Isospin chemical potential in holographic "QCD"

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Introduction

- AdS/CFT $\longrightarrow$ integrability $\longrightarrow$ "high precision tests"

  "purest"

  $N = 4$ a perfect generator of a huge # of integrable struct.

- non-AdS/non-CFT $\longrightarrow$ (direct) applications to realistic gauge theories
  - zero temperature and chemical potentials ($T = 0, \mu = 0$)
    - glueball spectra — Csaki et al.
    - masses of hadrons (mesons) — Karch & Katz...
    - hadron form factors — Polchinski & Strassler
  - finite-temperature and $\mu = 0$ theories (viscosity of quark-gluon plasma) — Son, Starinets, ...
  - Finite chemical potential $\longleftarrow$ THIS TALK !

Setup: (I) Pure Glue

- pure QCD — i.e. no matter → do not know geometry

instead, consider 4+1 dim max. susy YM
compactify on circle
impose anti-periodic bdy. cond. for fermions

→

in IR, reduces to
pure QCD, scalars
and fermions decouple

dual to near-horizon geometry
of non-extremal D4-brane, doubly Wick rotated

The geometry

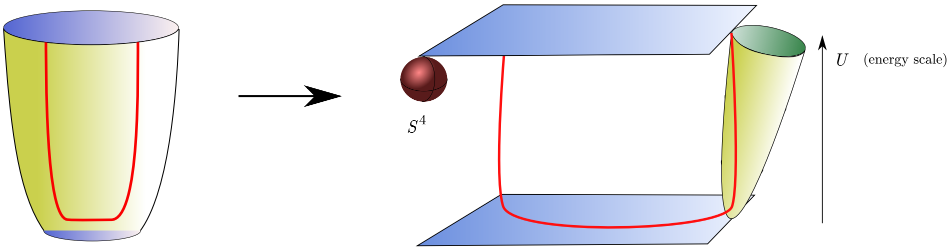
[Witten, Sakai & Sugimoto, . . .]

$$\mathrm{d}s^2 = \left(\frac{u}{R}\right)^{3/2}\left[\eta_{\mu\nu}\mathrm{d}X^\mu \mathrm{d}X^\nu + f(u)\mathrm{d}\theta^2\right] + \left(\frac{R}{u}\right)^{3/2}\left[\frac{\mathrm{d}u^2}{f(u)} + u^2 \mathrm{d}\Omega_4\right]$$

world-volume
our 3+1 world

$f(u) = 1 - \left(\frac{u_\Lambda}{u}\right)^3$
θ is a compact
Kaluza-Klein circle

u: radial direction
bounded from
below $u \geq u_\Lambda$



Several remarks

- Solution characterised by two parameters:

$$\begin{cases} R_{D4}: & R_{D4}^3 = \pi g_s l_s^3\, N_c \\ R: & R = \dfrac{2\pi}{3}\left(\dfrac{R_{D4}^3}{u_\Lambda}\right)^{1/2} \to M_\Lambda = \dfrac{2\pi}{R} \end{cases}$$

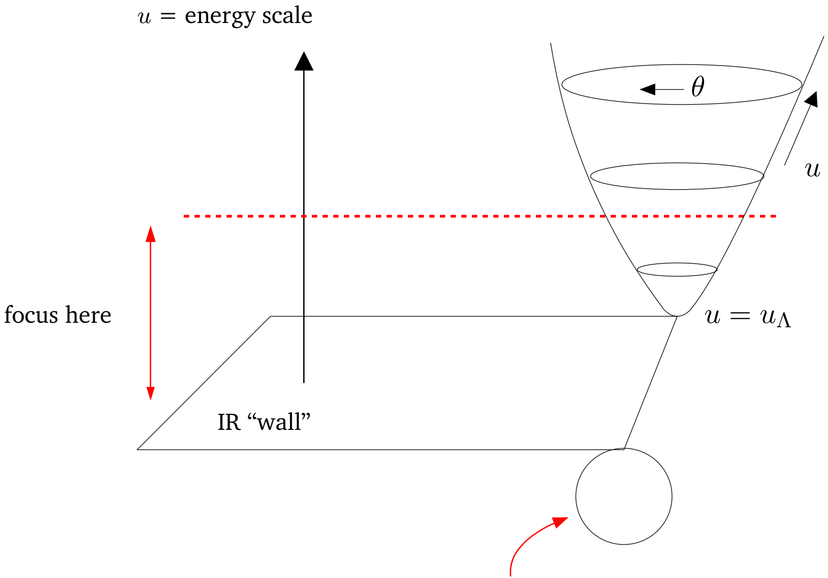
non-extremality of D-brane: angle θ identified with period R to avoid conical singularity

- Relation to gauge-theory parameters:
  - size of S^1 on D4 (i.e. M_{KK}) set by R
  - $\lambda \equiv g_{\mathrm{YM}}^2\, N_c = \dfrac{R_{D4}^3}{\alpha'\, R}$

- Regime of validity:
  - sugra OK if $\mathcal{R}^2 \equiv R_{D4}^3/R \gg \alpha'$ $\longrightarrow$ $\lambda \gg 1$
    (max curvature at the wall)
  - valid as long as $e^{\phi} = g_s(u/R_{D4})^{3/4} < 1$
    (min coupling at the wall)

- Problem : $M_{\mathrm{KK}} \sim M_{\text{glueball}} \sim M_{\text{meson}} \sim M_\Lambda \sim 1/R$ $\longrightarrow$ cannot decouple KK modes !

Overview



$$\left(\frac{R_{\rm D4}}{u_\Lambda}\right)^{3/2} u_\Lambda^2 = R_{\rm D4}^{3/2} u_\Lambda^{1/2}$$

Setup : (II) Introducing matter–Sakai-Sugimoto model

- Add D8 flavour (probe) branes to D4 stack

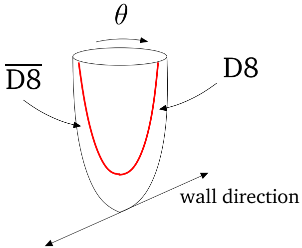
$\longrightarrow$ strings between flavour & colour branes in fund. rep. of flavour & colour group

- Solve for the shape of the D8

$$D4: \quad 0 \;\; 1 \;\; 2 \;\; 3 \quad \theta \quad - \;\; - \;\; - \;\; - \;\; -$$

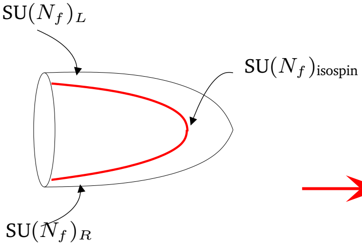
$$D8: \quad \underbrace{0 \;\; 1 \;\; 2 \;\; 3}_{\text{flat}} \quad \underbrace{-}_{\text{curve } u(\theta)} \quad \underbrace{5 \;\; 6 \;\; 7 \;\; 8 \;\; 9}_{\text{coord. sys. adapted to D8 } \theta,\, \Omega^4}$$

Solution to the 1st order equation
gives embedding $u(\theta)$



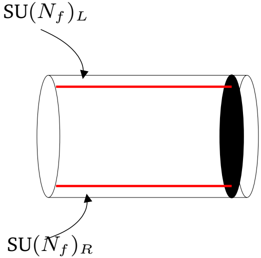
Symmetry encoded in geometry

- Asymptotically exhibits full chiral symmetry $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$
- Bending of the brane encodes spontaneous symmetry breaking in gauge theory in a geometrical way



$\longrightarrow$ spectrum of fluctuations contains $(\pi^{\pm}, \pi^0)$ Goldstone bosons

- Brane geometry also reproduces chiral symmetry restoration above $T > T_c$

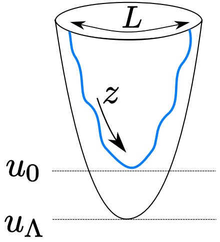


Low spin mesons

- Spectrum is known only in the limits:

Low-spin mesons:

fluctuations on and of the flavour brane



- Fluctuations governed by Dirac-Born-Infeld action of the flavour brane

$$S = V_{S^4} \int \mathrm{d}^5x\, e^{-\phi} \sqrt{-\det\left(g_{\mu\nu} + 2\pi\alpha' F_{\mu\nu}\right)} + S_{\text{Wess-Zumino}}$$

$$= V_{S^4} \int \mathrm{d}^4x\, \mathrm{d}z\, \sqrt{-g}\, F_{\mu\nu} F_{\rho\lambda}\, g^{\mu\rho} g^{\nu\lambda} + \ldots$$

- Expand world-volume fields in modes $\longrightarrow$ meson spectrum & action

Effective action for light mesons

- Decompose the gauge fields

$$F_{\mu\nu} = \sum_n G^{(n)}_{\mu\nu}(x)\,\psi_{(n)}(u)\,,$$

$$F_{u\mu} = \sum_n B^{(n)}_{\mu}(x)\,\partial_u\psi_{(n)}(u)\,,$$

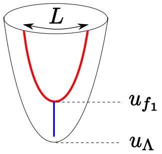
- Fourier transform & factor out polarisation vectors,

$$\int \mathrm{d}^4k\,\tilde{B}^{(m)}_{\mu}\,\tilde{B}^{(n)}_{\mu}\,\underbrace{\left[u^{-1/2}\gamma^{1/2}(\omega^2-\vec{k}^{\,2})\psi_{(n)}-\partial_u\left(u^{5/2}\gamma^{-1/2}\partial_u\psi_{(n)}\right)\right]}\,=0\,.$$

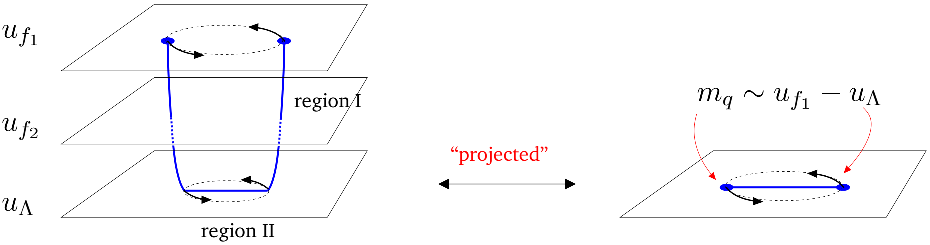
$\rightarrow$ a Sturm-Liouville problem
mass spectrum of mesons

High spin mesons

- Spectrum is known only in the limit:
- Sigma model (semiclass) $\longrightarrow$ high-spin glueballs (closed) & mesons (open)



$q\bar{q}$ meson:



N.B. High spin mass $M_{\text{high}} \sim \sqrt{\lambda} M_\Lambda$ vs. low spin mass $M_{\text{low}} \sim M_\Lambda \sim M_{KK}$

Part II:

Turning on an *isospin* chemical potential Chiral Langrangian

Isospin vs Baryon chemical potential

- Why isospin chemical potential is easier in holographic models than baryon chemical potential:

  - large N_c $\longrightarrow$ baryons much heavier than at finite N_c
    mesons closer to the real-world
  - baryons complicated solitons, mesons elementary fields
  - so far only singular solitons known
  - potentially comparable with the lattice (no sign problem)

Bad feature: Artificial, no pure isospin systems exist in nature (weak decays)
neutron stars

Chiral Lagrangian

- At small μ_I $\longrightarrow$ chiral Lagrangian (with $m_q = 0$) to get a feeling what happens

$$\mathcal{L}_{\text{chiral}} = \frac{f_\pi^2}{4} \mathrm{Tr}(D_\nu U D^\nu U^\dagger)\,, \qquad U \in \mathrm{U}(N_f)\,.$$

$$U \equiv e^{\frac{i}{f_\pi}\pi_a(x)T^a} \qquad T_a - -U(N_f) \text{ generators}$$

  Invariant under *separate*

$$U \to g_L^{-1} U\,, \quad U \to U g_R$$

- The vacuum $U = I$ preserves the vector-like $U(N_f)$ symmetry,

$$U \to g_L U g_R^{-1} \qquad \longrightarrow \qquad g_L = g_R\,.$$

  In $U = I$ want to turn on a vector chemical potential $\mu_L = \mu_R$.

- Other global transformations move us around on the moduli space of vacua,

$$\mathcal{M} = \frac{U(N_f) \times U(N_f)}{U(N_f)}$$

Chiral Lagrangian and $\mu \neq 0$

- As usual, chemical potentials via

$$D_\nu U = \partial_\nu U - \frac{1}{2}\delta_{\nu,0}(\mu_L U - U\mu_R) = \partial_\nu U - \frac{1}{2}\delta_{\nu,0}([\mu_V, U] - \{\mu_A, U\})$$

$(\mu_L = \mu_V - \mu_A,\ \mu_R = \mu_V + \mu_A)$.

$\longrightarrow$

$$V_\chi = \frac{f_\pi^2}{4}\,\mathrm{Tr}\Big(([\mu_V, U] - \{\mu_A, U\})([\mu_V, U^\dagger] + \{\mu_A, U^\dagger\})\Big)$$

- From V_χ minima:

(1) $\mu_V = 0$, μ_A-any $\longrightarrow$ V_χ-const. $\rho_A \sim f_\pi^2 \mu_A$

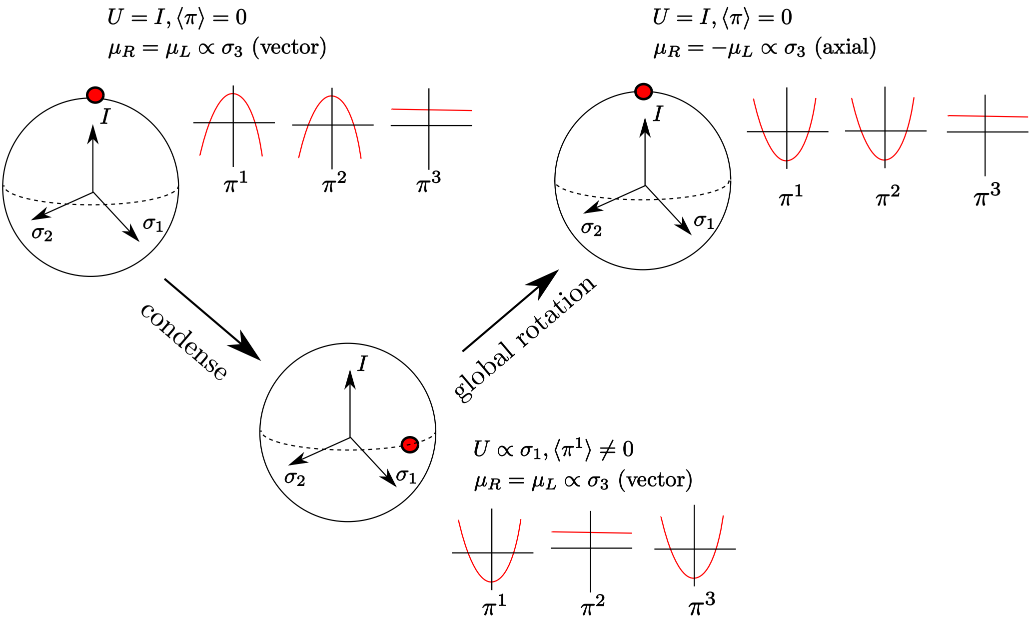
(2) $\mu_A = 0$, $\mu_V = \mu_I \sigma_3/2$ $\longrightarrow$

$U_{\max} = e^{i\alpha}(\cos(\beta) I + i\sin(\beta)\sigma_3)$ and $U_{\min} = e^{i\alpha}(\cos(\beta)\sigma_1 + \sin(\beta)\sigma_2)$

in the $U_{\min}$: $\rho_V \sim f_\pi^2 \mu_I$ $\rho_{A,I} = 0$.

(3) $\mu_V = \mu_I\sigma_3/2$, $\mu_A = \mu_{A,I}\sigma_3/2$: $\begin{cases} \mu_{A,I}^2 < \mu_V^2 & \longrightarrow\ U_{\min} \text{ as in (2)} \\ \mu_{A,I}^2 > \mu_V^2 & \longrightarrow\ U_{\min} \text{ opposite} \end{cases}$

Vectorial isospin potential

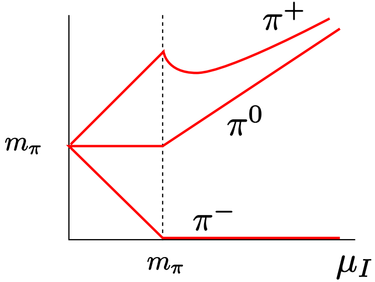


Effects of μ_V in $U = U_{\min}$ $\Leftrightarrow$ effects of μ_A in $U = I$ vacuum

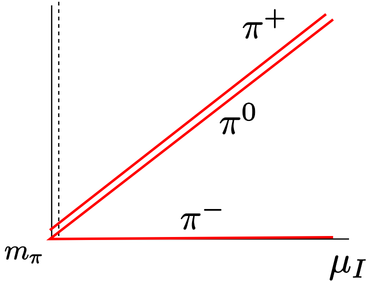
Aside: non-zero pion mass

- The chiral Lagrangian gives us the behaviour of the pions for small μ_I,

Son, Splittorf, Stephanov



- However, Sakai-Sugimoto has $m_\pi = 0$, so we will at small μ_I see



Beyond Chiral Langrangian

- Chiral Langrangian, valid up to the first massive vector meson,

$$\mu_I \ll m_\rho$$

- Other operators are relevant, e.g. Skyrme term

$$\mathcal{L}_{\text{Skyrme}} = \frac{1}{32e^2} \,\mathrm{Tr}\left(\left[U^{-1}\partial_\mu U, U^{-1}\partial_\nu U \right]^2 \right) .$$

  This leads to a dispersion relation for pions

$$-\omega^2 + k^2 + \mu_I^2 - \frac{k^2 \mu_I^2}{e^2 f_\pi^2} = 0 .$$

  This suggests massive pions eventually become unstable.

  But, does not explain what the ρ does.

- Sakai-Sugimoto has pions and fixed couplings to other mesons.

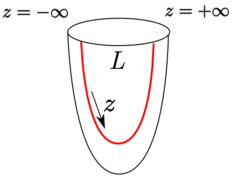
  → Study π's and ρ in this model as function of μ_I.

Part III:

Holographic *isospin* chemical potential

Beyond Chiral Langrangian $\mu_I = 0$

- Cigar-shaped subspace with D8's embedded,



$$u = (1 + z^2)^{1/3}$$

- No chemical potential $\longrightarrow$ no background field, trivial $A_\mu = 0$ vacuum.

- Meson massess from linearised DBI action around trivial vacuum.

$$A_\mu(x^\mu, z) = U^{-1}(x)\partial_\mu U(x)\psi_+(z) + \sum_{n\geq 1} B_\mu^{(n)}(x)\psi_n(z)\,,$$

$$A_z = 0$$

- Can go beyond χ-perturbation theory: have χ-Langrangian interacting with infinite tower of massive modes.

Beyond Chiral Langrangian $\mu_I = 0$

- Effective action we use come from the truncated string effective action

$$S = \tilde{T} \int \mathrm{d}^4x \, \mathrm{d}u \left[u^{-1/2} \gamma^{1/2} \, \mathrm{Tr}(F_{\mu\nu} F^{\mu\nu}) + u^{5/2} \gamma^{-1/2} \, \mathrm{Tr}(F_{\mu u} F^{\mu}{}_{u}) \right] + \ldots$$

where ignored DBI corrections to the YM, $((l_s^2 F)^n)$ and beyond $\mathcal{O}(l_s^3 \partial F)$

- For eg., just for pion this gives

$$F_{z\mu} = U^{-1} \partial_\mu U \, \phi_{(0)}(z) + \text{B-stuff}$$

$$F_{\mu\nu} = [U^{-1}\partial_\mu U, U^{-1}\partial_\nu U] \, \psi_+(z)\big(\psi_+(z) - 1\big) + \text{B-stuff} \, .$$

which gives chiral Lagrangian plus Skyrme term,

$$S = \int \mathrm{d}^4x \; \mathrm{Tr} \left(\frac{f_\pi^2}{4} (U^{-1}\partial_\mu U)^2 + \frac{1}{32e^2} \left[U^{-1}\partial_\mu U, U^{-1}\partial_\nu U \right]^2 \right) + \text{``}\pi \leftrightarrow B''$$

$$f_\pi^2 \sim \lambda N_c M_{KK}^2 \, , \quad e^2 \sim \frac{1}{\lambda N_c} \, , .$$

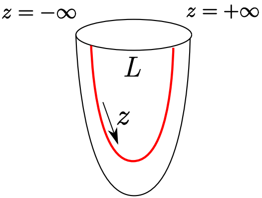
Sakai-Sugimoto and chiral symmetry

- In Sakai-Sugimoto, global symmetry is realised as large gauge transformation of bulk field,

$$A_\mu \to g A_\mu g^{-1} + i g \partial_\mu g^{-1}$$

$$\lim_{z\to-\infty} g(z,x^\mu) = g_L \in SU(N_f)_L\,,$$

$$\lim_{z\to+\infty} g(z,x^\mu) = g_R \in SU(N_f)_R\,.$$



Sakai-Sugimoto and chiral symmetry

- And changes holonomy

$$U = P \exp\left(i \int_{-\infty}^{\infty} \mathrm{d}z\, A_z\right) \to g_L g_R^{-1}\,.$$

changes the pion expectation value, since

$$U = \exp\left(i\pi_a(x)\sigma^a / f_\pi\right)\,.$$

- So if start with trivial vacuum $A_\mu = A_z = 0$, the vectorial transformation $g_L = g_R$ preserves vaccum, does not change U
- If $g_L \neq g_R$, does not preseve vacuum

i.e. changes holonomy $\longrightarrow$ χ-symmetry breaking

Turning on $\mu_I \neq 0$

- For SS model, bulk field $A_\mu(x,u)$

$$A_\nu(x,u) \to B_\nu(x)\left(1+\mathcal{O}(\frac{1}{u})\right) + \rho_\nu(x)u^{-3/2}\left(1+\mathcal{O}(\frac{1}{u})\right).$$

here

- $B_\mu(x) \leftrightarrow$ source term for gauge theory current $J^\nu(x)$ $\quad (\int d^4x B_\mu J^\nu(x))$
- $\rho_\nu(x) \leftrightarrow$ vev of J^μ

$\longrightarrow$ To add vectorial/axial chemical potential, solve for the even/odd bulk field with b.c. :

$$A_\mu(x, z \to -\infty) = \mu_L \delta_{\mu,0}$$
$$A_\mu(x, z \to +\infty) = \mu_R \delta_{\mu,0}$$

Isotropic & homogenious solution

- **First ansatz**, *assume* that condensate is x-independent $\longrightarrow$ $A_0(z), A_i = 0$

Isospin chemical potential background satisfies 5d YM equation (in $A_u = 0$ gauge),

$$\partial_z \left[(1 + z^2) \partial_z A_0^{(3)} \right] = 0 \quad \longrightarrow \quad \begin{cases} \text{V:} \quad A_0^{(3)} = \mu_V \,, \\ \text{A:} \quad A_0^{(3)} = \mu_A \arctan z \,. \end{cases}$$

N.B Soln to YM action, neglect DBI corrections, i.e. valid for

$$\mu_I \ll \lambda / L$$

- Spectrum around vectorial soln (V) tachyonic

$\longrightarrow$ *i.e. free energy is unaffected, but fluctuations are affected!*

roll down to $\langle \pi^{(1)} \rangle \neq 0$, then rotate back to trivial vacuum

$\longrightarrow$ Effectively work with axial solution (A)

- Properties of new vacuum:

f_π unmodified, two massive and one massless pion

Instability of isotropic solution

- Soln found is unique **isotropic** soln: pions condensed.

> What about ρ et al?
>
> Are there any other ground states which dominate for higher μ_I?

Analyse general stability of soln

- For $\mu_I \ll \lambda_5/L^2$ can still use just nonabelian YM $\longrightarrow$

  expand YM around axial solution $\bar{A}_0$ in $U = I$ vacuum

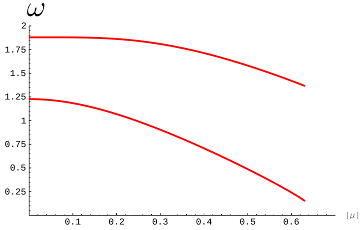
$$A_0 = \bar{A}_0(u) + \delta A_0^{(a)}(\omega, \vec{k}, u)\, \sigma_a\, e^{i\omega t + i\vec{k}\cdot\vec{x}}\,,$$

$$A_i = \delta A_i^{(a)}(\omega, \vec{k}, u)\, \sigma_a\, e^{i\omega t + i\vec{k}\cdot\vec{x}}\,,$$

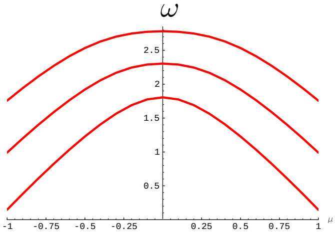
$$A_u = 0$$

Transverse vectors and scalars

- The transverse vectors ($\delta A_0 = 0, \partial_i \delta A^i = 0$) develop an instability: at $\vec{k} = 0$ the dispersion relation is



- The scalars (fluations transverse to the brane) are unstable too, but only for much larger μ,



- The main question: what about the pions & longitudinal vectors **?**

Pions and longitudinal vectors

- Both pions and longitudinal vectors are governed by $A_i \equiv i k_i A_T$ and A_0.
- Equations diagonal for

$$\delta A_i^{(1)} = \pm i \delta A_i^{(2)} \,, \qquad \delta A_0^{(1)} = \pm i \delta A_0^{(2)} \,.$$

- The difference is the boundary conditions
  - The pion is "pure large gauge", so impose $F_{0i} = 0$,

$$A_T(z \to +\infty) = \frac{\pi}{2} + \frac{c_3}{z} + \dots$$

$$A_0(z \to +\infty) = (\omega + \pi\mu)\frac{\pi}{2} + \left(\frac{c_3\, k^2}{\omega + \pi\mu} - \pi\mu\right)\frac{1}{z} + \dots$$

  - Vectors asymptote to zero at $z \to \pm\infty$,

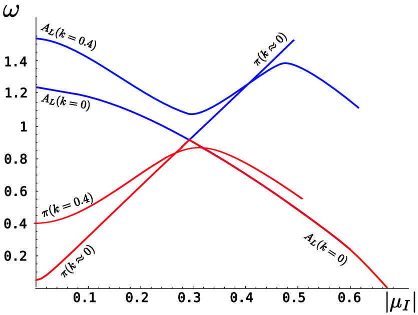
$$A_T(z \to +\infty) = \frac{1}{z} + \dots$$

$$A_0(z \to +\infty) = \frac{k^2}{\omega + \pi\mu}\frac{1}{z} + \dots$$

N.B $\mu_I = 0$ recover Lorentz inv. rels. ($\delta A_0 = \omega \delta A_T$ pion and $\delta A_0 = k^2/\omega A_T$, long. vec.)

Pions and longitudinal vectors cont.

- Similarly, imposing appropriate b.c. at $z = -\infty$ fixes $\omega(\mu, k)$. So the spectrum is



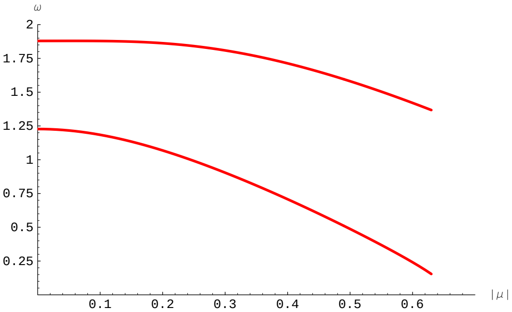
- π's, for small μ_I mass up (as from $\chi - L$)
- modes change "nature"
- no-crossing for $k \neq 0$
- $k = 0$ special $\longrightarrow$ crossing of ρ and π $\longrightarrow$ ρ condenses

Vector instability

- The value of μ_{crit} the same as for *transverse* ρ

all components of ρ vector for $k = 0$ condense at $\mu_{crit} \approx 1.7 m_\rho$

transverse:



$\longrightarrow$ ρ meson instability.

Finding a new ground state

- What is the new ground state?
  - Ansatz (inspired by linear analysis):

$$A_3^{(1)}(z) = \pm i A_3^{(2)}(z)\,, \quad A_i^{(1)}(z) = A_i^{(2)}(z) = 0 \quad (i = 1, 2)\,,$$
$$A_\mu^{(3)} = \delta_{\mu,0} A_0^{(3)}(z) \quad A_u = 0\,,$$

with b.c.

$$A_0^{(3)}(z = \pm\infty) = \pm\mu_I/2\,, \quad A_3^{(1)}(z = \pm\infty) = 0$$

  - Solution of the nonlinear equations

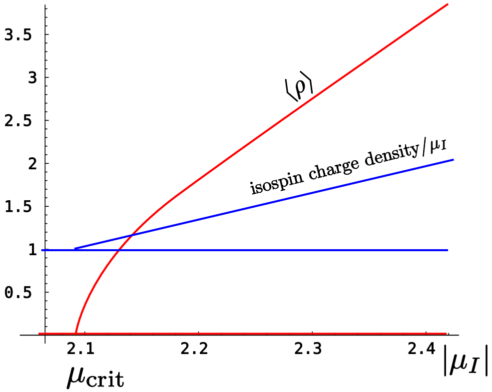
$$\partial_u \left[u^{5/2}\gamma^{-1/2}\partial_u A_0^{(3)}\right] = 4(A_3^{(1)})^2 A_0^{(3)}\, u^{-1/2}\gamma^{1/2}\,,$$
$$\partial_u \left[u^{5/2}\gamma^{-1/2}\partial_u A_3^{(1)}\right] = -4(A_0^{(3)})^2 A_3^{(1)}\, u^{-1/2}\gamma^{1/2}\,.$$

  - Have two solutions

$$\begin{cases} \mu < \mu_{\text{crit}} : & A_3^{(1)} = 0 \quad A_0^{(3)} = \frac{\mu_I}{\pi} arctan\left(\frac{z}{u_\Lambda}\right) \\ \mu > \mu_{\text{crit}} : & A_3^{(1)} \neq 0 \quad A_0^{(3)} \neq 0\,. \end{cases}$$

The new ground state

- A numerical solution yields:



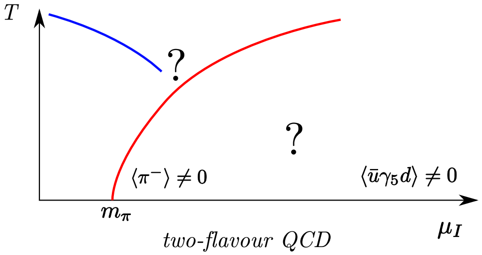
$$\mu_{\rm crit} \approx 1.7\, m_\rho\,, \qquad \langle\rho\rangle \propto \sqrt{\mu - \mu_{\rm crit}}\,.$$

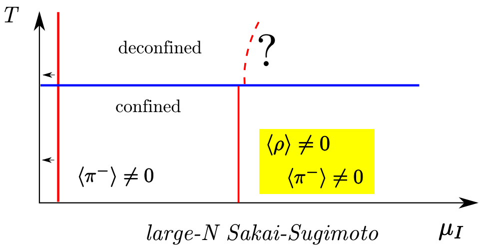
$\longrightarrow$ ρ-meson condensate forms:

- breaking rotational SO(3) $\longrightarrow$ SO(2)
- breaking the residual flavour U(1)

(in addition, the pion condensate remains present)

Summary and todo





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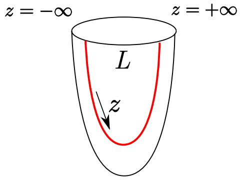

- Can we include the pion mass (using tachyon) ?
- How does this depend on L (constituent quark masses) ?
- Are there further instabilities at even higher μ ?

  Corrections due to DBI and Chern-Simons ?
- Behaviour in deconfined phase, as function of temperature ?