSIONS.
- FINITE-LENGTH EFFECTS: HAMILTONIAN?
- LESS SUPERSYMMETRIC THEORIES?