

A Model of Anyons in One Dimension. Correlation Functions.

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Thank you

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Outline

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One-Dimensional Anyons

- Lieb-Liniger Gas of Anyons
- Quantum Mechanics, Bethe Ansatz and Yang Thermodynamics

2

Correlation Functions

- Determinant Representation
- Differential equations
- Matrix Riemann-Hilbert Problem
- Large Distance Asymptotics

Lieb-Liniger Gas of Anyons

$$H = \int dx \left([\partial_x \Psi_A^\dagger(x)] [\partial_x \Psi_A(x)] + c \Psi_A^\dagger(x) \Psi_A^\dagger(x) \Psi_A(x) \Psi_A(x) - h \Psi_A^\dagger(x) \Psi_A(x) \right)$$

$$\Psi_A(x_1) \Psi_A^\dagger(x_2) = e^{-i\pi\kappa\epsilon(x_1-x_2)} \Psi_A^\dagger(x_2) \Psi_A(x_1) + \delta(x_1 - x_2)$$

$$\Psi_A^\dagger(x_1) \Psi_A^\dagger(x_2) = e^{i\pi\kappa\epsilon(x_1-x_2)} \Psi_A^\dagger(x_2) \Psi_A^\dagger(x_1)$$

- **Statistics parameter** $\kappa \in [0, 1]$ ($\kappa = 0$ bosons, $\kappa = 1$ fermions)
- $\epsilon(x) = |x|/x = \text{sign of } x$,
- Limit $c \rightarrow \infty$ describes impenetrable case

⇒ Algebraic definition of fractional statistics in one dimension

- Girardeau, Batchelor, Khundu: consistency of algebra
- Averin, Nesteroff : Coulomb blockade in a coherent transport in systems of multiple antidots of FQHE
- Calabrese, Mintchev, Santachiara, Cabra: important results

Quantum Mechanics of 1D Anyons

Eigenstates of the Hamiltonian

$$|\Psi_N\rangle = \frac{1}{\sqrt{N!}} \int dz^N \chi_N(z_1, \dots, z_N | \lambda_1, \dots, \lambda_N) \Psi_A^\dagger(z_N) \cdots \Psi_A^\dagger(z_1) |0\rangle$$

$$\chi_N(\dots, z_i, z_{i+1}, \dots) = e^{i\pi\kappa\epsilon(z_i - z_{i+1})} \chi_N(\dots, z_{i+1}, z_i, \dots)$$

Self-consistent boundary conditions

$$\chi_N(0, z_2, \dots, z_N) = \chi_N(L, z_2, \dots, z_N),$$

$$\chi_N(z_1, 0, \dots, z_N) = e^{i(2\pi\kappa)} \chi_N(z_1, L, \dots, z_N),$$

$$\vdots$$

$$\chi_N(z_1, z_2, \dots, 0) = e^{i(2\pi(N-1)\kappa)} \chi_N(z_1, z_2, \dots, L).$$

Patu, Korepin, Averin . arXiv:0707.4520

Bethe Ansatz and Yang Thermodynamics

$$\chi_N = \frac{e^{+i\frac{\pi\kappa}{2} \sum_{j < k} \epsilon(z_j - z_k)}}{\sqrt{N!}} \prod_{j > k} \epsilon(z_j - z_k) \sum_{\pi \in S_N} (-1)^\pi e^{i \sum_{n=1}^N z_n \lambda_{\pi(n)}}.$$

Bethe equations = boundary conditions

$$e^{i\lambda_j L} = (-1)^{N-1} e^{i\bar{\eta}}, \quad \bar{\eta} = -\pi\kappa(N-1)$$

Twist $\sim N$. Momentum in the ground state.

Thermodynamic limit $N \rightarrow \infty$, $L \rightarrow \infty$, N/L is fixed.

Yang-Yang thermodynamics

$$\vartheta(\lambda, \mathbf{h}, \mathbf{T}) = \frac{1}{(1 + e^{(\lambda^2 - \mathbf{h})/\mathbf{T}})}$$

$$D = \frac{1}{2\pi} \int_{-\infty}^{\infty} \vartheta(\lambda, h, T) d\lambda, \quad E = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lambda^2 \vartheta(\lambda, h, T) d\lambda$$

Correlation Functions

$$\langle \Psi_A^\dagger(x) \Psi_A(0) \rangle_T = \frac{\text{Tr} \{ e^{-\frac{H}{T}} \Psi_A^\dagger(x) \Psi_A(0) \}}{\text{Tr} \{ e^{-\frac{H}{T}} \}}$$

- Method of A. Its, A. Izergin, V.Korepin and N.Slavnov: Journal of Modern Physics B. **4**, 1003 (1990).
- Represent correlation function as Fredholm determinant.
Integrable integral operators.
- Differential equations.
- Riemann-Hilbert problem \rightarrow large distance asymptotic.
- Textbook: QUANTUM INVERSE SCATTERING METHOD AND CORRELATION FUNCTIONS, by A. Izergin, V. Korepin, N. Bogoliubov, Cambridge University Press, 1993

Important developments: V. Cheianov, M. Zvonarev; F. Colomo, A. Pronko; N. Kitanine, K. Kozlowski, J. Maillet, N. Slavnov, V. Terras; F. Essler, F. Latremoliere.

Determinant Representations

$$\langle \Psi_A^\dagger(x) \Psi_A(0) \rangle_T = B_{++} \det(1 - \gamma \hat{K}_T)$$

The kernel of \hat{K}_T

$$K_T(\lambda, \mu) = \frac{e_+(\lambda)e_-(\lambda) - e_-(\lambda)e_+(\mu)}{2i(\lambda - \mu)}, \quad e_\pm(\lambda) = \sqrt{\vartheta(\lambda)} e^{\pm i\lambda x}$$

$$B_{lm} \equiv \gamma \int_{-\infty}^{+\infty} e_l(\lambda) f_m(\lambda) d\lambda, \quad l, m = \pm$$

$$f_\pm(\lambda) - \gamma \int_{-\infty}^{+\infty} K_T(\lambda, \mu) f_\pm(\mu) d\mu = e_\pm(\lambda), \quad \gamma = \frac{(1 + e^{i\pi\kappa})}{\pi}$$

The functions $f_\pm(\lambda)$ appear during inversion of the operator $(\hat{I} - \gamma \hat{K}_T)$.

$$\langle \Psi_A^\dagger(0) \Psi_A(x) \rangle_T = \overline{\langle \Psi_A^\dagger(x) \Psi_A(0) \rangle_T}$$

Differential Equations for Potentials

$$\partial_\beta B_{+-} = x + \frac{1}{2} \frac{\partial_x \partial_\beta B_{++}}{B_{++}}, \quad \partial_x B_{+-} = B_{++}^2$$

initial conditions

$$B_{++} = \gamma d(\beta) + [\gamma d(\beta)]^2 x, \quad x \rightarrow 0,$$

$$B_{+-} = \gamma d(\beta) + [\gamma d(\beta)]^2 x, \quad x \rightarrow 0$$

$$d(\beta) = \int_{-\infty}^{+\infty} \vartheta(\lambda) d\lambda, \quad \vartheta(\lambda, h, T) = \frac{1}{(1 + e^{(\lambda^2 - h)/T})}, \quad \gamma = \frac{(1 + e^{i\pi\kappa})}{\pi}$$

$$B_{++} = B_{+-} \rightarrow 0, \quad \beta \rightarrow -\infty.$$

Differential equations for $\sigma = \ln \det(1 - \gamma \hat{K}_T)$

$$\begin{aligned}\partial_x \sigma &= -B_{+-}, & \partial_x^2 \sigma &= -B_{++}, \\ \partial_\beta \sigma &= -x \partial_\beta B_{+-} + \frac{1}{2} (\partial_\beta B_{+-})^2 - \frac{1}{2} (\partial_\beta B_{++})^2,\end{aligned}$$

Integrable nonlinear partial differential equation for σ

$$(\partial_\beta \partial_x^2 \sigma)^2 + 4(\partial_x^2 \sigma)[2x \partial_\beta \partial_x \sigma + (\partial_\beta \partial_x \sigma)^2 - 2\partial_\beta \sigma] = 0$$

initial conditions depends on statistics:

$$\begin{aligned}\sigma &= -\gamma d(\beta)x - [\gamma d(\beta)]^2 \frac{x^2}{2} + O(x^3), & x &\rightarrow 0 \\ \sigma &= 0, & \beta &\rightarrow -\infty,\end{aligned}$$

Differential equation for σ at $T \rightarrow 0$

$$\xi = xh^{1/2} \text{ and } \sigma_0 = \xi(\tilde{\sigma}_0)'$$

$$(\xi\sigma_0'')^2 + 4(\xi\sigma_0' - \sigma_0)[4\xi\sigma_0 - 4\sigma_0 + (\sigma_0')^2] = 0$$

This Painlevé V equation was first derived by Jimbo, Miwa, Sato, Mori for impenetrable bosons. For anyons **initial conditions are different**.

$$\sigma_0 = -2\gamma\xi - 4\gamma^2\xi^2 + O(\xi^3)$$

Agrees with Santachiara and Calabrese Toeplitz determinant approach.

Matrix Riemann-Hilbert Problem

Matrix function $\chi(\lambda)$ is analytic in the upper and lower half-plane and

$$\begin{aligned}\chi_-(\lambda) &= \chi_+(\lambda)\mathbf{G}(\lambda), & \chi_{\pm}(\lambda) &= \lim_{\epsilon \rightarrow 0^+} \chi(\lambda \pm i\epsilon), \quad \lambda \in \mathbb{R} \\ \chi(\infty) &= I\end{aligned}$$

Conjugation matrix

$$\mathbf{G}(\lambda) = \begin{pmatrix} 1 + \pi\gamma \mathbf{e}_+(\lambda)\mathbf{e}_-(\lambda) & -\pi\gamma \mathbf{e}_+^2(\lambda) \\ \pi\gamma \mathbf{e}_-^2(\lambda) & 1 - \pi\gamma \mathbf{e}_+(\lambda)\mathbf{e}_-(\lambda) \end{pmatrix}$$

here $\mathbf{e}_{\pm}(\lambda) = \sqrt{\vartheta(\lambda)} \mathbf{e}^{\pm i\lambda x}$

$$\lim_{\lambda \rightarrow \infty} \chi(\lambda) = I + \frac{1}{2i\lambda} \begin{pmatrix} B_{+-} & -B_{++} \\ B_{++} & -B_{+-} \end{pmatrix} + O\left(\frac{1}{\lambda^2}\right)$$

Asymptotics

$$\langle \Psi_A^\dagger(x) \Psi_A(0) \rangle_T = e^{-x\sqrt{T}C(h/T, \kappa)/2} \left(c_0 e^{ix\sqrt{T}\lambda_0} + c_1 e^{ix\sqrt{T}\lambda_1} \right),$$

$$C(\beta, \kappa) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \ln \left(\frac{e^{\lambda^2 - \beta} + 1}{e^{\lambda^2 - \beta} - e^{i\pi\kappa}} \right) d\lambda,$$

$$\lambda_0 = \frac{1}{\sqrt{2}} \left(\beta + \sqrt{\beta^2 + \pi^2 \kappa^2} \right)^{1/2} + \frac{i}{\sqrt{2}} \left(-\beta + \sqrt{\beta^2 + \pi^2 \kappa^2} \right)^{1/2}, \quad \beta = h/T$$

$$\lambda_1 = -\frac{1}{\sqrt{2}} \left(\beta + \sqrt{\beta^2 + \pi^2 [\kappa - 2]^2} \right)^{1/2} + \frac{i}{\sqrt{2}} \left(-\beta + \sqrt{\beta^2 + \pi^2 [\kappa - 2]^2} \right)^{1/2}$$

Conformal limit

For small temperature the expression in the exponent becomes linear function of temperature

- For $0 < \kappa < 2/3$: $\Im\lambda_1 \leq 2\Im\lambda_0$

$$\langle \Psi_A^\dagger(x) \Psi_A(0) \rangle_T \simeq c_0 e^{-x \frac{\pi T}{v_F} \left(\frac{\kappa^2}{2} + \frac{1}{2} \right)} e^{ixk_F \kappa},$$

- For $2/3 < \kappa < 1$: $\Im\lambda_1 \geq 2\Im\lambda_0$

$$\langle \Psi_A^\dagger(x) \Psi_A(0) \rangle_T \simeq c_0 e^{-x \frac{\pi T}{v_F} \left(\frac{\kappa^2}{2} + \frac{1}{2} \right)} e^{ixk_F \kappa} + c_1 e^{-x \frac{\pi T}{v_F} \left[2 \left(\frac{\kappa}{2} - 1 \right)^2 + \frac{1}{2} \right]} e^{ixk_F (\kappa - 2)}$$

Agrees with harmonic fluid approach of Calabrese and Mintchev 2006.

Large time and distance asymptotics

Back to higher temperatures.

Determinant representations and Riemann-Hilbert approach also work for time and temperature dependent correlations.

Asymptotic $x \rightarrow \infty, \quad t \rightarrow \infty, \quad \{x/t \text{ is fixed}\}$

$$\langle \Psi_A(x_2, t_2) \Psi_A^\dagger(x_1, t_1) \rangle_T \sim \exp \left\{ \frac{1}{\pi} \int_{-\infty}^{+\infty} |x - 2t\lambda| \ln \left| \frac{e^{\lambda^2 - \beta} - e^{i\pi\kappa}}{e^{\lambda^2 - \beta} + 1} \right| d\lambda \right\}$$

$$x = \frac{1}{2}(x_1 - x_2)\sqrt{T}, \quad t = \frac{1}{2}(t_2 - t_1)T, \quad \beta = \frac{h}{T}$$

λ is momentum, κ is statistics parameter ($\kappa \neq 1$) and h is chemical potential.

