Holomorphic Parafermions in Loop Models and SLE

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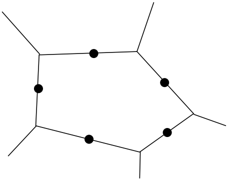
Discretely holomorphic functions



- $\mathcal{G}$ is a planar graph embedded in $\mathbf{R}^2$, vertices $\{z_j\}$
- $F(z_{jk})$ is a function defined on the mid-points $z_{jk} = \frac{1}{2}(z_j + z_k)$ of the edges (jk) of $\mathcal{G}$
- F is *discretely holomorphic* if

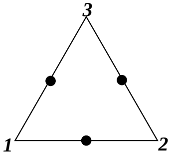
$$\sum_{(jk)\in\mathcal{F}} F(z_{jk})(z_j - z_k) = 0$$

for each face $\mathcal{F}$.

- discrete version of Cauchy's theorem
- depends on how $\mathcal{G}$ is embedded in $\mathbf{R}^2$

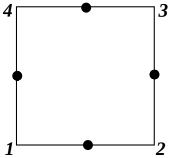
Examples

- regular triangular lattice



$$F(z_{12}) + \omega F(z_{23}) + \omega^2 F(z_{31}) = 0 \qquad (\omega = e^{2\pi i/3})$$

- regular square lattice



$$F(z_{12}) + iF(z_{23}) + i^2 F(z_{34}) + i^3 F(z_{41}) = 0$$

Warning!

- there are only N_{faces} linear equations for N_{edges} unknowns – not enough to determine $F(z_{jk})$
- additional arguments or assumptions [e.g. that $F(z)$ is continuous] are needed to assert that F becomes an analytic function in the continuum limit
- compare with *discretely harmonic* functions

Importance of discretely holomorphic observables

- in the scaling limit they are expected to converge to correlators

$$F(z) = \langle \psi_s(z) \prod_j \mathcal{O}_j(z_j, \bar{z}_j) \rangle$$

  of holomorphic fields $\psi_s(z)$
  - typically these have *fractional conformal spin s* and are called parafermionic fields
- they have simple power law correlations

$$\langle \psi_s(z_1)\psi_s(z_2) \rangle \sim (z_1 - z_2)^{-2s}$$

  which indicate that the lattice model is *critical*.
- they can often be used as building blocks for the whole CFT
- if this describes the ground state of a 2+1-dimensional quantum model, they correspond to excitations with *fractional statistics*
- if $F(z)$ can be related to a suitable observable of a *curve* in the original lattice model, holomorphicity can be used to argue that its scaling limit must be SLE [Smirnov].

Examples

- for many models with a *duality* symmetry, parafermionic observables can be defined by

$$\psi_s(z_{rR}) \sim e^{is\theta_{rR}} \underbrace{\sigma(r)}_{\text{order}} \underbrace{\mu(R)}_{\text{disorder}}$$

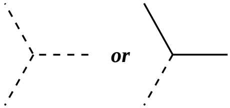
  where $\theta_{rR} = \arg(z_r - z_R)$ and $2\pi s$ is the phase $s(r)$ picks up as it goes once around $\mu(R)$.
- for Z_N lattice models these are discretely holomorphic precisely at the Fateev-Zamolodchikov critical points [Rajabpour & JC, 2007]
- for the FK representation of the Potts model at its critical point [Riva & JC, Smirnov, 2006]
- however duality is *not* necessary..

O(n) model on honeycomb lattice [Smirnov]

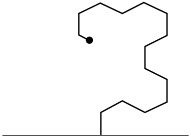
- nested set of loops,

$$Z = \sum x^{\text{total length}}\, n^{\text{number of loops}}$$

- at each vertex



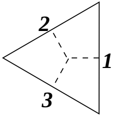
- in addition to the closed loops, suppose there is an open curve γ from a fixed point on the boundary to a point r mid-way along an edge:



$$F(r) \equiv \langle x^{|\gamma|} \, \lambda^{|\text{left turns}|} \, \bar{\lambda}^{|\text{right turns}|} \rangle$$

where $\lambda = e^{-is(\pi/3)}$.

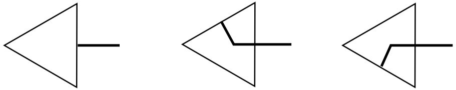
- consider neighbouring points $r \in \{z_1, z_2, z_3\}$



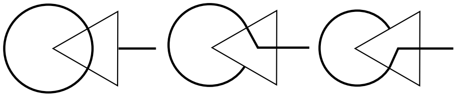
- then we want to show that

$$F(z_1) + \omega F(z_2) + \omega^2 F(z_3) = 0$$

where $\omega = e^{2\pi i/3}$.
- in fact this is true, term by term, for a fixed configuration outside the triangle
- suppose γ first hits the triangle at z_1
- possibilities are



$$1 + x\bar{\lambda}\omega + x\lambda\omega^2$$



$$nx + x\bar{\lambda}^4\omega + x\lambda^4\omega^2$$

Setting these =0 gives

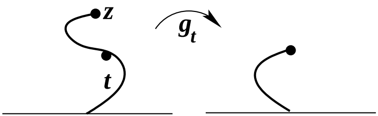
$$x = \left(2 \pm \sqrt{2-n}\right)^{-1/2}$$

(in agreement with Nienhuis' conjecture [1982]), and

$$s = (6-\kappa)/2\kappa \qquad \text{where} \quad n = -2\cos(4\pi/\kappa)$$

- in CFT language, the parafermion has conformal dimensions $(h_{2,1}, 0)$
- holomorphic operators have the same bulk and boundary scaling dimensions, so the operator which creates the curve at the boundary must have dimension $h_{2,1}$
- this is consistent with the scaling limit of γ being SLE_κ

Connection to SLE



- as $z \rightarrow$ positive(negative) real axis, winding $\rightarrow \pm\pi$, so

$$\langle\psi_s(z)\rangle_{\mathrm{UHP}} \propto z^{-2s}$$

- under Loewner map $z \longrightarrow g_t(z)$, so

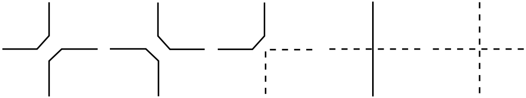
$$\langle\psi_s(z)\rangle = g_t'(z)^s \langle\psi_s\big(g_t(z)\big)\rangle = \left(\frac{g_t'(z)}{g_t^2(z)^2}\right)^s$$

  for *all* $t < t_z$

- use Loewner equation $dg_t = 2dt/g_t^2 - dW_t$ for large z to show that $\mathbf{E}(W_t) = 0$ and $\mathbf{E}\big((W_t - W_{t'})^2\big) = \kappa|t - t'|$, so that W_t is Brownian motion with

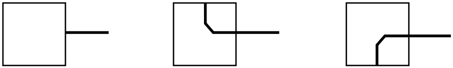
$$s = (6 - \kappa)/2\kappa$$

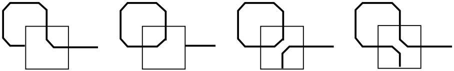
$\mathrm{O}(n)$ model on square lattice [Ikhlef, JC, Fendley]



with weights $(w, w, u, v, 1)$ (below we take $v = 0$ for simplicity)

$$F(z_1) + iF(z_2) + i^2 F(z_3) + i^3 F(z_4) = 0$$





$$\begin{aligned} 1 + u\bar{\lambda} i + u\lambda i^3 &= 0 \\ u\lambda^2 i^2 + nu + nw\lambda i^3 + w\lambda i^3 &= 0 \end{aligned}$$

where $\lambda = e^{-is(\pi/2)}$.

- solution for $n = 1$, $w = \frac{1}{2}$, and $u = \big(2\sin(\pi s/2)\big)^{-1} \geq \frac{1}{2}$ (agrees with integrable case found by Nienhuis [1989])
- this lattice model can be mapped exactly onto the critical phase $|\Delta| \leq 1$ of the 6-vertex model, so is a $c = 1$ critical theory
- for $v \neq 0$ the model maps onto the 19-vertex model of Izergin and Korepin [1981] and discrete holomorphicity = integrability
- the model appears to give examples of SLEs in theories with both $c < 1$ and $c > 1$

Summary

- parafermionic observables can be identified in a number of lattice spin and loop models
- they are discretely holomorphic at *integrable, critical* points
  - however discrete holomorphicity gives simple linear equations in the weights and doesn't require introduction of a spectral parameter
- they are conjectured to correspond to holomorphic parafermions in CFT
- they can be used to show that lattice curves converge to SLE
- in 2+1 dimensions they obey fractional statistics