Trace anomaly $(\varepsilon - 3p)$ in the deconfinement phase

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Introduction

Classically, YM theory is conformally (or scale) invariant under $A^a_\mu \to \sigma A^a_\mu(\sigma x) \Rightarrow$ the dilatation current $s_\mu = x_\nu \Theta_{\mu\nu}$ is conserved:

$$\partial_\mu s_\mu = \Theta_{\mu\mu} = 0, \tag{1}$$

where $\Theta_{\mu\nu}$ is the traceless energy-momentum tensor: $\Theta_{\mu\mu} = 0$.

At the quantum level,

- the scale invariance is broken by the dimensionful UV cutoff a, and $\Theta_{\mu\mu} \neq 0$ owing to loop effects – the trace anomaly;
- equality (1) still holds in the weak sense:

$$\langle \mathcal{O}[A] \partial_\mu s_\mu \rangle = \langle \mathcal{O}[A] \Theta_{\mu\mu} \rangle \,,$$

where the functional $\mathcal{O}[A]$ is gauge-invariant;

- the trace anomaly originates from the noninvariance of the quantum measure $\langle \ldots \rangle$ under the scale transformations (K. Fujikawa, '79, '80)$\Rightarrow$

$$\partial_\mu s_\mu(x) = -\lim_{a\to 0} \mathrm{tr} \, \langle x | \frac{1}{1 + a^2 (iD^2_\mu)} | x \rangle \quad \Rightarrow$$

Introduction

$$\langle \Theta_{\mu\mu} \rangle = \frac{\beta(g)}{2g} \left\langle (F^a_{\mu\nu})^2 \right\rangle \simeq -\frac{b}{32\pi^2} g^2 \left\langle (F^a_{\mu\nu})^2 \right\rangle ,$$

where $b = 11N_c/3$.

At $T = 0$, the nonperturbative vacuum energy density $\varepsilon_{\rm vac} = \frac{1}{4} \langle \Theta_{\mu\mu} \rangle$ is negative-definite.

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At $T > T_c$, the difference

$$\langle \Theta_{\mu\mu} \rangle_T - \langle \Theta_{\mu\mu} \rangle_{T=0} = \varepsilon - 3p,$$

characterizes the measure of deviation of the gluon plasma from the free Stephan-Boltzmann gas, for which $\varepsilon - 3p = 0$.

(Cf. $\mathcal{N} = 4$ SYM, which is also conformally invariant.)

The "nonideality" $\frac{\varepsilon - 3p}{T^4}$ in SU(3) YM theory has been measured on the lattice (F. Karsch *et al.*, '96).

Introduction

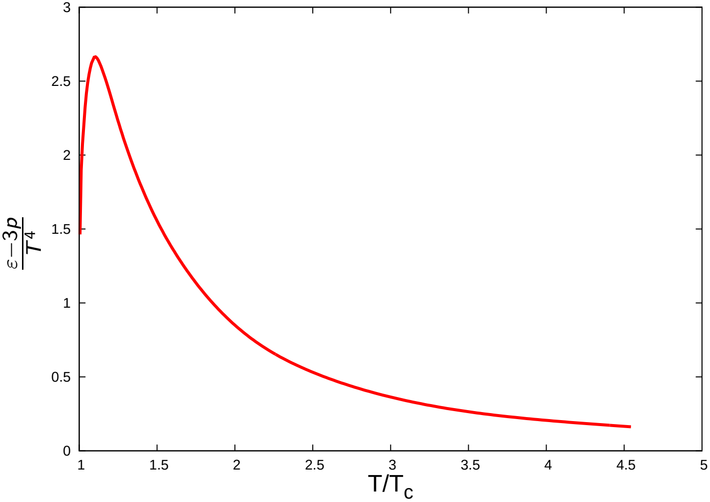

Figure: Lattice values on $\frac{\varepsilon - 3p}{T^4}$ in SU(3) YM theory extrapolated to the continuum limit (courtesy of F. Karsch).

Introduction

A recent application is to the calculation of the bulk viscosity of the gluon plasma

$$\zeta = \frac{\pi}{9} \lim_{\omega \to 0} \frac{\rho(\omega, \mathbf{0})}{\omega}.$$

D. Kharzeev and K. Tuchin ('07) obtained an exact "sum rule"

$$\int_0^\infty d\omega \frac{\rho(\omega, \mathbf{0})}{\omega} = \frac{T^5}{2} \frac{\partial}{\partial T} \frac{\varepsilon - 3p}{T^4} - 8\varepsilon_{\rm vac}.$$

Assuming the Ansatz

$$\frac{\rho(\omega, \mathbf{0})}{\omega} = \frac{9\zeta}{\pi} \frac{\omega_0^2}{\omega^2 + \omega_0^2}$$

with $\omega_0(T) = 1.4(T/T_c)\,{\rm GeV}$, they found $\zeta/({\rm entropy\ density})$ falling off from infinity to zero at $T = T_c$.

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The main goal of this project is to reproduce the lattice values on $\frac{\varepsilon - 3p}{T^4}$ theoretically.

Evaluation of the trace anomaly

• since the chromo-magnetic condensate does not vanish at $T > T_c$, large spatial Wilson loops exhibit the area law $\Rightarrow$

• in the course of their spatial motion, gluonic fluctuations in the stochastic chromo-magnetic background (= valence gluons) attract pairwise; the color-octet spatial string tension $\sigma(T) = \frac{9}{8}\sigma_{\rm fund}(T)$.

• $\left\langle E_i^a(x) B_k^b(x')\right\rangle \ll \left\langle B_i^a(x) B_k^b(x')\right\rangle$ (A. Di Giacomo *et al.*, '97) $\Rightarrow$ the Wilson loop of the valence gluon

$$\langle W[z_\mu]\rangle = \frac{1}{N_c}\left\langle {\rm tr}\,\mathcal{P}\exp\left[ig\int_\Sigma (d\sigma_k B_k + d\sigma_{k4} E_k)\right]\right\rangle$$

factorizes:

$$\langle W[z_\mu]\rangle \simeq \langle L^n\rangle \langle W[\mathbf{z}]\rangle\,.$$

Evaluation of the trace anomaly

$\Rightarrow$ The free-energy density of valence gluons:

$$F \simeq -(N_c^2 - 1) \cdot 2 \sum_{n=1}^{\infty} \int_0^{\infty} \frac{ds}{s} \mathrm{e}^{-\frac{\beta^2 n^2}{4s}} \int \mathcal{D}\mathbf{z}\, \mathrm{e}^{-\frac{1}{4}\int_0^s d\tau \dot{\mathbf{z}}^2} \langle W[\mathbf{z}] \rangle \times$$

$$\times \int \mathcal{D}z_4\, \mathrm{e}^{-\frac{1}{4}\int_0^s d\tau \dot{z}_4^2} \langle L^n \rangle .$$

The Polyakov loop

$$\langle L \rangle \equiv L(T) = \frac{1}{N_c} \left\langle \mathrm{tr}\, \mathcal{T} \exp\left(ig \int_0^{\beta} dz_4 A_4 \right) \right\rangle$$

does not depend on $z_4(\tau)$ $\Rightarrow$ the path integral over $z_4(\tau)$ is the free one:

$$\int \mathcal{D}z_4\, \mathrm{e}^{-\frac{1}{4}\int_0^s d\tau \dot{z}_4^2} = \frac{1}{\sqrt{4\pi s}}.$$

Evaluation of the trace anomaly

Instead, the spatial Wilson loop DOES depend on $\mathbf{z}(\tau)$:

$$\langle W[\mathbf{z}]\rangle = \begin{cases} \exp\left[-\frac{N_c}{24(N_c^2-1)} g^2 \left\langle (F_{\mu\nu}^a)^2 \right\rangle_T \cdot S^2\right], \ \text{at} \ \sqrt{S} \lesssim \mu^{-1}(T) \\ \exp\left[-\sigma(T)\cdot S\right], \ \text{at} \ \sqrt{S} \gtrsim \mu^{-1}(T) \end{cases},$$

where $\mu(T)$ is the inverse vacuum correlation length.

We will fix $N_c = 3$.

Combining the two laws into a single formula by introducing the dimensionless proper time $t = \mu^2(T)s$:

$$\langle W[\mathbf{z}]\rangle = \exp\left[-G(t)S^2 - \sigma(t)S\right],$$

where

$$G(t) = \frac{g^2 \left\langle (F_{\mu\nu}^a)^2 \right\rangle_T}{64} \cdot \theta(1-t),$$

$$\sigma(t) = \sigma \cdot \theta(t-1),$$

θ is the step function.

Evaluation of the trace anomaly

The path integral over $\mathbf{z}(\tau)$ becomes analytically calculable by

- introducing an einbein λ as

$$\langle W[\mathbf{z}]\rangle = \int_0^\infty \frac{d\lambda}{\sqrt{\pi\lambda}} \exp\left[-\lambda - \left(\frac{\sigma^2(t)}{4\lambda} + G(t)\right) S^2\right],$$

- using an equality in the formula

$$S = \int_0^s d\tau |\mathbf{z}\times\dot{\mathbf{z}}| \geq \sqrt{\left[\int_0^s d\tau(\mathbf{z}\times\dot{\mathbf{z}})\right]^2}$$

(no-backtracking approximation),

- applying the Hubbard-Stratonovich (Gaussian, "bosonization",...) trick

$$\exp\left(-C\mathbf{f}^2\right) = \frac{1}{(4\pi C)^{3/2}} \int d^3H \exp\left(-\frac{\mathbf{H}^2}{4C} + i\mathbf{H}\mathbf{f}\right).$$

Here $C > 0 \Rightarrow$ from this point on, the model describes the valence gluons only, and not the free ones.

Evaluation of the trace anomaly

The integral over $\mathbf{z}(\tau)$ becomes the bosonic Euler-Heisenberg Lagrangian:

$$\int \mathcal{D}\mathbf{z} \mathrm{e}^{-\frac{1}{4}\int_0^s d\tau \dot{\mathbf{z}}^2} \exp\left[i\mathbf{H}\int_0^s d\tau(\mathbf{z}\times\dot{\mathbf{z}})\right] = \frac{1}{(4\pi s)^{3/2}}\frac{Hs}{\sinh(Hs)}.$$

This trick reduces the YM-average to the average over the Gaussian action of the auxiliary magnetic field $\mathbf{H}$.

In each of the intervals $t \in [0,1]$ and $t \in [1,\infty)$, the λ-integrations can be performed analytically.

Evaluation of the trace anomaly

$\Rightarrow$ the final result for the free-energy density:

$$F(T)\Big|_{n=1} = -\frac{4\mu^{10}(T)L(T)}{\pi^3}\times$$

$$\times\left\{\frac{64\sqrt{\pi}}{[g^2\left\langle(F_{\mu\nu}^a)^2\right\rangle_T]^{3/2}}\int_0^1\frac{dt}{t^2}\mathrm{e}^{-\frac{\mu^2(T)\beta^2}{4t}}\int_0^\infty\frac{dhh^3}{\sinh(ht)}\exp\left[-\frac{16\mu^4(T)h^2}{g^2\left\langle(F_{\mu\nu}^a)^2\right\rangle_T}\right]+\right.$$

$$\left.+\sigma(T)\int_1^\infty\frac{dt}{t^2}\mathrm{e}^{-\frac{\mu^2(T)\beta^2}{4t}}\int_0^\infty\frac{dhh^3}{\sinh(ht)\cdot[\mu^4(T)h^2+\sigma^2(T)]^2}\right\},$$

where $\mathbf{h}=\mathbf{H}/\mu^2(T)$.

Evaluation of the trace anomaly

The parameters:

- $\mu(T) = 1.044 g^2(T) T$ as suggested by F. Karsch *et al.* with the one-loop

$$g^2(T) = \frac{8\pi^2}{11 \ln \frac{T}{0.104 T_c}} ,$$

$T_c = 270\,\mathrm{MeV}$. This $\mu(T)$ reproduces correctly $\mu(T_c) = 894\,\mathrm{MeV}$ found by A. Di Giacomo *et al.*

- $\sigma_{\mathrm{fund}}(T) = [0.566 g^2(T) T]^2$ (F. Karsch *et al.*) $\Rightarrow$
$\sigma_{\mathrm{fund}}(T_c) = (485\,\mathrm{MeV})^2$ is larger than $\sigma_{\mathrm{fund}}(0) = (440\,\mathrm{MeV})^2$.

- $g^2 \left\langle (F^a_{\mu\nu})^2 \right\rangle_T = \frac{72}{\pi} \mu^2(T) \sigma_{\mathrm{fund}}(T)$ (H.-J. Pirner *et al.*) $\Rightarrow$
$g^2 \left\langle (F^a_{\mu\nu})^2 \right\rangle_{T_c} = 4.30\,\mathrm{GeV}^4$ is larger than $g^2 \left\langle (F^a_{\mu\nu})^2 \right\rangle_0 = 3.55\,\mathrm{GeV}^4$.

Evaluation of the trace anomaly

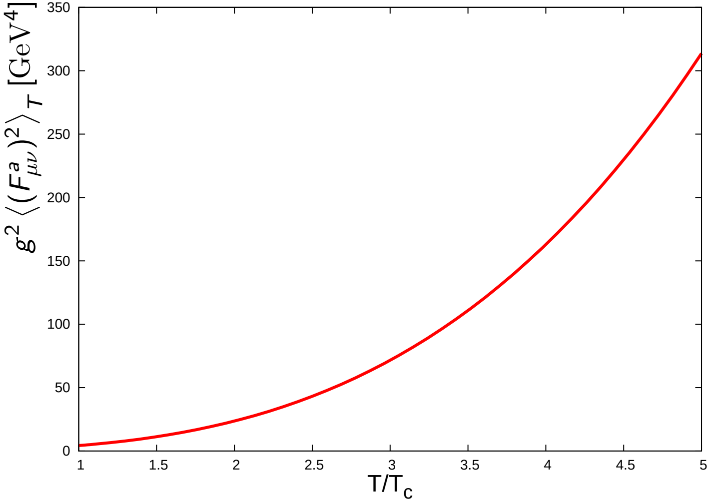

Figure: The chromo-magnetic gluon condensate as a function of temperature.

Evaluation of the trace anomaly

Setting the Polyakov loop $L(T)$ equal to 1, one can evaluate the deviation of $F(T)$ from the free-energy density of the Stephan-Boltzmann gas

$$F_{\rm free}(T)\Big|_{n=1} = -\frac{16}{\pi^2}T^4.$$

The same can be done for the entropy density $S(T) = -\partial F_{\rm free}(T)/\partial T$ and the internal-energy density $U(T) = F_{\rm free}(T) + TS$.

Evaluation of the trace anomaly

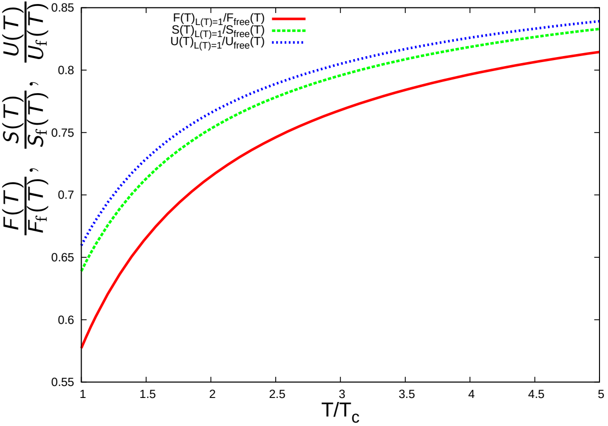

Figure: The ratios of the densities of free energies, entropies, and internal energies of valence gluons to the same quantities for the free gluons. The Polyakov loop $L(T)$ is set equal to 1, therefore the curves illustrate the deviations produced entirely by the spatial confinement.

Accounting for the Polyakov loop in the $(n = 1)$-term by fitting the corresponding lattice data in SU(3) YM (F. Karsch *et al.*, '02) by the two-parameter function

$$L(T) = 1.066 - \frac{1.174}{\exp\left(\frac{T - T_c}{113\,\mathrm{MeV}}\right) + 1}.$$

This fit is better than the fit by the one-parameter function

$$L(T) = 1 - \exp\left(-\frac{T - T_c}{43\,\mathrm{MeV}}\right).$$

Evaluation of the trace anomaly

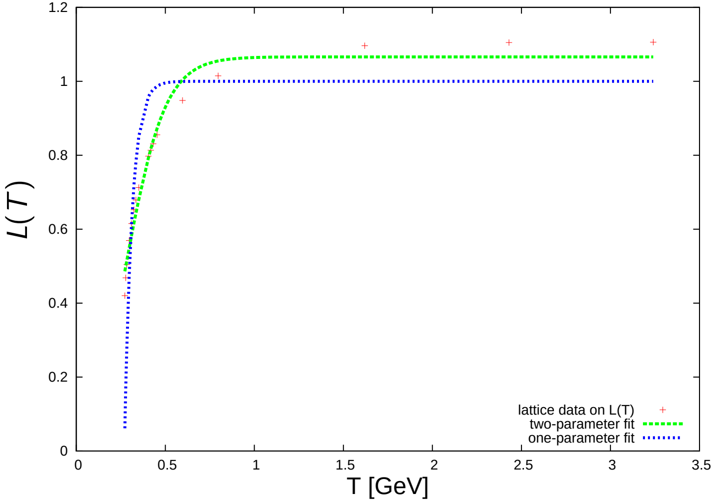

Figure: The Polyakov loop $L(T)$ according to the lattice simulations and fits to it.

The results

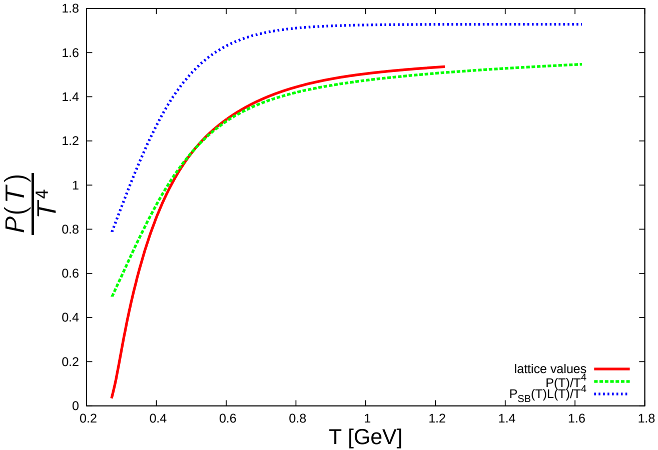

Figure: Lattice data on the ratio of the pressure density over T^4 compared to the predictions of our model and the corresponding values for the Stephan-Boltzmann gas with the Polyakov loop fitted by the two-parameter fit.

The results

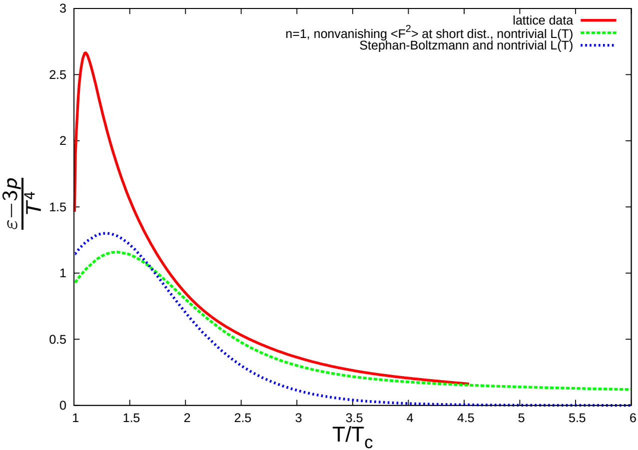

Figure: The trace anomaly $\frac{\varepsilon - 3p}{T^4} = \frac{4F(T) - T\frac{\partial F(T)}{\partial T}}{T^4}$ with the Polyakov loop fitted by the two-parameter fit for i) valence spatial gluons and ii) free spatial gluons, compared to the lattice data.

The results

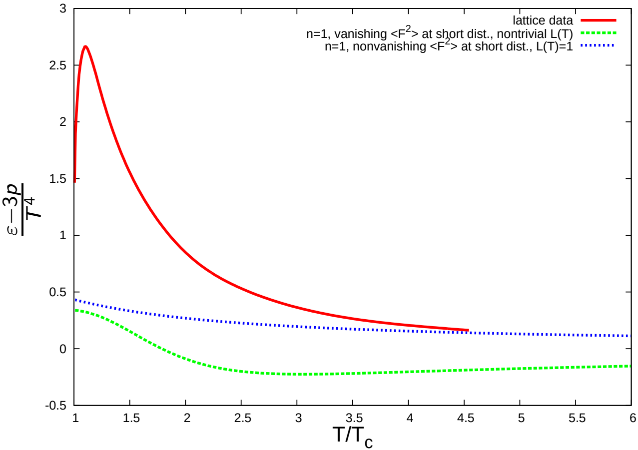

Figure: The trace anomaly $\frac{\varepsilon - 3p}{T^4}$ compared to the lattice data for valence spatial gluons with i) the contribution of the chromo-magnetic condensate at distances smaller than the vacuum correlation length disregarded and ii) the Polyakov loop set to 1.

The results

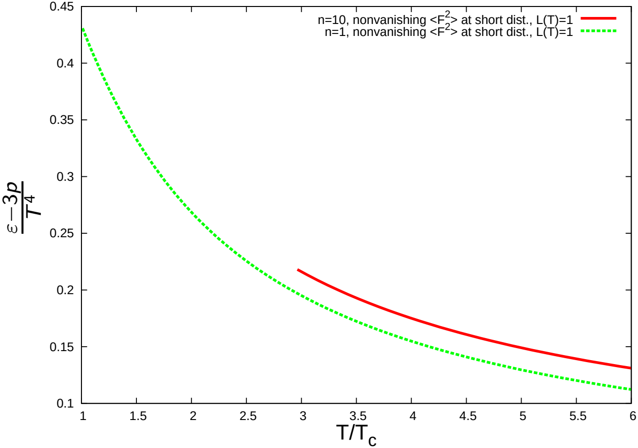

Figure: The trace anomaly $\frac{\varepsilon - 3p}{T^4}$ with the Polyakov loop set to 1, for i) 10 first winding modes at high temperatures, where the Polyakov loop is indeed very close to 1, and ii) the 1st winding mode alone, as in the previous figure.

The results

In search for a "bag constant": An attempt fails to fit by const$/T^4$ the ratio $\frac{F(T)-F_{\rm SB}(T)}{T^4}$, where $F_{\rm SB}(T)=F_{\rm free}(T)L(T)$.

Instead, at $T>1.7T_c$, we find a good fit by a polynomial quartic in β:

$$\frac{F(T)-F_{\rm SB}(T)}{T^4}\simeq$$

$$\simeq 0.048+0.23\,{\rm GeV}\cdot\beta-(0.17\,{\rm GeV}\cdot\beta)^2+(0.17\,{\rm GeV}\cdot\beta)^3-(0.26\,{\rm GeV}\cdot\beta)^4.$$

The results

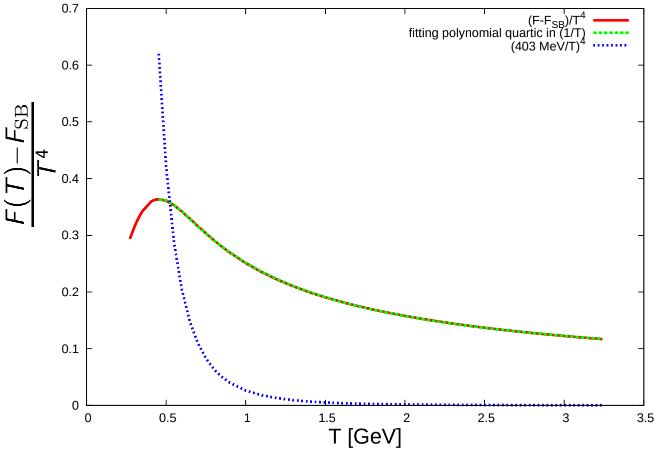

Figure: The difference of the free-energy densities in our model and that of free gluons with the nontrivial Polyakov loop, divided by T^4, a fit to it by the polynomial quartic in β, and an attempt of a fit by $\mathrm{const}\cdot\beta^4$ alone.

Conclusions

• Valence spatial gluons reproduce the lattice pressure density and the trace anomaly much better than the free spatial gluons.

• Only accounting for the chromo-magnetic gluon condensate at the distances smaller than the vacuum correlation length, can one obtain the correct overall scale (and sign) of the trace anomaly.

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• As was first noticed in 1995 by H.-J. Pirner, H.G. Dosch, and Yu.A. Simonov, it is critical behavior of the Polyakov loop, which is liable for the rapid increase of $\frac{\varepsilon-3p}{T^4}$ at $T \to T_c + 0$. The first-order critical behavior of the Polyakov loop in SU(3) YM theory is not sufficient alone to yield the lattice values of $\frac{\varepsilon-3p}{T^4}$ at $T \lesssim 1.7\,T_c$.

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• The effect of winding modes higher than the first one is negligible even at $T > 3T_c$, where $L(T) = 1$.

• At $T > 1.7T_c$, one finds a good fit to $\frac{F(T)-F_{\rm SB}}{T^4}$ by the polynomial quartic in β, and not by $\mathrm{const} \cdot \beta^4 \Rightarrow$ no bag constant.