GGI, 23 May 2007

Large N & SUSY: some new ideas and results

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Part I: Planar Equivalence

* Old and new large-N QCD
* Orientifold planar equivalence

Part II: Planar Quantum Mechanics

* Hamiltonian Planar QM
* An intriguing SUSY matrix model

Part I based on

- A. Armoni, M. Shifman, GV, hep-th/0302163, 0307097, 0309013, 0403071, 0412203, 0701229;
- A. Armoni, G.M. Shore, GV, hep-ph/0511143

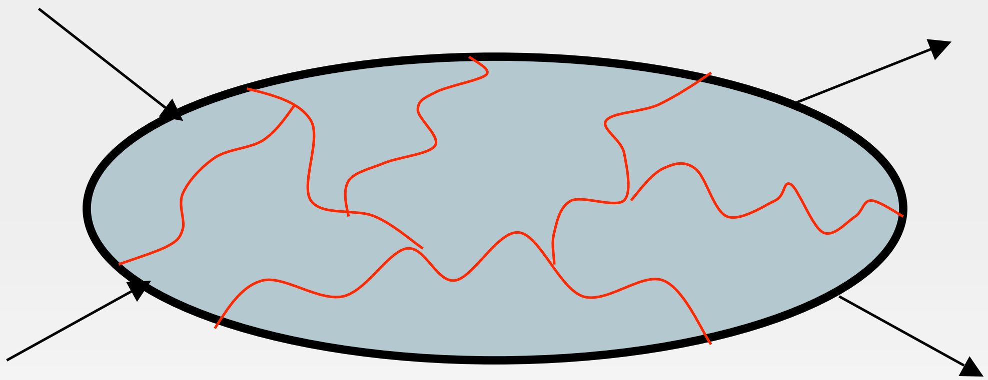
Part II based on

- J. Wosiek & GV hep-th/0512301, 0603045, 0607198, 0609210 (cond-mat);
- E. Onofri, J.Wosiek & GV math-ph/0603082

Large-N expansions in QCD

1. Planar & quenched limit ('t Hooft, 1974)

$1/N_c$ expansion @ fixed $\lambda = g^2 N_c$ and N_f
Leading diagrams



Corrections: $O(N_f/N_c)$ from q-loops,
$O(1/N_c^2)$ from higher genus diagrams

Properties at leading order

1. Resonances have zero width +
2. U(1) problem not solved, WV @ NLO - ?
3. Multiparticle production not allowed -

Theoretically (if not phenomenologically) appealing: should give the tree-level of some kind of string theory

Proven hard to solve, except in D=2....

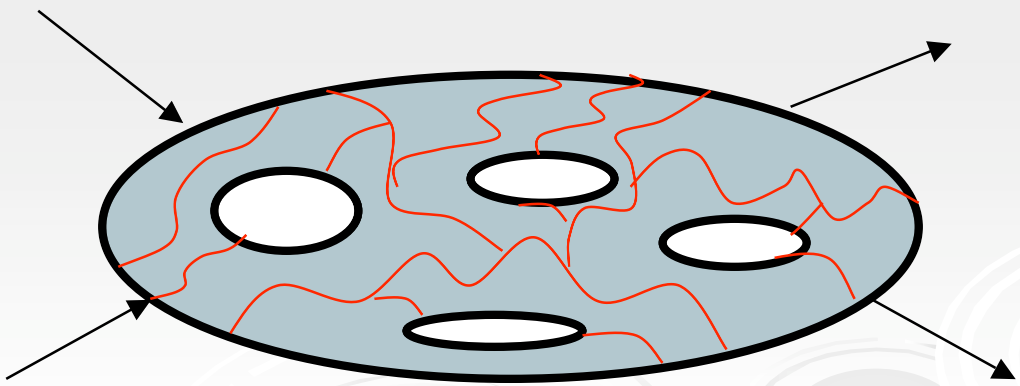
❷ Planar unquenched limit = Topological Expansion (GV '74--'76)

$1/N$ expansion at fixed g^2N and N_f/N_c

Leading diagrams include "empty" q-loops

Corrections:

$O(1/N^2)$ from non-planar diagrams



Properties

1. Widths are O(1) -
2. U(1) problem solved to leading order, no reason for WV to be good + ?
3. Multiparticle production allowed +
   => Bare Pomeron & Gribov's RFT

Perhaps phenomenologically more appealing than 't Hooft's but even harder to solve…

But there is a third possibility…

③ Generalize QCD to $N \neq 3$ ($N = N_c$ hereafter) in other ways by playing with matter rep. The conventional way, QCD_F, is to keep the quarks in $N + N^*$ rep.

Another possibility, called for stringy reasons*) QCD_{OR}, is to assign quarks to the 2-index-antisymm. rep. of SU(N) (+ its c.c.)**)

As in 't Hooft's exp. (and unlike in TE), N_f is kept fixed ($N_f < 6$, or else AF lost at large N)

NB: For N = 3 this is still good old QCD!

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*) see e.g. P.Di Vecchia et al. hep-th/0407038

**) Pioneered by Corrigan and Ramond (1979) for very different reasons

Leading diagrams are planar, include “filled” q-loops since there are $O(N^2)$ quarks

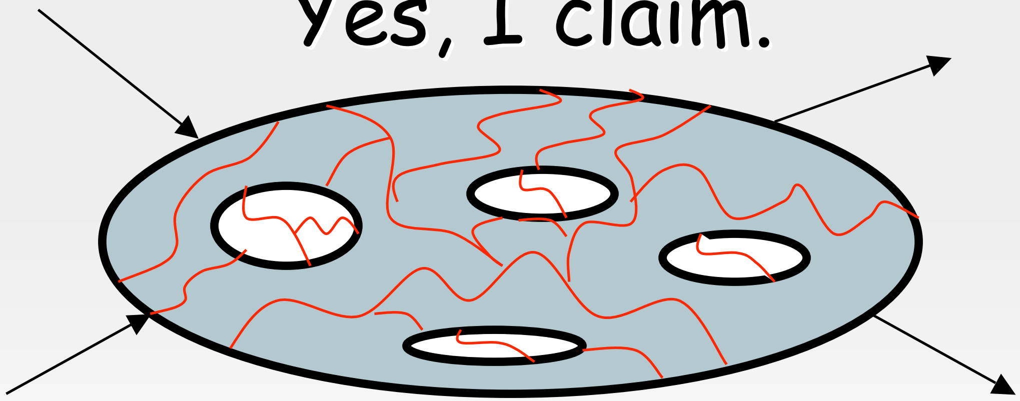
Widths are zero, U(1) problem solved, no p.pr.

Phenomenologically interesting?

Don’t know.

Better manageable?

Yes, I claim.



Numerology of QCD_F vs. QCD_{OR}

Th / coeff	YM	QCD_F	QCD_{OR}	Large-N, $N_f=1$
β_0	$11N/3$	$(11N-2N_f)/3$	$(11N-2(N-2)N_f)/3$	$3N$
$3\beta_1$	$17N^2$	$17N^2 - 3N_f \times (13N/6 - 1/2N)$	$17N^2 - N_f (N-2) \times (5N + 3(N-2)(N+1)/N)$	$9N^2$
γ_0	X	$3(N^2-1)/2N$	$3(N-2)(N+1)/N$	$3N$

QCD_{OR} as an interpolating theory:
Coincides with pure YM (AS fermions decouple) @ N=2
Coincides with QCD @ N=3
… and at large N?

ASV claim of Planar Equivalence

At large-N a bosonic sector of QCD_{OR} is equivalent to a corresponding sector of QCD_{Adj} i.e. of QCD with N_f Majorana fermions in the adjoint representation

If true, important corollary:

For $N_f = 1$ and $m = 0$, QCD_{OR} is planar-equivalent to supersymmetric Yang-Mills (SYM) theory

Some properties of the latter should show up in one-flavour QCD … if N=3 is large enough

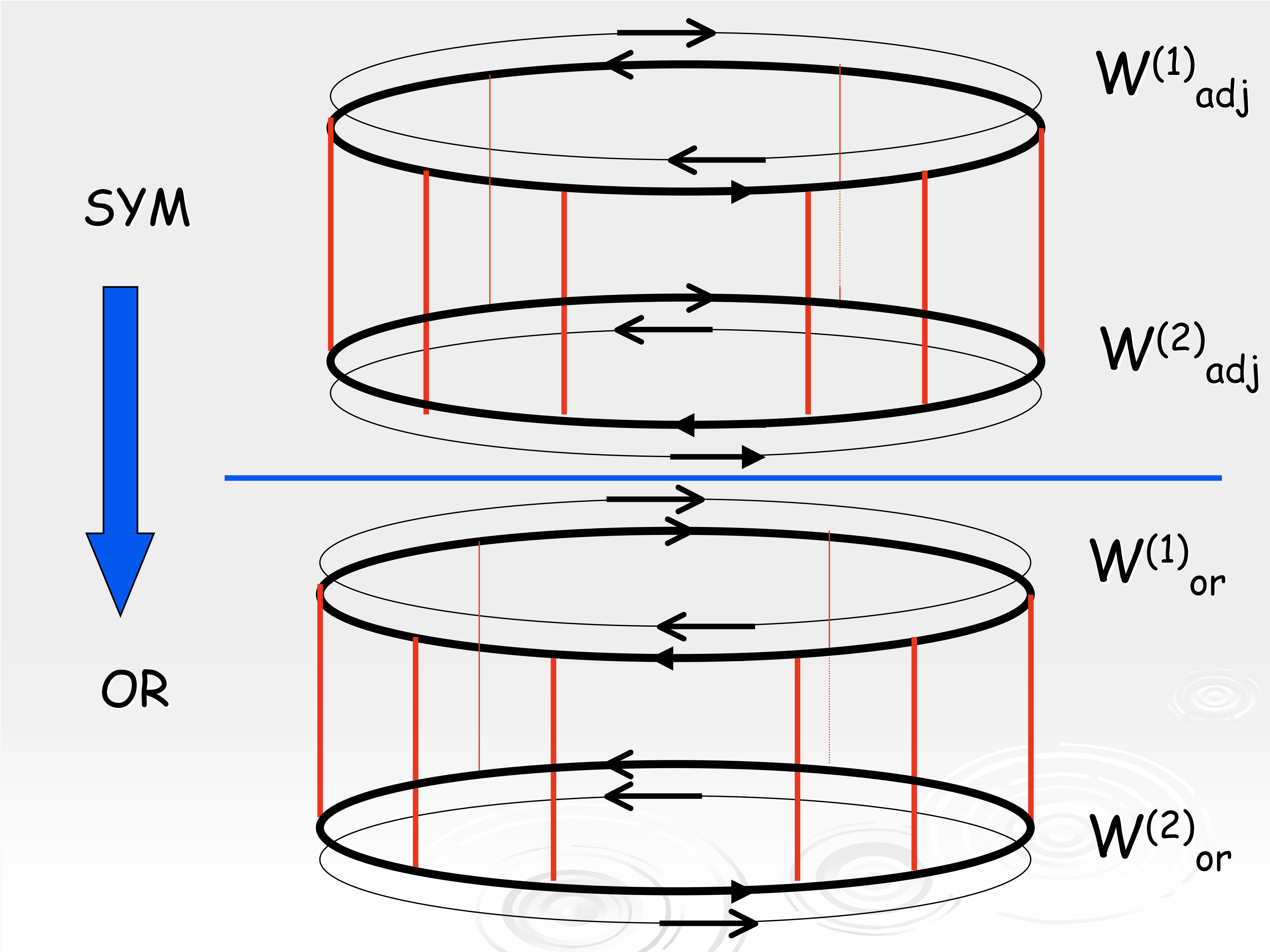
NB: Expected accuracy 1/N

ASV gave both perturbative and NP arguments

Sketch of non-perturbative argument
(ASV '04, A. Patella, '05)

- Integrate out fermions (after having included masses, bilinear sources)
- Express Trlog($\not{D}$+m+J) in terms of Wilson-loops using world-line formulation
- Use large-N to write adjoint and AS Wilson loop as products of fundamental and/or antifundamental Wilson loops (e.g. $W_{adj} = W_F \times W_{F*} + O(1/N^2)$)
- Use symmetry relations between F and F* Wilson loops and their connected correlators

An example: $\langle W^{(1)}\ W^{(2)} \rangle_{conn}$



SYM
OR
$W^{(1)}_{adj}$
$W^{(2)}_{adj}$
$W^{(1)}_{or}$
$W^{(2)}_{or}$

Key ingredient is C!

- Clear from our NP proof that C-invariance is necessary. Kovtun, Unsal and Yaffe have argued that it is also sufficient
- U&Y (see also Barbon & Hoyos) have also shown that C is spontaneously broken if the theory is put on $R^3 x S^1$ w/ small enough S^1. PE doesn't (was never claimed to) hold in that case
- Numerical calculations (De Grand and Hoffmann) have confirmed this, but also shown that, as expected on some general grounds (see e.g. ASV), C is restored for large radii and in particular on R^4
- Lucini, Patella & Pica have shown (analyt.lly & numer.lly) that SB of C is also related to a non-vanishing Lorentz-breaking F#-current generated at small R but disappearing as well as R is increased

➡ Overwhelming evidence for PE on R^4?

An interesting proposal

Kovtun, Unsal and Yaffe ('07) have also made the claim that QCD_{adj} , unlike QCD_F and QCD_{OR} , suffers no phase transition as a volume-reducing process a la Eguchi-Kawai is performed at large-N

If this is indeed the case, we could get properties of QCD_{adj} at small volume by numerical methods and use them at large volume where the connection to QCD_{OR} can be established (C being OK there)

Finally, one would make semi-quantitative predictions for QCD itself (at different values of N_f) by extrapolating down to N=3

For the moment, we shall try to use instead the connection with a SUSY theory

SUSY relics in one-flavour QCD

① Approximate **bosonic parity doublets**:

$m_S = m_P = m_F$ in SYM => $m_S \sim m_P$ in QCD*)

Looks ~ OK if can we make use of:

i) WV for m_P ($m_P \sim \sqrt{2}(180)^2/95$ MeV ~ 480 MeV),

ii) Experiments for m_S (σ @ 600MeV including quark masses)

Recent lattice work by Keith-Hynes & Thacker also support this approximate degeneracy

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*) Composite-**fermions NOT related**.

Interesting aspects of baryons in QCD_{OR} have been discussed by S. Bolognesi (hep-th/0605065) and by A. Cherman and T. D. Cohen (hep-th/0607028)

② Approximate absence of "activity" in certain chiral correlators

In SYM, a well-known WI gives

$$\langle\lambda\lambda(x)\lambda\lambda(y)\rangle = const.\,,\quad \langle\lambda\lambda(x)\bar{\lambda}\bar{\lambda}(y)\rangle \neq const.$$

PE then implies that, in the large-N limit:

$$\langle\bar{\Psi}_R\Psi_L(x)\bar{\Psi}_R\Psi_L(y)\rangle = const.\,,\quad \langle\bar{\Psi}_R\Psi_L(x)\bar{\Psi}_L\Psi_R(y)\rangle \neq const.$$

Of course the constancy of the former is due to an exact cancellation between intermediate scalar and pseudoscalar states.

The quark condensate in N_f=1 QCD

Using $$\langle\bar{\lambda}\lambda\rangle_\mu = -\frac{9}{2\pi^2}\mu^3\lambda_\mu^{-2} exp\left(-\frac{1}{\lambda_\mu}\right)$$ $$\lambda_\mu = \alpha_s(\mu)N/2\pi$$

and vanishing of quark cond. at N=2, we get

$$\langle\bar{\psi}\psi\rangle_\mu = -\frac{3}{2\pi^2}\mu^3\lambda_\mu^{-1578/961} exp(-\frac{27}{31\lambda_\mu})\ k(1/3)$$

which can be rewritten as

1±0.3?

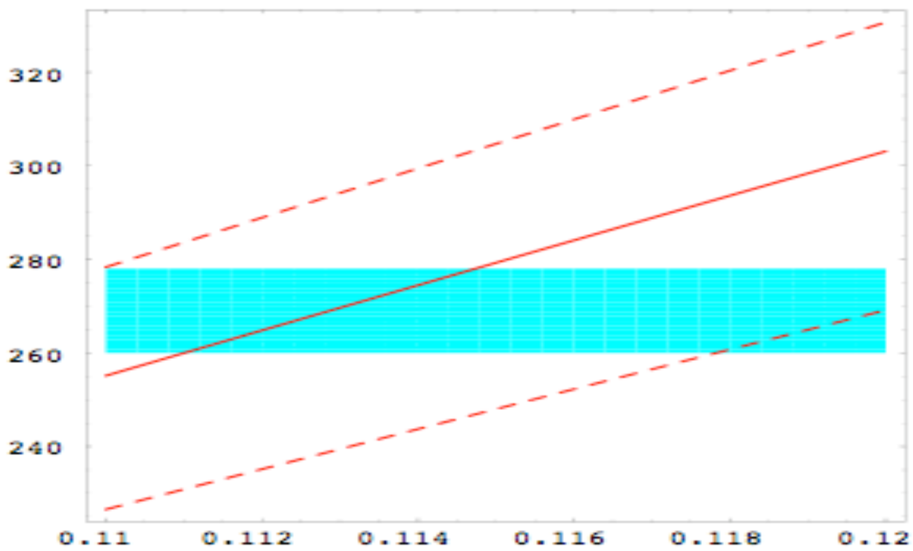
$$<(g^2)^{12/31}\bar{\psi}\psi> = -1.1k(1/3)\Lambda_{st}^3$$

$$\Lambda_{st} = \mu\ exp\left[-\frac{N}{\beta_0\lambda_\mu}\right]\left(\frac{2N}{\beta_0\lambda_\mu}\right)^{\beta_1/\beta_0^2}$$

N_f=1 condensate "measured"?

DeGrand, Hoffmann , Schaefer & Liu,
hep-th/0605147
(using dynamical overlap fermions and distribution of low-lying eigenmodes)

$\left(\langle\bar{\psi}\psi\rangle_{2GeV}\right)^{1/3}$



$3\alpha_s(2GeV)/2\pi$

Exact meaning of agreement still to be fully understood

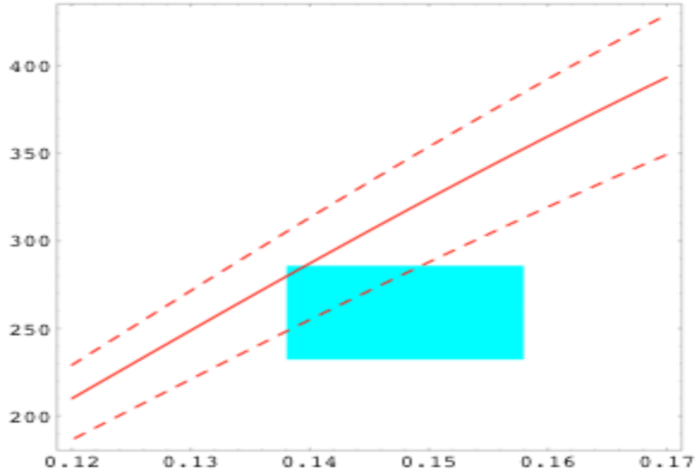
Extension to $N_f > 1$ (Armoni, G. Shore and GV, '05)

- Take OR theory and add to it n_f flavours in N+N* .
- At N=2 it's n_f-QCD, @ N=3 it's $N_f(=n_f+1)$-QCD.
- At large N cannot be distinguished from OR (fits SYM β-functions even better at $n_f = 2$: e.g. same β_0)
- Vacuum manifold, NG bosons etc. are different!
- Some correlators should still coincide in large-N limit. In above paper it was argued how to do it for the quark condensate

Quark condensate (ren. @ 2 GeV) vs α_s(2GeV) for N_f=3

Very encouraging!

$(\langle\bar{\psi}\psi\rangle_{2GeV})^{1/3}$



$3\alpha_s(2GeV)/2\pi$

all in $\overline{MS}$

$$\langle\bar{\psi}\psi\rangle_\mu = -\frac{3}{2\pi^2}\mu^3\lambda_\mu^{-\frac{44}{27}} exp\left(-\frac{1}{\lambda_\mu}\right)$$

Cf. $$\langle\bar{\lambda}\lambda\rangle_\mu = -\frac{9}{2\pi^2}\mu^3\lambda_\mu^{-2} exp\left(-\frac{1}{\lambda_\mu}\right)$$

Conclusions part I

- The orientifold large-N expansion is arguably the first example of large-N considerations leading to quantitative analytic predictions in D=4, strongly coupled, non-supersymmetric gauge theories
- Since its proposal, progress has been made on

  - Tightening the NP proof of PE
  - Providing numerical checks (more is coming!)

but more work is still needed for:

- Estimating the size of 1/N corrections
- Extending the equivalence in other directions

II. Planar quantum mechanics: an intriguing SUSY matrix model

- Original motivation: check planar equivalence and compute its accuracy at finite N in a simple QM case: not done yet!
- On the way, J. Wosiek and I stumbled on an amusing model with unexpected properties and possible implications for HE and CM physics as well as for Maths.
- New motivation: following KUY's suggestion, such QM excercises, once suitably extended, may become relevant for QCD itself!

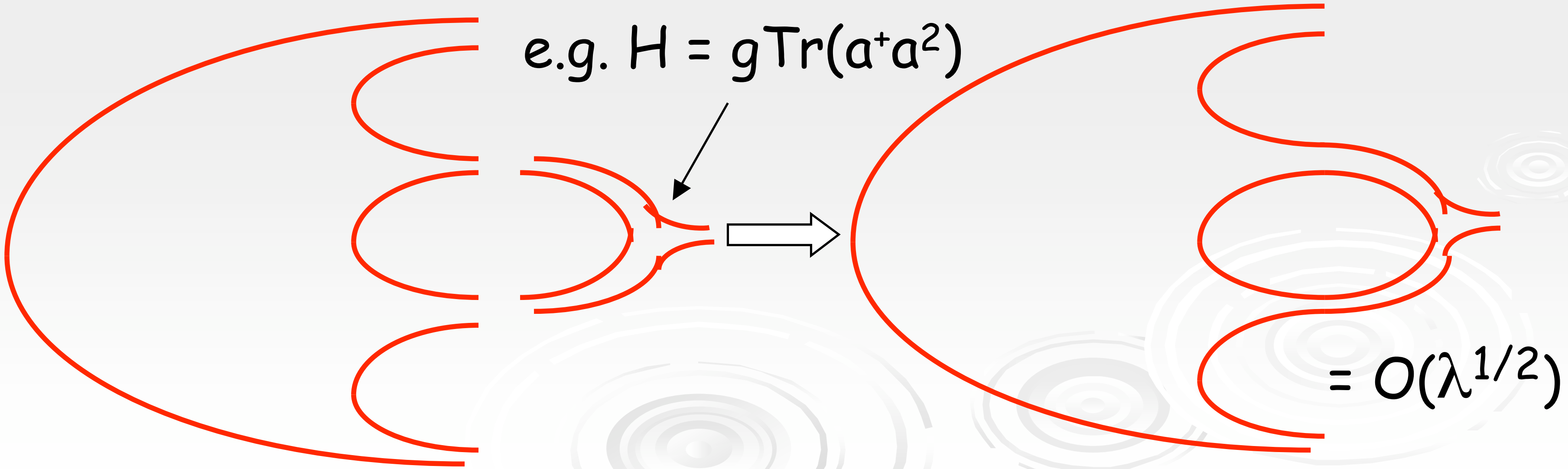
- Consider the large-N limit of a U(N) matrix theory
- With some qualifications relevant singlet states are given by single-trace operators
- In SUSY-QM with a single bosonic matrix ***a*** and a single fermionic matrix ***f*** planar Hilbert space spanned by

$$|\{n_i, m_j\}\rangle \propto Tr[a^{n_1} f^{m_1} \dots a^{n_k} f^{m_k}]^\dagger |0\rangle$$

where |0 > is the usual empty Fock vacuum

Hamiltonians are taken to be single-trace normal-ord. operators, a trace with n factors being multiplied by g^{n-2}. With some qualifications, the Hamiltonian acting on a single-trace state gives, to leading order, a combination of single-trace states w/coefficients that depend only on 't Hooft's $\lambda = g^2N$

$$|\{n_i, m_j\}\rangle \sim N^{-(\sum n_i + \sum m_j)/2} Tr[a^{n_1} f^{m_1} \ldots a^{n_k} f^{m_k}]^\dagger |0\rangle$$



Take the SUSY charges to be simply:

$$Q = Tr\left(f(a^\dagger + g a^{\dagger 2})\right)\ ,\ Q^2 = 0$$

$$H = \{Q^\dagger, Q\}\ ,\ C = [Q^\dagger, Q]\ ,\ C^2 = H^2\ ,\ F = Tr(f^\dagger f)$$

$$H = H_B + H_F$$

$$H_B = Tr[a^\dagger a + g a^\dagger (a + a^\dagger) a) + g^2 a^{2\dagger} a^2]$$

$$H_F = Tr[f^\dagger f + g(f^\dagger f(a+a^\dagger) + f^\dagger (a+a^\dagger) f) + g^2 (f^\dagger a f a^\dagger + f^\dagger a a^\dagger f + f^\dagger f a^\dagger a + f^\dagger a^\dagger f a)]$$

- Trivial E=0 vacuum: |0> => SUSY is unbroken
- E > 0 states must be organized in SUSY doublets w/ same $C_F = (-1)^F C$
- Dependence on λ highly non-trivial

$$H = H_B + H_F \qquad H_B = Tr[a^\dagger a + g a^\dagger (a + a^\dagger) a) + g^2 a^{2\dagger} a^2]$$

$$H_F = Tr[f^\dagger f + g(f^\dagger f(a + a^\dagger) + f^\dagger (a + a^\dagger) f) + g^2 (f^\dagger a f a^\dagger + f^\dagger a a^\dagger f + f^\dagger f a^\dagger a + f^\dagger a^\dagger f a)]$$

Two extreme limits

❶ λ -> 0 : the theory becomes free

$$Q = Tr\left(f(a^\dagger + g a^{\dagger 2})\right) \to Tr\left(f a^\dagger\right) \qquad H \to Tr\left(a^\dagger a + f^\dagger f\right)$$

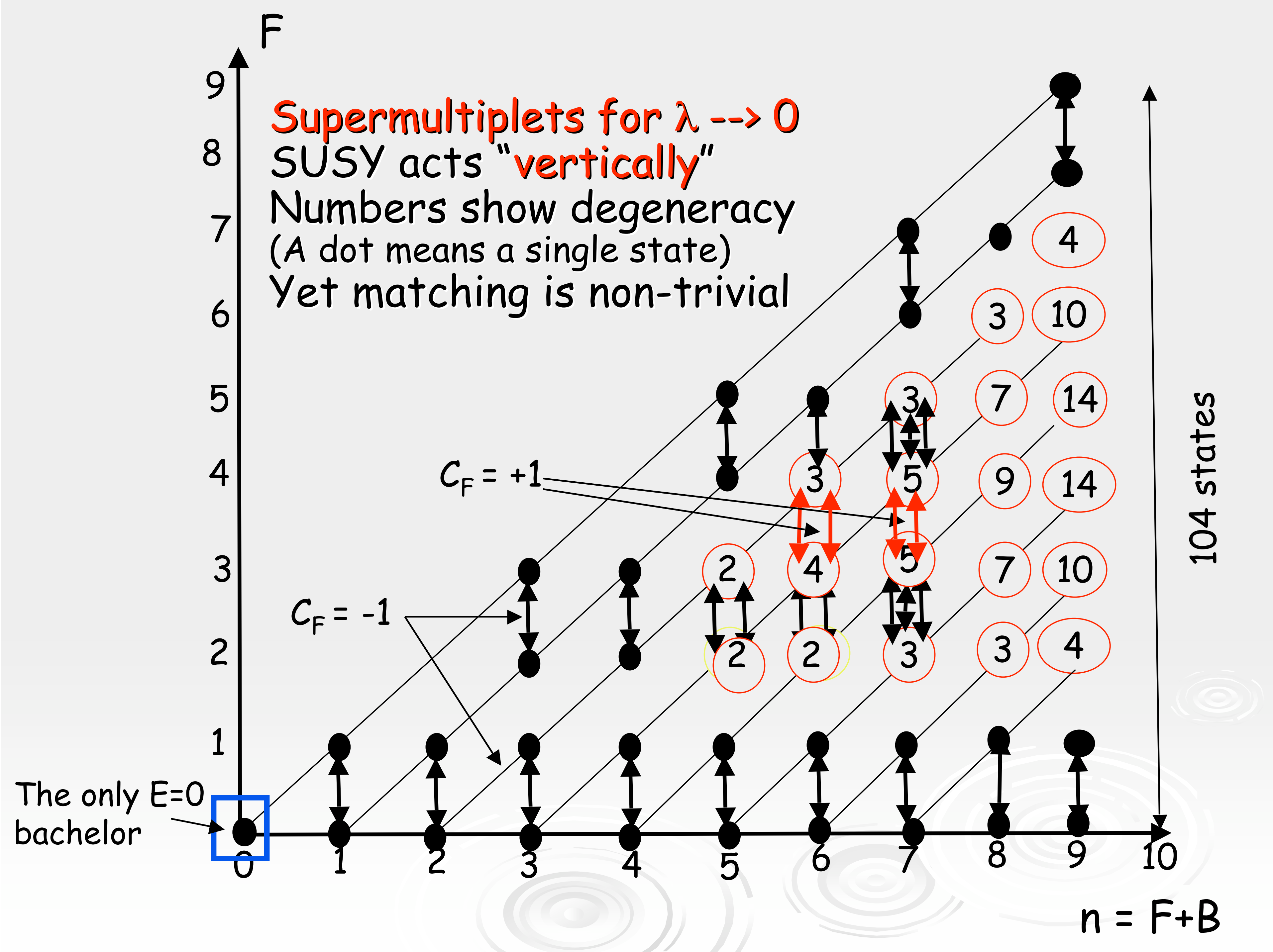
❷ λ -> ∞ : H (better: H/λ) simplifies again

$$Q = \to g\, Tr\left(f a^{\dagger 2}\right) \qquad H_B \to g^2 Tr[a^{\dagger 2} a^2]$$

$$H_F \to g^2 Tr[(f^\dagger a f a^\dagger + f^\dagger a^\dagger f a) + N f^\dagger f]$$

H conserves B & F separately => block-diagonal

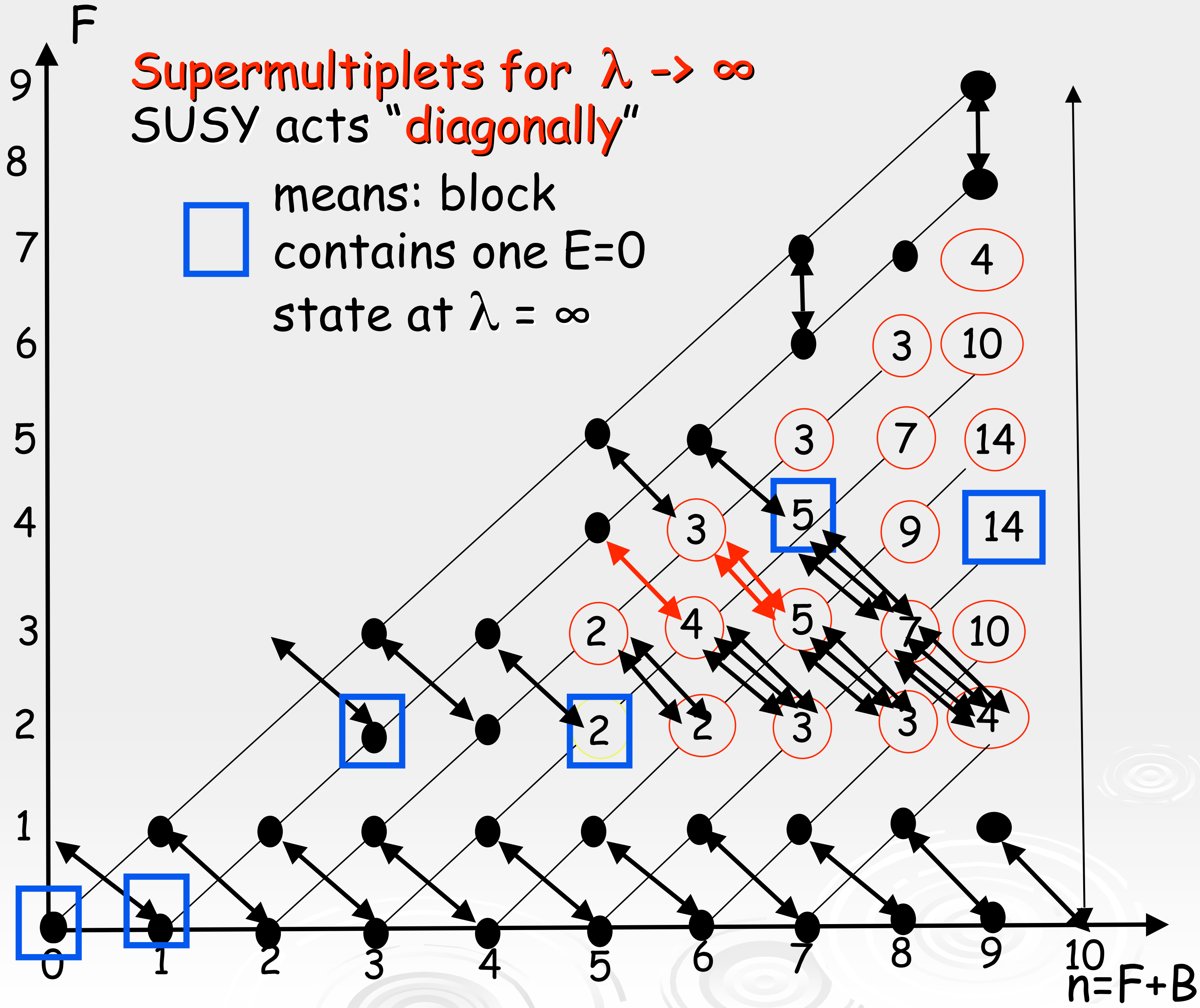
Qs: How does SUSY act in the two limits? How is it implemented? And what happens at generic λ?



Supermultiplets for λ --> 0
SUSY acts "vertically"
Numbers show degeneracy
(A dot means a single state)
Yet matching is non-trivial
F
9
8
7
6
5
4
3
2
1
C_F = +1
C_F = -1
The only E=0 bachelor
0
1
2
3
4
5
6
7
8
9
10
n = F+B
104 states
4
3
10
3
7
14
3
5
9
14
2
4
5
7
10
2
2
3
3
4

Supermultiplets for $\lambda \to \infty$

SUSY acts "diagonally"



The null states appear to form an infinite staircase!

- At $\lambda \ll 1$ it is trivial to solve for the spectrum: yet, this has non-trivial implications on the combinatorics of binary necklaces
- As $\lambda \Rightarrow \infty$ mathematical results on the combinatorics of binary necklaces have implications on the spectrum of the model and on how SUSY is realized

Emerging picture

- At $\lambda \ll 1$ there is perfect matching of bosonic and fermionic states with the single exception of the bosonic Fock vacuum: $W(\lambda \ll 1) = 1$
- As $\lambda \Rightarrow \infty$ many other bosonic states can't find a fermionic partner => they must all have E=0!
- Necessarily, W must jump between $\lambda = 0$ and $\lambda = \infty$! Since unpaired states occur at any even F, we can look for this jump numerically in low-F sectors (this is actually how we found the phenomenon in the first place!)
- Cutoff (in n), needed for numerical studies, breaks SUSY, but SUSY is recovered fast (at generic λ) as cutoff is increased

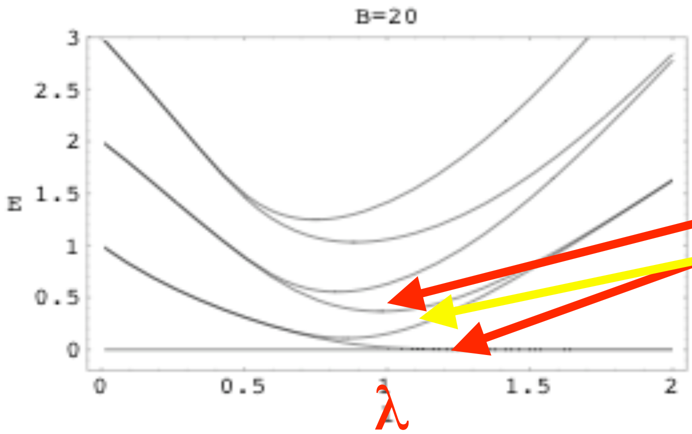
Results in F = 0, 1, 2, 3 sectors

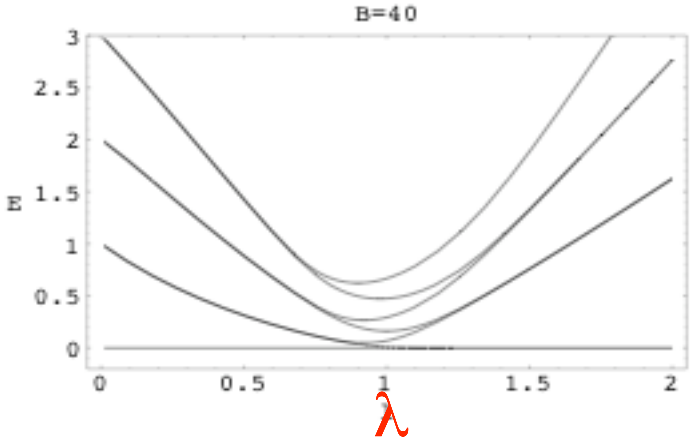
- There is a phase transition at λ =1: the weak-coupling energy gap disappears at λ=1 for all F
- The spectrum becomes discrete again for $\lambda > 1$; In the F=0,1 sectors the eigenvalues at λ are related to those at $1/\lambda$ by a strong-weak duality formula:

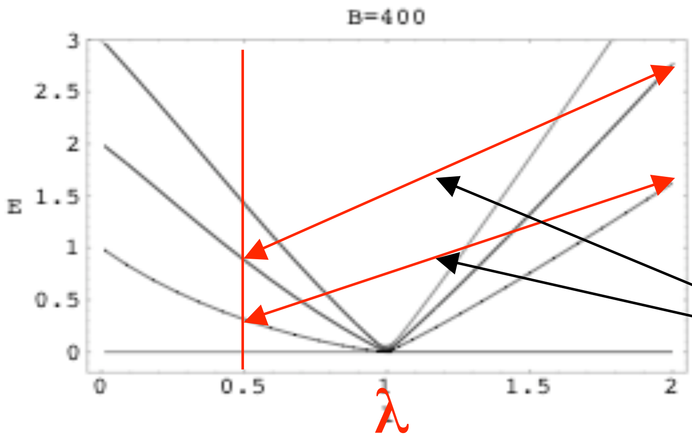
$$E(1/\lambda)+1=\lambda^{-2}(E(\lambda)+1)$$

- For F=0,1 the spectrum can be computed analytically in terms of the zeroes of an $_1F_2$ function. Duality and phase transition can be studied analytically

- At $\lambda > 1$ a second F=0, E=0 bosonic state pops up making Witten's index jump by one unit (within the F=0, 1 sectors).
- First found numerically. The analytic form of the 2nd ground state can be formally given at all λ but is only normalizable for $\lambda > 1$
- In the F=2 sector two more E=0 states pop up at $\lambda > 1$: Witten index jumps by two more units
- For finite cutoff (=> SUSY expl.ly broken) supermultiplets rearrange around $\lambda = 1$ by a sort of "partner swapping" mechanism. At infinite cutoff, these new "couples" emerge already "remarried" from an infinitely degenerate state

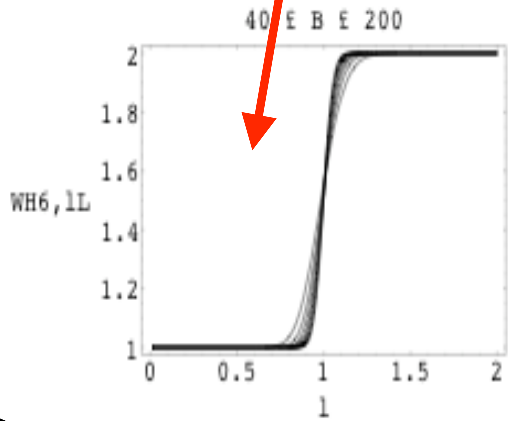


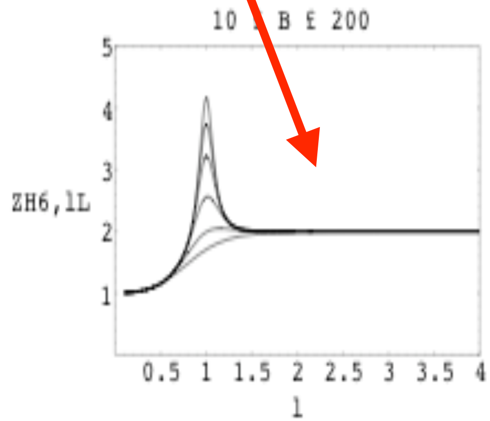




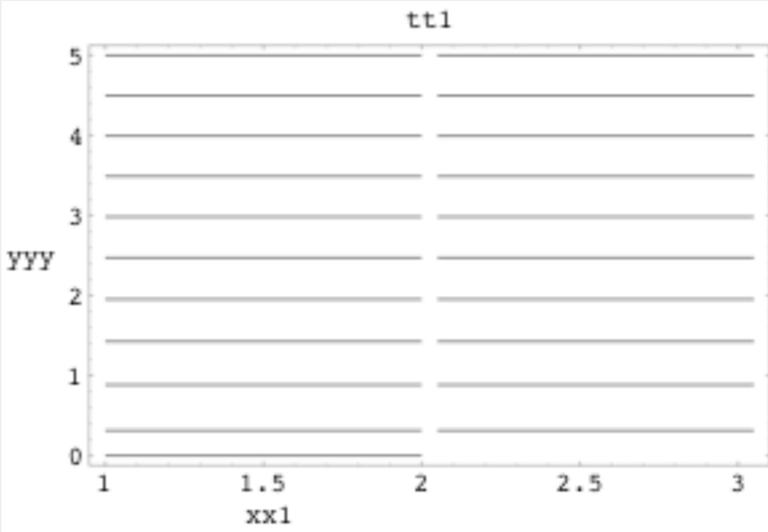
Lowest bosonic and fermionic states as a funtion of λ for different values of the cutoff (NB swapping of SUSY partners at finite cutoff)

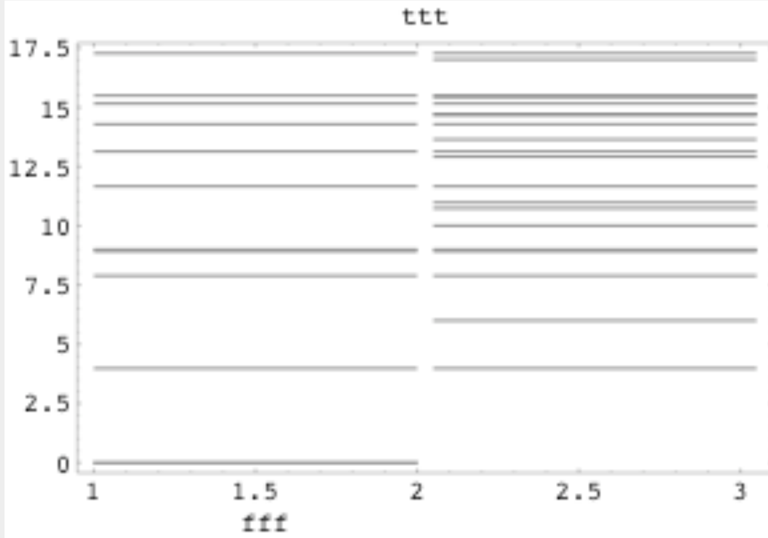
Witten index and free energy as functions of λ



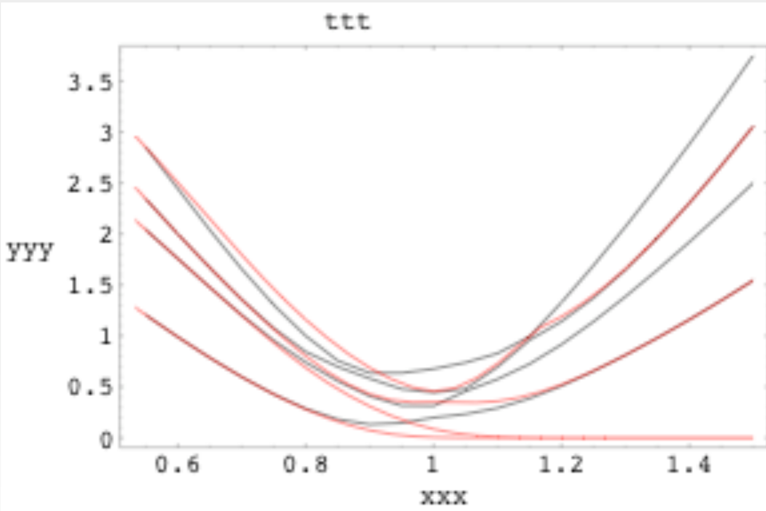


Energy related by $\lambda^2\,(E(1/\lambda)+1) = E(\lambda)+1$

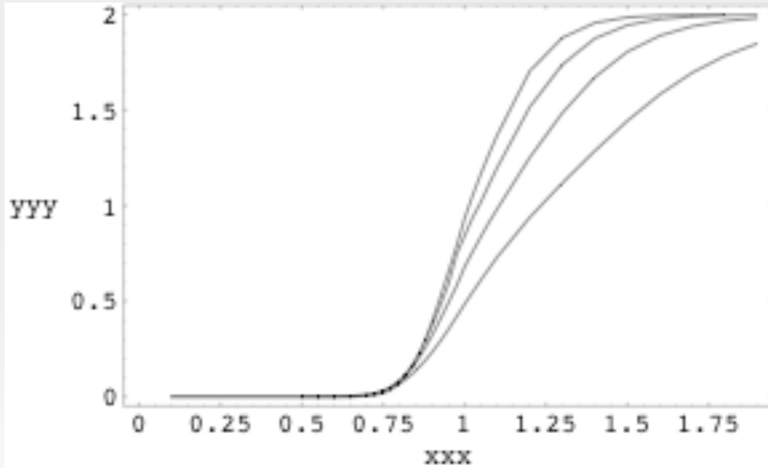




F=0 F=1 F=2 F=3



F=2,3



F=2,3

Connection with

Binary Necklaces (BNL)

(E.Onofri, J.Wosiek & GV,math-ph/0603082)

- Having constructed, counted, and paired the states in SUSY doublets, we searched for something similar known in maths.
- Naturally, we looked for a possible connection with binary necklaces, necklaces with two kinds of beads, zeros and ones (or bosons and fermions)
- Their number (see the on-line encyclopedia of integer sequences):
- A000031(n) = Number of n-bead necklaces with 2 colours when turning over is not allowed (cyclic and anticyclic are distinct) is given by Mac Mahon's formula (see below).

But there was a problem:

- The number of binary necklaces w/ even and odd # of fermions is, in general, different! Example (n=2)

  (aa), (ff), (af) = (fa) => 2 bosons, 1 fermion, ..

  and indeed the numbers did not match..

- Q: How can supersymmetry work if $n_B \neq n_F$?
- A: Pauli's exclusion principle kills some BNL giving back the balance between bosons and fermions

  $N(n) = N_{PAN}(n)$ (PAN = Pauli-allowed necklaces)

Binary Necklaces, Pauli Necklaces

n= B+F

B \ F	even	odd
even	PFN	
even	PAN $\neq$ BNL	PAN = BNL
odd	PAN = BNL	PAN = BNL

SUSY

$$N_{BNL}(n) = \frac{1}{n}\sum_{d|n} \varphi(d)\, 2^{n/d}$$

$$N_{PAN}(n) = \frac{1}{n}\sum_{\substack{d|n \\ d\,odd}} \varphi(d)\, 2^{n/d}$$

$$N_{PFN}(n) = \frac{1}{n}\sum_{\substack{d|n \\ d\,even}} \varphi(d)\, 2^{n/d}$$

ϕ(d) is Euler's "totient" function counting the number of prime numbers ($\leq$ d) relative to d

If B and F are not both even we have a more detailed counting:

$$N_{BNL}(B,F) = N_{PAN}(B,F) = \sum_{d/F,B} \frac{\phi(d)}{(B+F)} \binom{(B+F)/d}{B/d}$$

giving back the previous formula for B+F odd if one sums over B at fixed n=B+F.

When B & F are even we have proven a simple formula for PFN

$$N_{PFN}(B,F) = N_{BNL}(B/2^k, F/2^k) = N_{PAN}(B/2^k, F/2^k)$$

(see table) where k is the unique +ve integer (if it exists) for which $F/2^k$ is odd and $B/2^k$ is integer (otherwise n_{PFN} is zero).

$$N_{PAN}(B,F) = N_{BNL}(B,F) - N_{BNL}(B/2^k, F/2^k)$$

$$N_{BNL}(B,F) = N_{PAN}(B,F) + N_{PAN}(B/2^k, F/2^k)$$

N_{PFN} fluctuates a lot!

																				$F \rightarrow$
$B \downarrow$	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40
0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
4	0	1	1	2	0	3	1	4	0	5	1	6	0	7	1	8	0	9	1	10	0
6	0	1	0	4	0	7	0	12	0	19	0	26	0	35	0	46	0	57	0	70	0
8	0	1	1	5	1	14	2	30	0	55	3	91	1	140	4	204	0	285	5	385	1
10	0	1	0	7	0	26	0	66	0	143	0	273	0	476	0	776	0	1197	0	1771	0
12	0	1	1	10	0	42	4	132	0	335	7	728	0	1428	12	2586	0	4389	19	7084	0
14	0	1	0	12	0	66	0	246	0	715	0	1768	0	3876	0	7752	0	14421	0	25300	0
16	0	1	1	15	1	99	5	429	1	1430	14	3978	2	9690	30	21318	0	43263	55	82225	3
18	0	1	0	19	0	143	0	715	0	2704	0	8398	0	22610	0	54484	0	120175	0	246675	0
20	0	1	1	22	0	201	7	1144	0	4862	26	16796	0	49742	66	130752	0	312455	143	690690	0
22	0	1	0	26	0	273	0	1768	0	8398	0	32066	0	104006	0	297160	0	766935	0	1820910	0
24	0	1	1	31	1	364	10	2652	0	14000	42	58786	4	208012	132	643856	0	1789515	335	4552275	7
26	0	1	0	35	0	476	0	3876	0	22610	0	104006	0	400024	0	1337220	0	3991995	0	10855425	0
28	0	1	1	40	0	612	12	5538	0	35530	66	178296	0	742900	246	2674440	0	8554275	715	24812400	0
30	0	1	0	46	0	776	0	7752	0	54484	0	297160	0	1337220	0	5170604	0	17678835	0	54587280	0
32	0	1	1	51	1	969	15	10659	1	81719	99	482885	5	2340135	429	9694845	1	35357670	1430	115997970	14
34	0	1	0	57	0	1197	0	14421	0	120175	0	766935	0	3991995	0	17678835	0	68635478	0	238819350	0
36	0	1	1	64	0	1463	19	19228	0	173593	143	1193010	0	6653325	715	31429068	0	129644790	2704	477638700	0
38	0	1	0	70	0	1771	0	25300	0	246675	0	1820910	0	10855425	0	54587280	0	238819350	0	930138522	0
40	0	1	1	77	1	2126	22	32890	0	345345	201	2731365	7	17368680	1144	92798380	0	429874830	4862	1767263190	26

TABLE 1. N_{PFN}, the number of Pauli forbidden necklaces(entries with odd F and/or odd B vanish identically) .

A formula for the PAN generating function
(OVW(DZ) see also Bianchi, Morales & Samtleben)

$$\Phi_{PAN}(x,y;n) \equiv \sum_F N_{PAN}(n-F,F)\, x^{n-F} y^F = \frac{1}{n}\sum_{d/n} \phi(d) \left(x^d - (-y)^d\right)^{n/d}$$

leads immediately to formulae for Witten-like indices

$$W(n;m) \equiv \sum_{\substack{B+F=n \\ F \leq m}} (-1)^{F-m} N_{PAN}(B,F) \geq 0\ ,\ W(n;n) = 0$$

$$\tilde{W}(n;n) = \sum_F (-1)^F N_{PAN}(n-2F,F) = \delta_{n=1(mod6)} + \delta_{n=-1(mod6)}$$

Conjecture: as λ -> ∞ there is one and only one E=0 bosonic eigenstate in and only in each (B,F) block with |B-F| = 1

F=0

F=2

$B \downarrow$ \ $F \rightarrow$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	[illegible]	2	[illegible]	3	3	4	5	5	5	6	7	7	7	8	9	9	9	10	11
3	1	1	2	4	5	7	10	12	15	19	22	26	31	35	40	46	51	57	64	70	77
4	1	1	2	5	9	14	20	30	43	55	70	91	115	140	168	204	245	285	330	385	445
5	1	1	3	7	14	26	42	66	99	143	201	273	364	476	612	776	969	1197	1463	1771	2126
6	1	1	3	10	22	42	[illegible]	132	217	335	497	728	1038	1428	1932	2586	3399	4389	5601	7084	8866
7	1	1	4	12	30	66	132	246	429	715	1144	1768	2652	3876	5538	7752	10659	14421	19228	25300	
8	1	1	4	15	42	99	212	429	[illegible]	1430	2424	3978	6308	9690	14520	21318	30667	43263	60060		
9	1	1	5	19	55	143	335	715	1430	2704	4862	8398	14000	22610	35530	54484	81719	120175			
10	1	1	5	22	73	201	497	1144	2438	4862	[illegible]	16796	[illegible]	49742	81686	130752	204347				
11	1	1	6	26	91	273	728	1768	3978	8398	16796	32066	5878[illegible]	104006	178296	297160					
12	1	1	6	31	115	364	1028	2652	6310	14000	29372	58786	[illegible]	208012	[illegible]						
13	1	1	7	35	140	476	1428	3876	9690	22610	49742	104006	208012	[illegible]							
14	1	1	7	40	172	612	1932	5538	14550	35530	81686	178296	371[illegible]6								
15	1	1	8	46	204	776	2586	7752	21318	54484	130752	297160									
16	1	1	8	51	244	969	3384	10659	30666	81719	204248										
17	1	1	9	57	285	1197	4389	14421	43263	120175											
18	1	1	9	64	335	1463	5601	19228	60115												
19	1	1	10	70	385	1771	7084	25300													
20	1	1	10	77	[illegible]	[illegible]	[illegible]														
21	1	1	11	85	506	2530															
22	1	1	11	92	578																
23	1	1	12	100																	
24	1	1	12																		
25	1	1																			
26	1																				

742900

208012

1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786,

(block sizes = Catalan's numbers)

TABLE 1. $N_{\mathrm{PAN}}(B,F)$ as generated with the sieve method.

Connections with statistical mechanics

(J.Wosiek & GV hep-th/0609210)

1. XXZ spin chain

$$H^{(\Delta)}_{XXZ} = -\frac{1}{2}\sum_{i=1}^{n}\left(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \Delta\sigma_i^z\sigma_{i+1}^z\right)$$

With a cyclic symmetry: n+1 coincides with 1

We have proven the following equivalence between the XXZ chain at asymmetry parameter Δ and our (rescaled) SUSY QM at $\lambda = \infty$ (H_{SC})

If F is odd: $$H_{SC} \Leftrightarrow -H^{(+1/2)}_{XXZ} + \frac{3}{4}n$$ n = B+ F

If F is even and B is odd (includes magic stairway):

$$H_{SC} \Leftrightarrow H^{(-1/2)}_{XXZ} + \frac{3}{4}n$$

NB: SC SUSY connects these two cases for odd B

Non-trivial consequences of SUSY for XXZ

- We reinterpret the ground state of XXZ model at Δ = - 1/2 as the E=0 state of a SUSY theory: will this help proving (some of) the RS conjectures?

A. V. Razumov & Y. G. Stroganov, cond-mat/0012141
One conjecture: ratio of largest to smallest component of ground-state eigenvector = number of alternating sign matrices. If n=2m+1:

$$A_m = \prod_{j=0}^{m-1} \frac{(3j+1)!}{(m+j)!}$$

For m=8 this number is 10,850,216.
Math. gave this to 0.1 acc.(1430^2 mx)

- SUSY relates in a non-trivial way XXZ spectra at different asymmetry parameter and number of sites: spectrum for Δ = +1/2 contained in that of Δ = - 1/2 and vice versa (probably unnoticed so far)

Conjecture: as λ -> ∞ there is one and only one E=0 bosonic eigenstate in and only in each (B,F) block with |B-F| = 1

F=0

F=2

$B\downarrow$ / $F\rightarrow$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	[illegible]	2	[illegible]	3	3	4	5	5	5	6	7	7	7	8	9	9	9	10	11
3	1	1	2	4	5	7	10	12	15	19	22	26	31	35	40	46	51	57	64	70	77
4	1	1	2	5	9	14	20	30	43	55	70	91	115	140	168	204	245	285	330	385	445
5	1	1	3	7	14	26	42	66	99	143	201	273	364	476	612	776	969	1197	1463	1771	2126
6	1	1	3	10	22	42	[illegible]	132	217	335	497	728	1038	1428	1932	2586	3399	4389	5601	7084	8866
7	1	1	4	12	30	66	132	246	429	715	1144	1768	2652	3876	5538	7752	10659	14421	19228	25300	
8	1	1	4	15	42	99	212	429	[illegible]	1430	2424	3978	6308	9690	14520	21318	30667	43263	60060		
9	1	1	5	19	55	143	335	715	1430	2704	4862	8398	14000	22610	35530	54484	81719	120175			
10	1	1	5	22	73	201	497	1144	2438	4862	[illegible]	16796	[illegible]	49742	81686	130752	204347				
11	1	1	6	26	91	273	728	1768	3978	8398	16796	32066	58786	104006	178296	297160					
12	1	1	6	31	115	364	1028	2652	6310	14000	29372	58786	[illegible]	208012	[illegible]						
13	1	1	7	[illegible]	[illegible]	[illegible]	[illegible]	3876	9690	22610	49742	104006	208012	400024							
14	1	1	7	40	172	612	1932	5538	14550	35530	81686	178296	[illegible]								
15	1	1	8	46	204	776	2586	7752	21318	54484	130752	297160									
16	1	1	8	51	244	969	3384	10659	30666	81719	204248										
17	1	1	9	57	285	1197	4389	14421	43263	120175											
18	1	1	9	64	335	1463	5601	19228	60115												
19	1	1	10	70	385	1771	7084	25300													
20	1	1	10	77	[illegible]	[illegible]	[illegible]														
21	1	1	11	85	506	2530															
22	1	1	11	92	578																
23	1	1	12	100																	
24	1	1	12																		
25	1	1																			
26	1																				

TABLE 1. $N_{\mathrm{PAN}}(B, F)$ as generated with the sieve method.

B=9, F=8

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1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786,

Conclusions, Part II

- SUSY has implications about non-trivial combinatorial problems
- Combinatorial methods have non-trivial implications on the dynamics of SUSY models
- Extending the approach to (semi) realistic QFTs w/ or w/out SUSY remains the main physics goal of this (otherwise just amusing mathematical) game. Work in progress in D=2. However:
- Interesting connections to stat. mech. models have already emerged at infinite λ (Cf. AdS/CFT!)