Topological stability of q-deformed quantum spin chains

Eddy Ardonne

Nordita

Collaborators:
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ERLDQS, Firenze, 090908

Motivation

- What's a spin chain?
- Use a modular tensor category/quantum group!

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Outline

- The golden (fibonacci) chain
- Topological stability of criticality
- Perturbing the chain
- Generalizations

The Heisenberg model

Fusion rules for SU(2)$_3$

$$1 \times 1 = 1$$
$$1 \times \tau = \tau$$
$$\boxed{\tau \times \tau = 1 + \tau}$$

SU(2) spins

$$\frac{1}{2} \times \frac{1}{2} = 0 + 1$$

triplet

singlet

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Heisenberg Hamiltonian

$$H = J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j$$
$$= J \sum_{\langle ij \rangle} \vec{J_{ij}}^{\,2} - \vec{S_i}^{\,2} - \vec{S_j}^{\,2}$$

$$\vec{J}_{ij} = \vec{S}_i + \vec{S}_j$$

$$H = J \sum_{\langle ij \rangle} \prod_{ij}^{0}$$

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Heisenberg Hamiltonian for (Fibonacci) anyons

$$H = J \sum_{\langle ij \rangle} \prod_{ij}^{1}$$

"antiferromagnet" favors 1 $(J > 0)$

"ferromagnet" favors τ $(J < 0)$

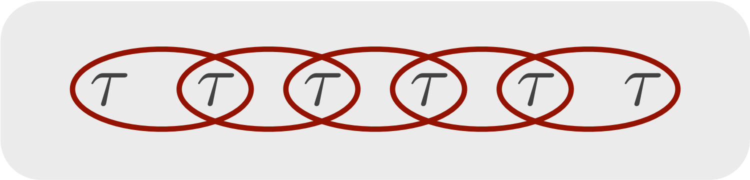
Heisenberg Hamiltonian

$$\begin{aligned} H &= J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j \\ &= J \sum_{\langle ij \rangle} \vec{J}_{ij}^{\,2} - \vec{S}_i^{\,2} - \vec{S}_j^{\,2} \end{aligned}$$

$$\vec{J}_{ij} = \vec{S}_i + \vec{S}_j$$

$$H = J \sum_{\langle ij \rangle} \prod_{ij}^{0}$$

The golden chain



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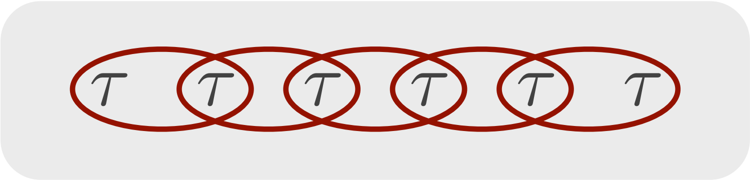

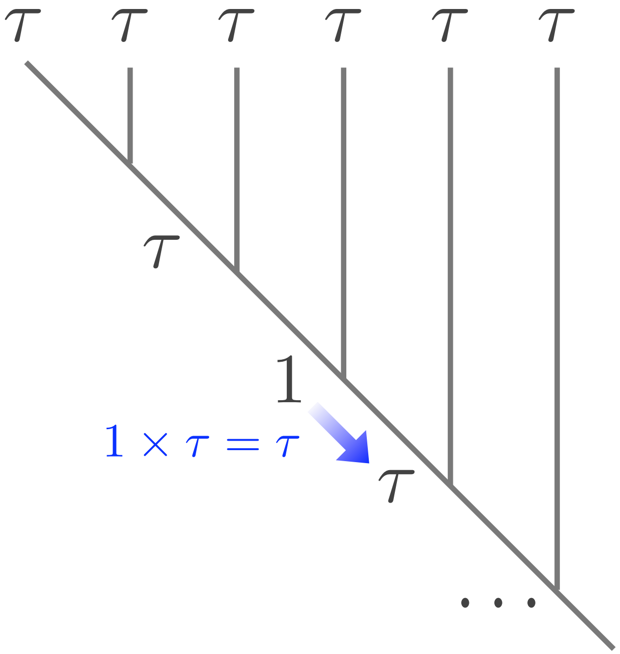

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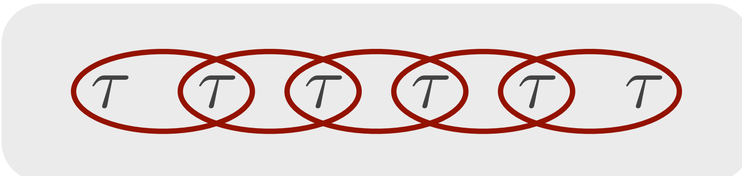
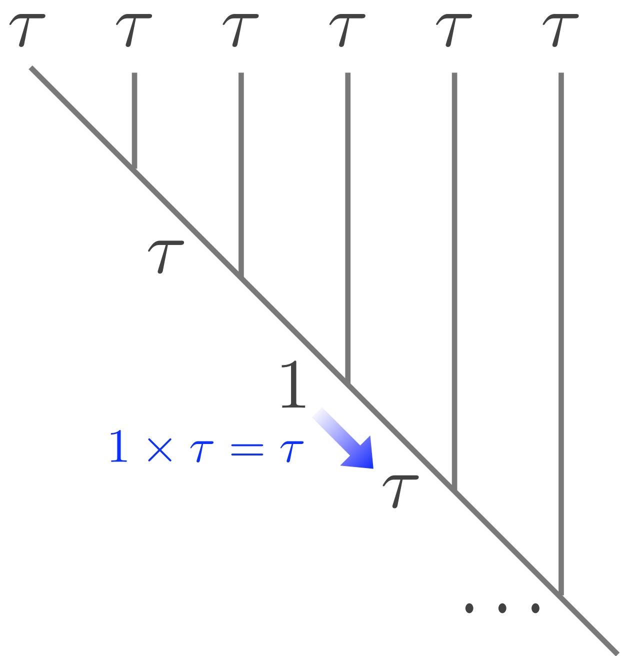
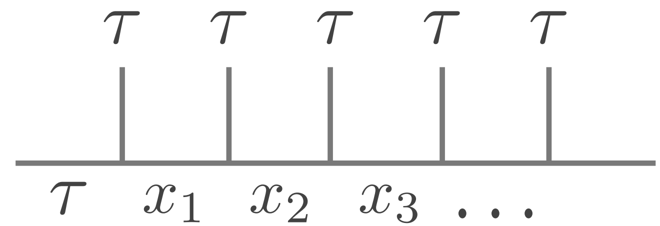
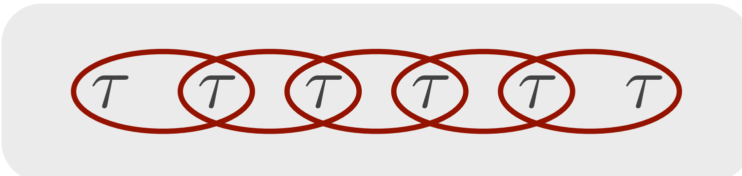


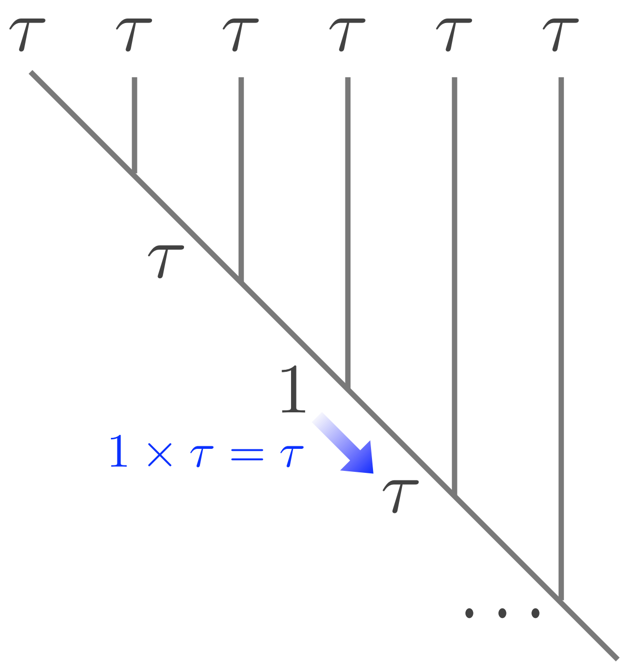


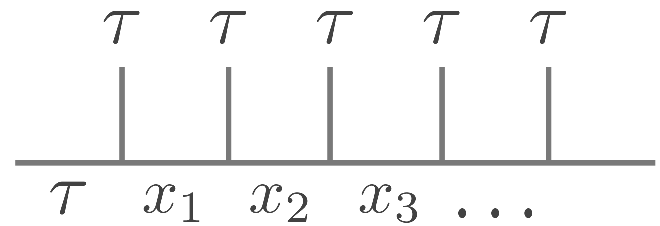


Hilbert space: $|x_1, x_2, x_3, \ldots\rangle$

The golden chain







Hilbert space: $|x_1, x_2, x_3, \ldots\rangle$

$$\dim_L = F_{L+1} \propto \phi^L$$

$$\phi = \frac{1+\sqrt{5}}{2} = 1.618\ldots$$

Hilbert space has **no natural decomposition** as tensor product of single-site states.

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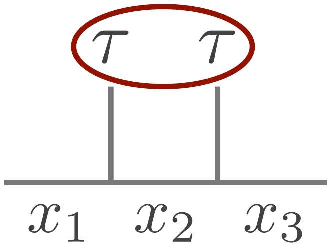

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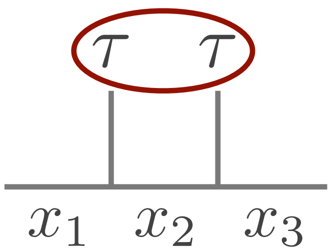

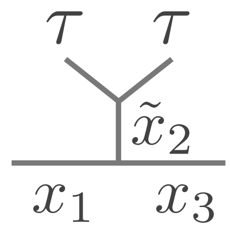

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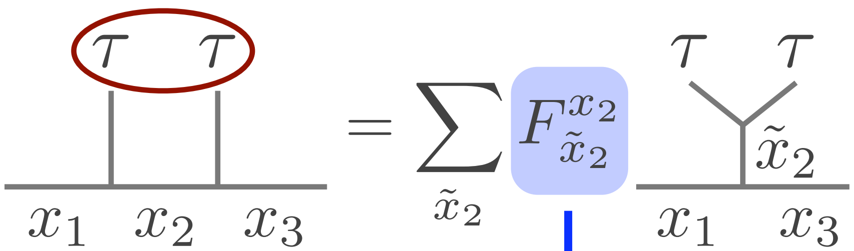

F-matrix

$$F = \begin{pmatrix} \phi^{-1} & \phi^{-1/2} \\ \phi^{-1/2} & -\phi^{-1} \end{pmatrix}$$

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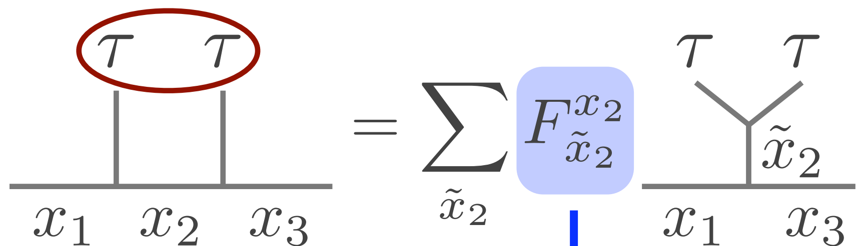

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Clebsch-Gordan coefficient

6-J symbol
E. Wigner 1940

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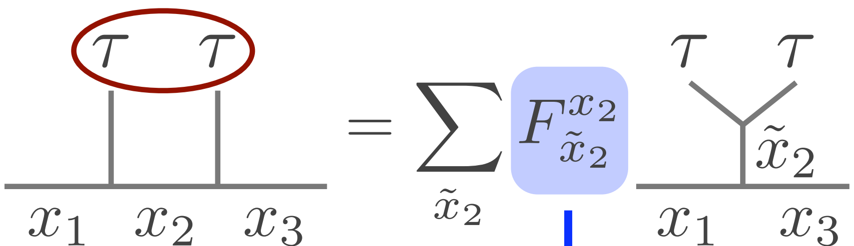

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$$H_i = -\begin{pmatrix} \phi^{-2} & \phi^{-3/2} \\ \phi^{-3/2} & \phi^{-1} \end{pmatrix}$$

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Explicit form:

$$\begin{aligned} H_i \quad = \quad & -\mathcal{P}_{1\tau 1} - \phi^{-2}\mathcal{P}_{\tau 1 \tau} - \phi^{-1}\mathcal{P}_{\tau\tau\tau} \\ & -\phi^{-3/2}\left(|\tau 1 \tau\rangle \langle \tau\tau\tau| + \text{h.c.}\right) \end{aligned}$$

off-diagonal matrix element

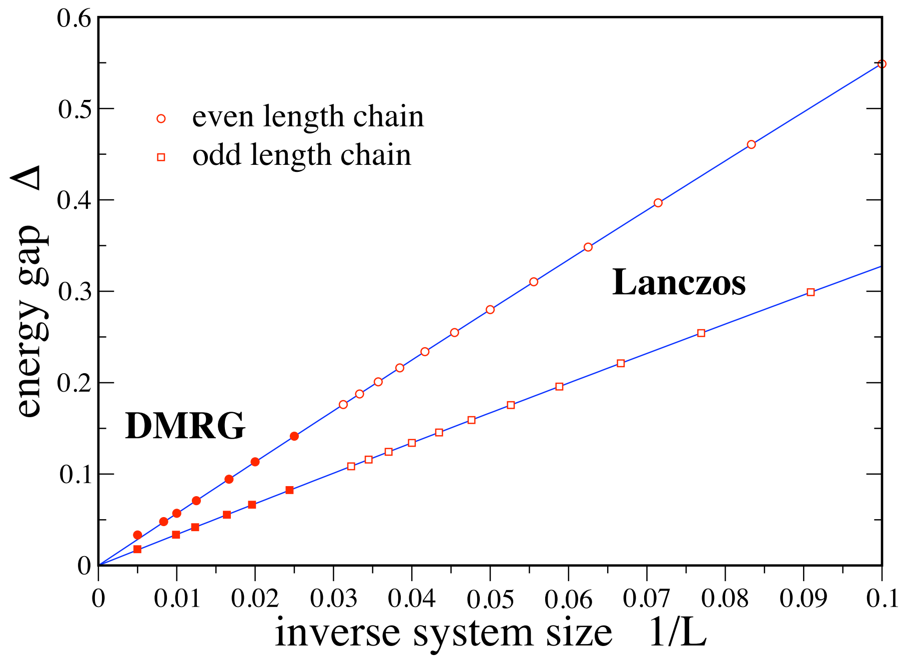
A. Feiguin et. al, PRL (2007)

Criticality

Finite-size gap

$$\Delta(L) \propto (1/L)^{\boxed{z=1}}$$

conformal field theory description



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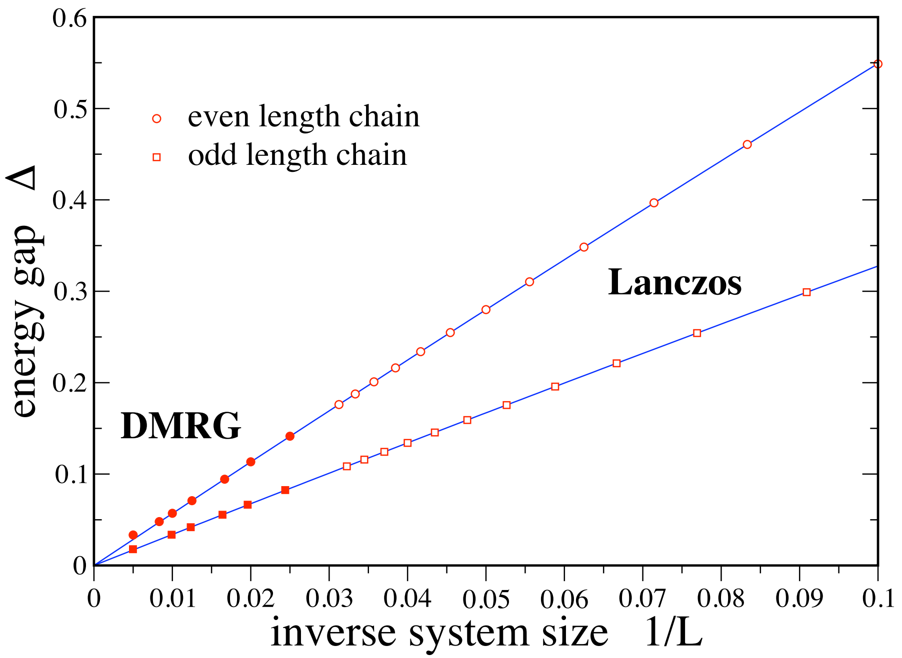
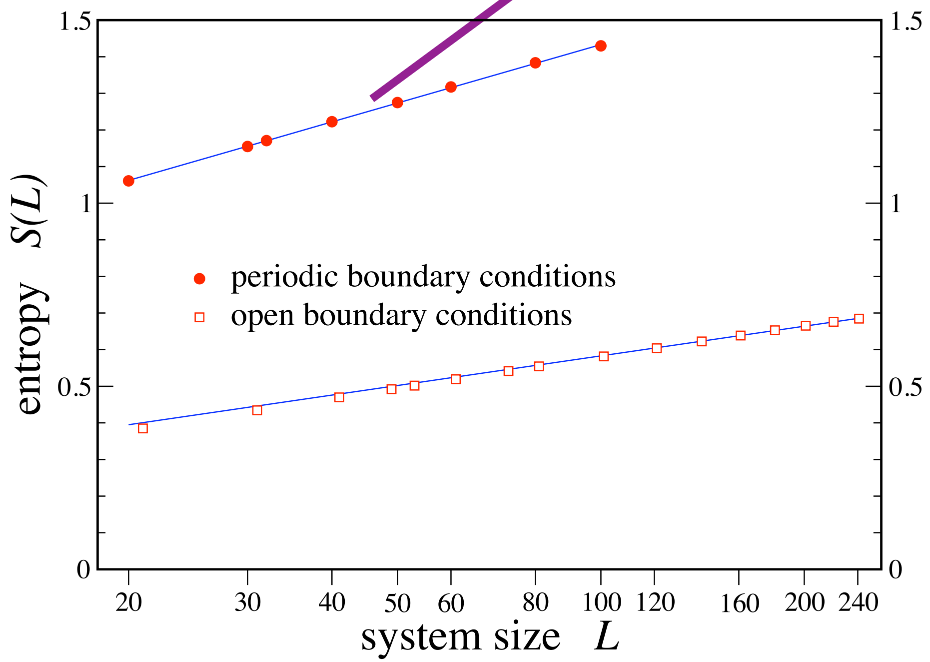


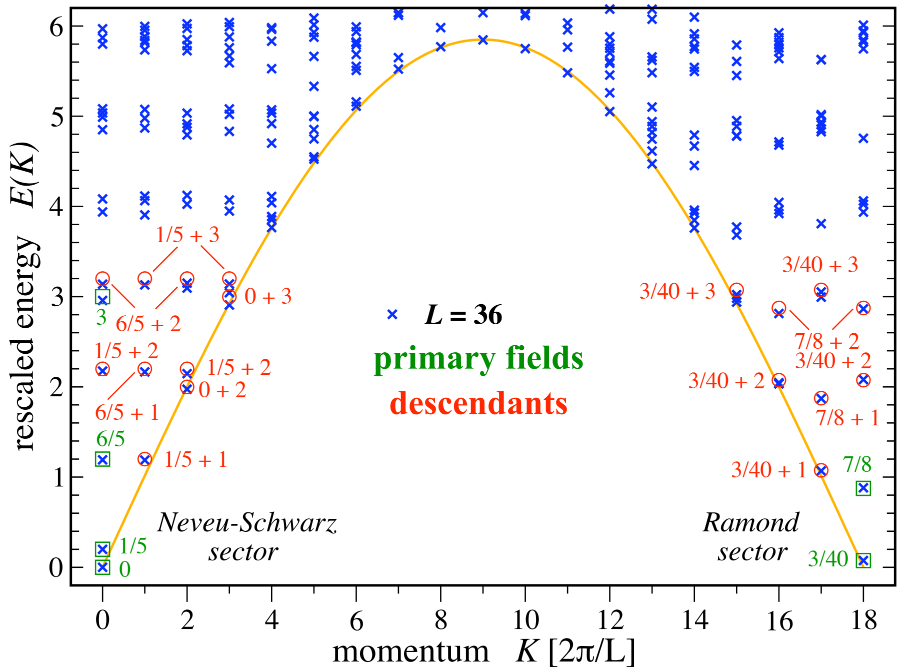
Entanglement entropy

$$S_{\mathrm{PBC}}(L) \propto \boxed{\frac{c}{3}} \log L$$

central charge
$c = 7/10$



Energy spectra



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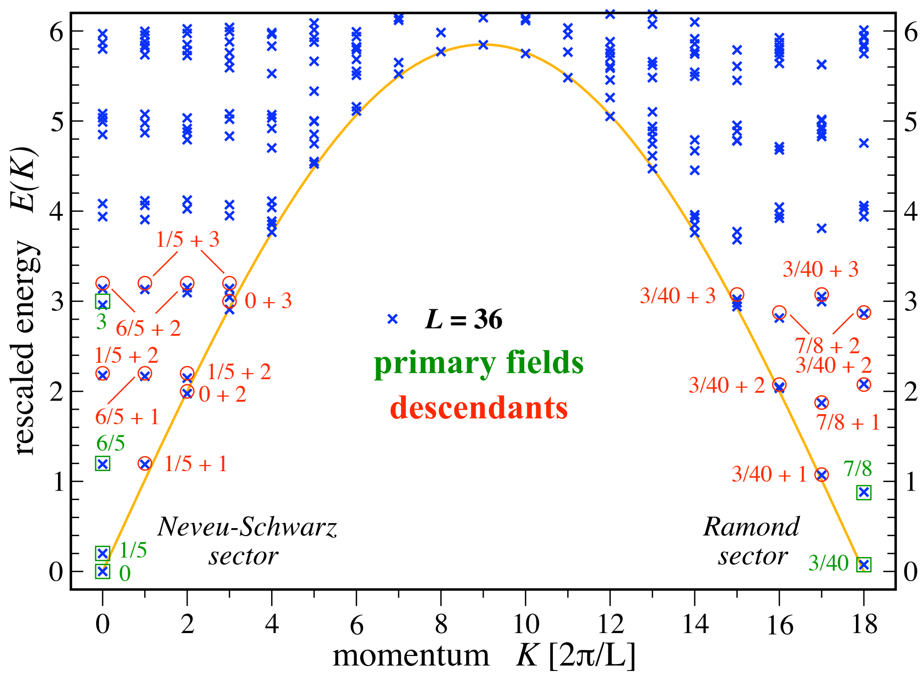

primary fields I ϵ ϵ' ϵ'' σ σ'

scaling dimensions $\underbrace{0 \quad 1/10 \quad 3/5 \quad 3/2}_{K=0}$ $\underbrace{3/80 \quad 7/16}_{K=\pi}$

Mapping to Temperly-Lieb

The operators $\mathrm{X}_i = -\phi H_i$ form a representation of the Temperly-Lieb algebra:

$$X_i^2 = dX_i \qquad X_i X_{i\pm1} X_i = X_i \qquad X_i, X_j = 0 \quad \text{when } |i-j| \geq 2$$

$d = \phi$ is the isotopy parameter

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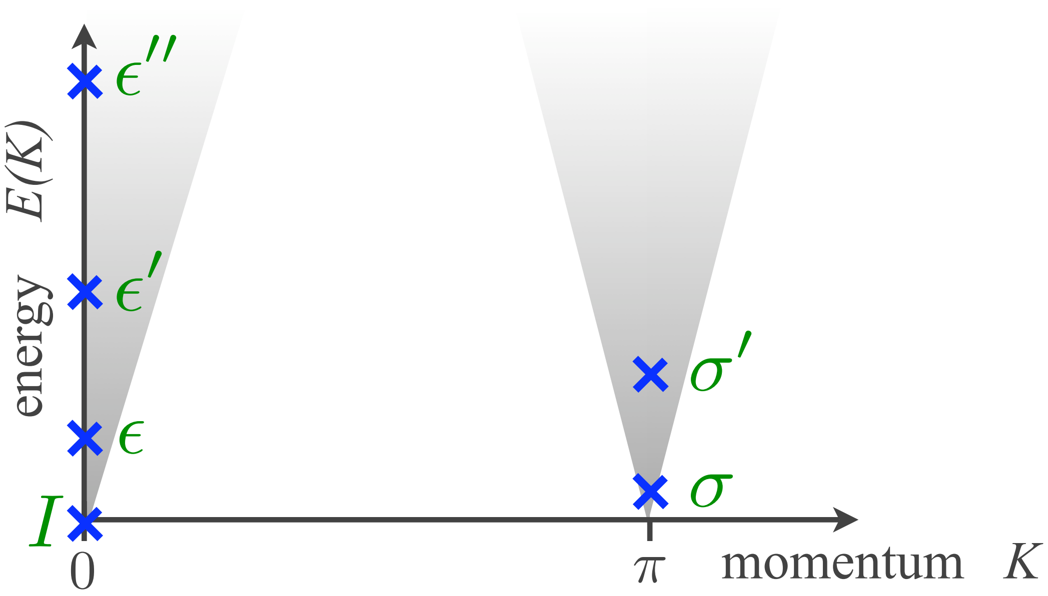
Get: Restricted solid on solid model (RSOS)

AFM interactions: M_k minimal model $c = 1 - \dfrac{6}{(k+1)(k+2)}$

FM interactions: Z_k parafermions $c = \dfrac{2(k-1)}{k+2}$

Topological symmetry



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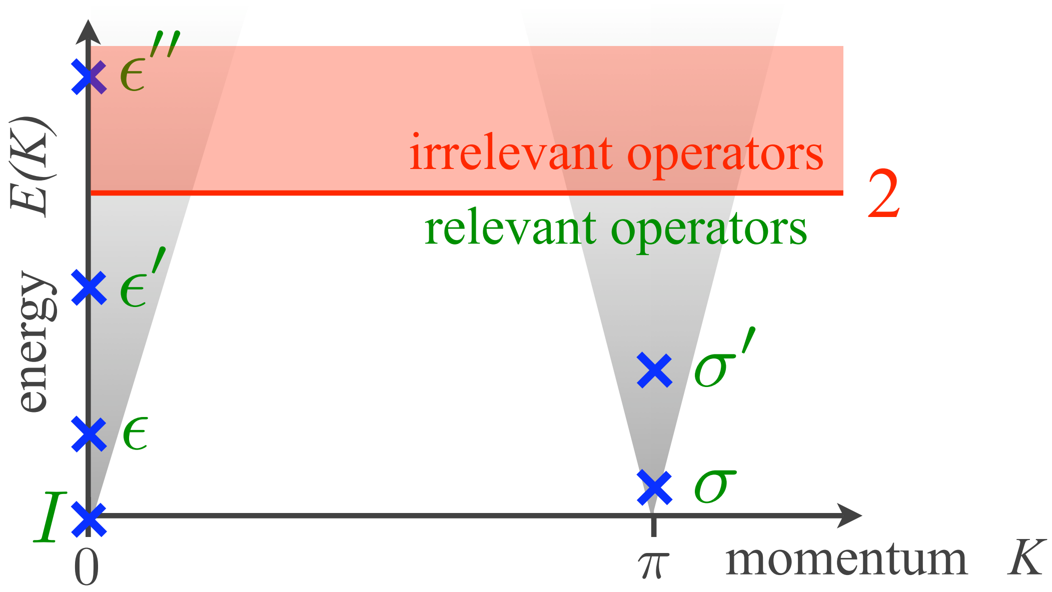

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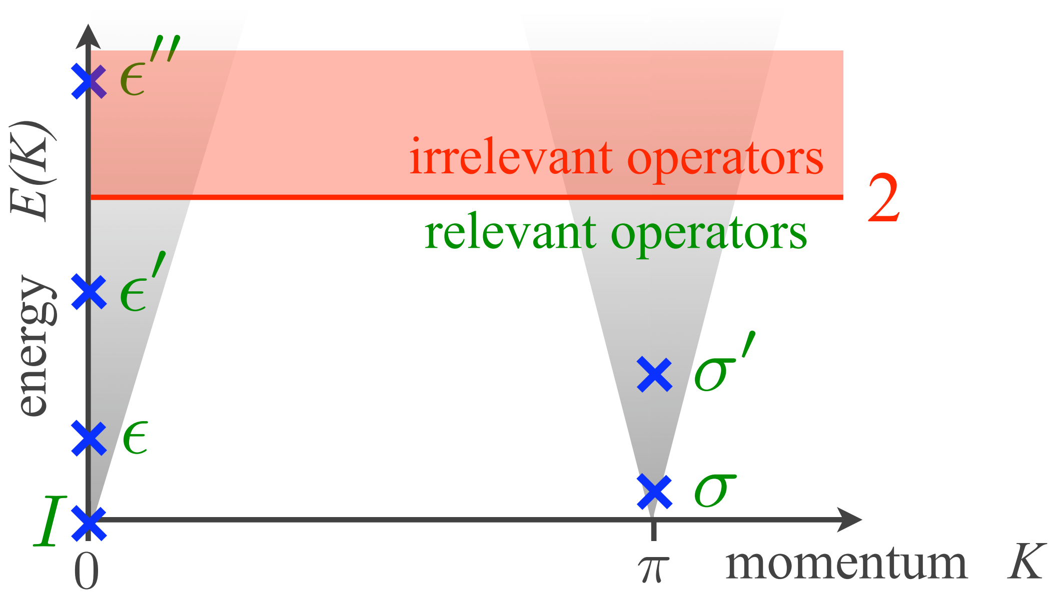
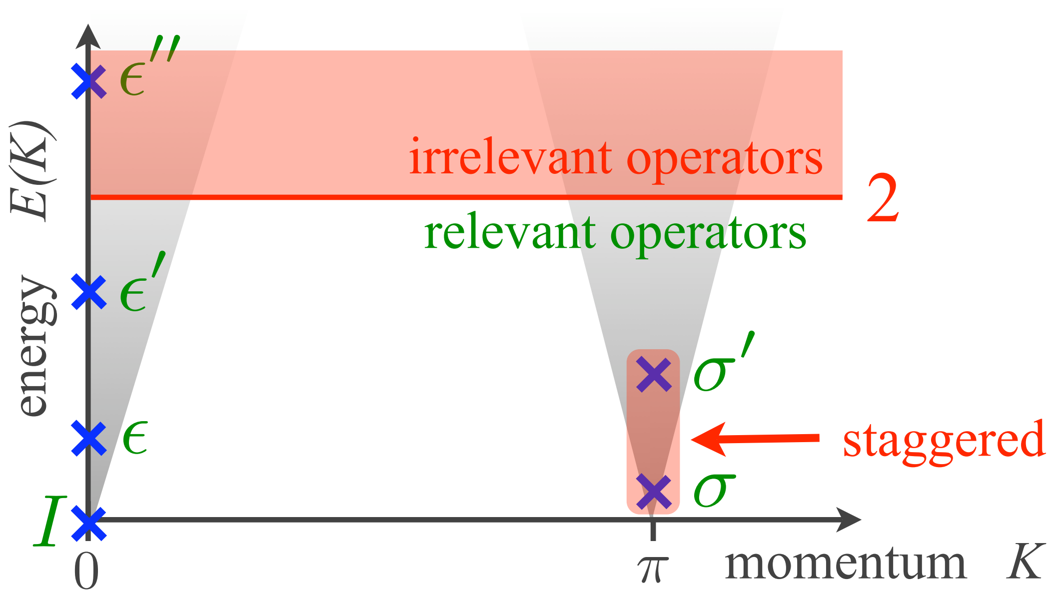


Relevant perturbations

σ σ'

ϵ ϵ'

Topological symmetry



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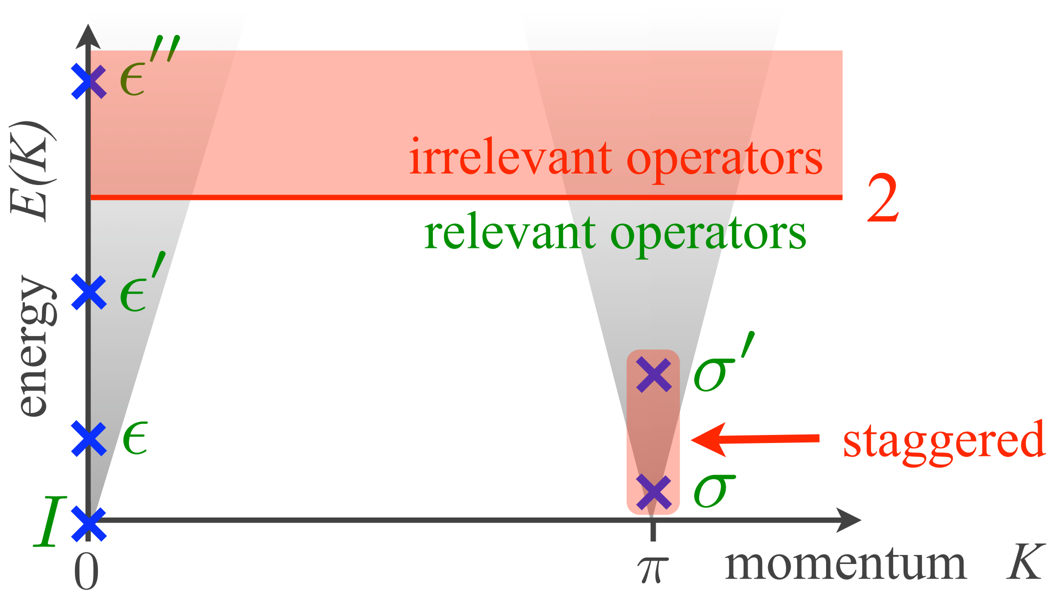
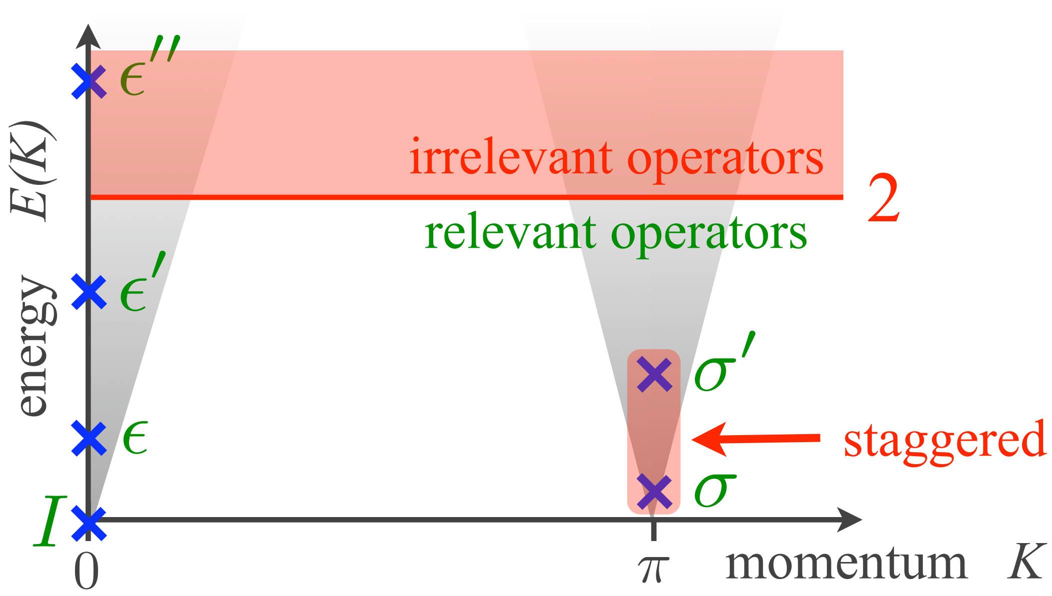


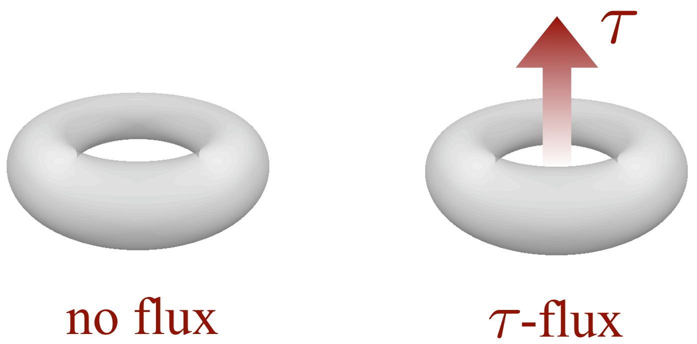
Relevant perturbations

$\cancel{\sigma}$ $\cancel{\sigma'}$ prohibited by **translational symmetry**

ϵ ϵ'

Topological symmetry





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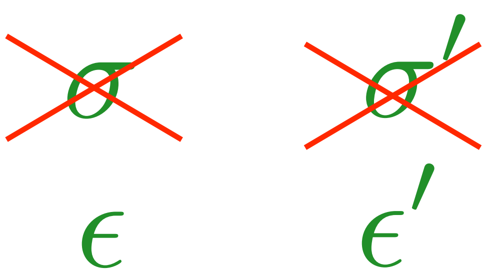

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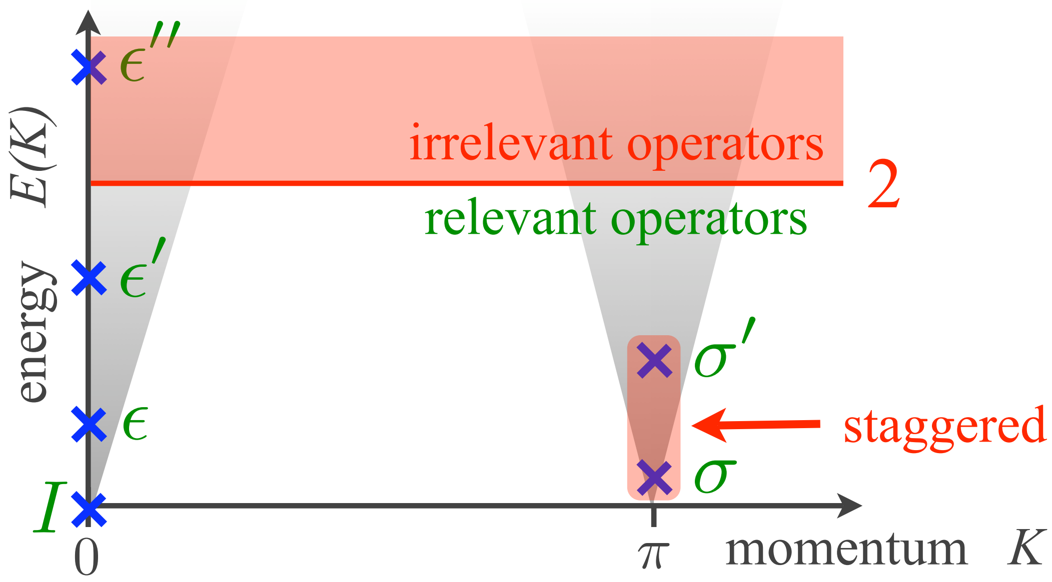

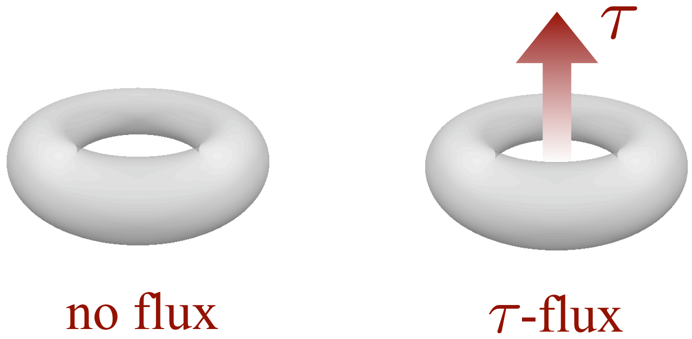

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Symmetry operator

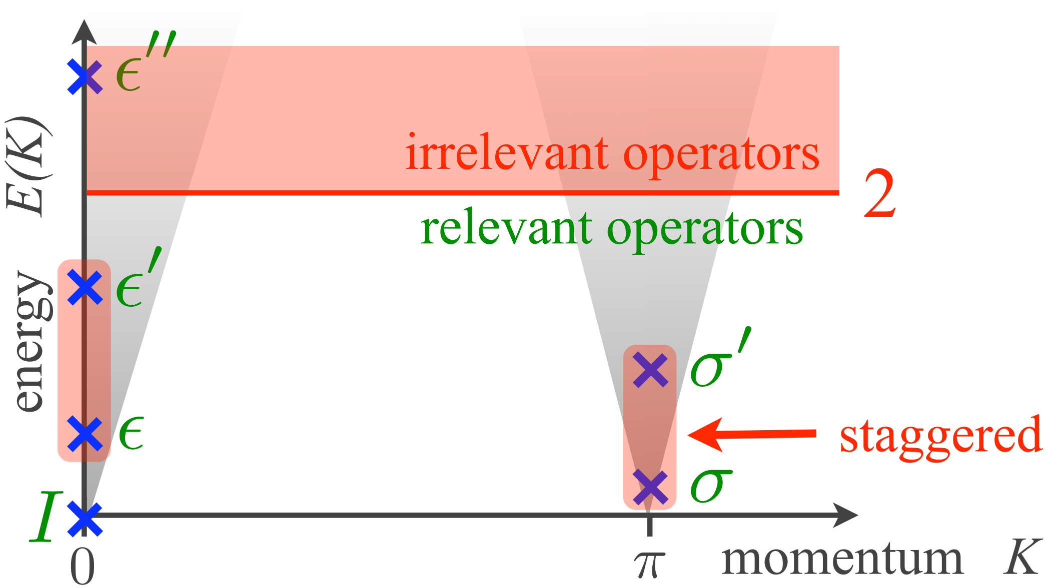
$$\langle x'_1, \ldots, x'_L | Y | x_1, \ldots, x_L \rangle = \prod_{i=1}^{L} \left(F^{x'_{i+1}}_{\tau x_i \tau} \right)^{x'_i}_{x_{i+1}}$$

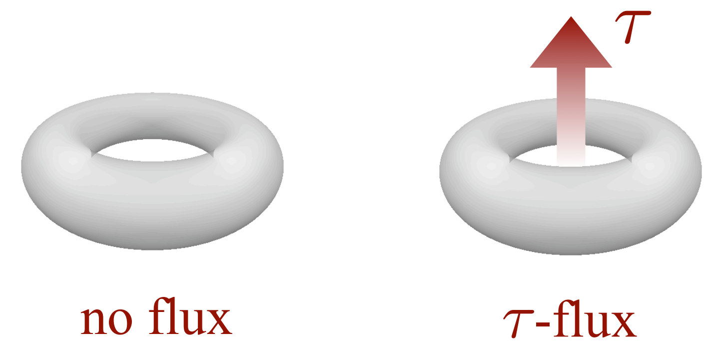
with eigenvalues

$$S_{\tau-\text{flux}} = \phi \quad S_{\text{no flux}} = -\phi^{-1}$$

$$[H, Y] = 0$$

Topological symmetry





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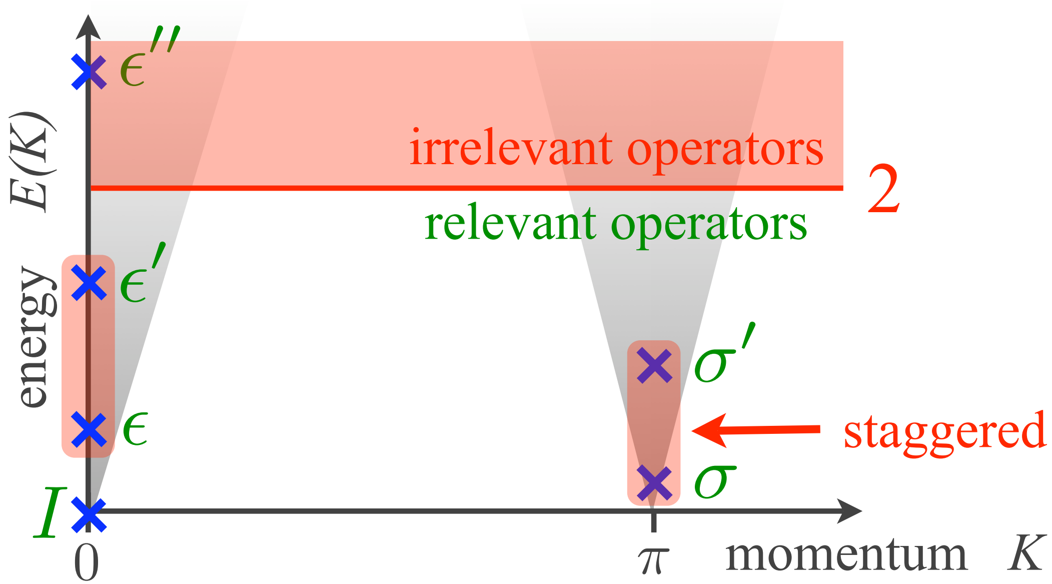

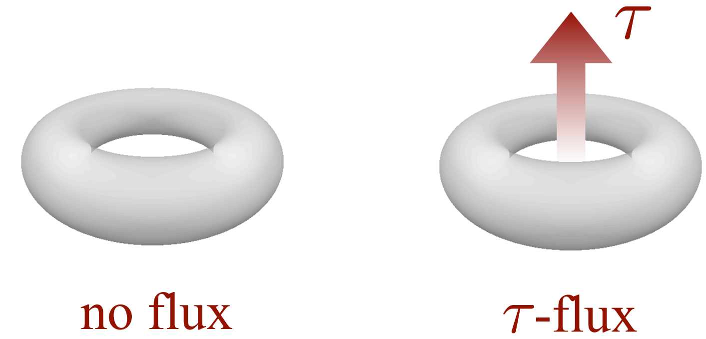

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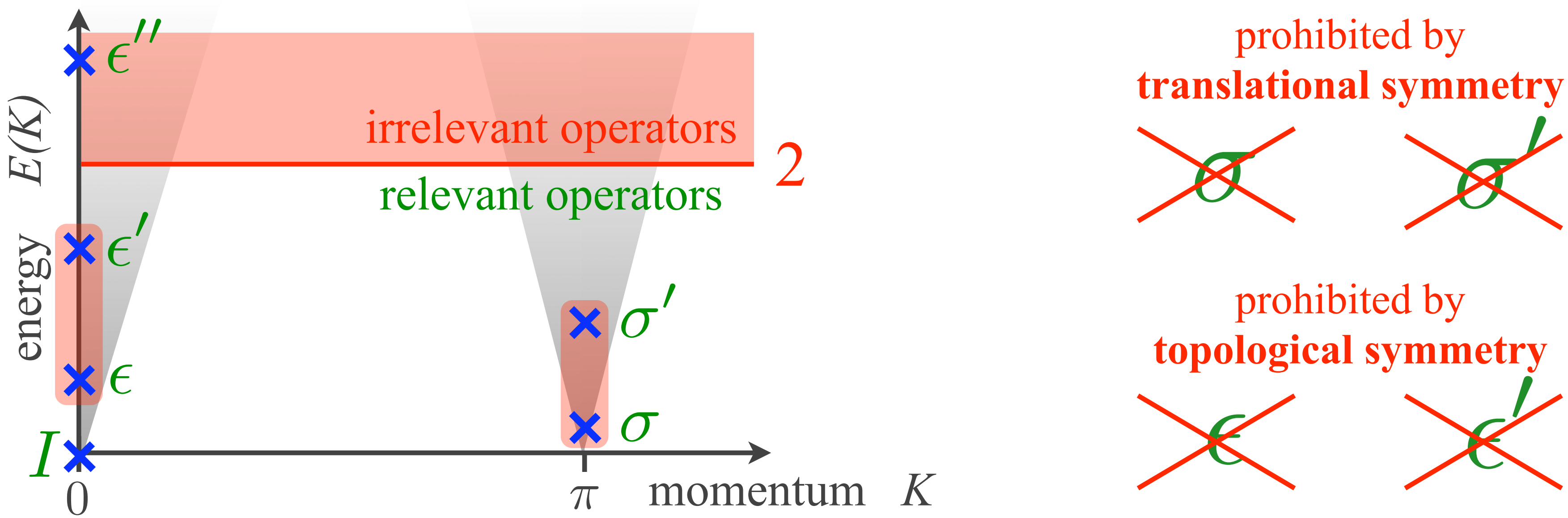
Topological stability



The **criticality** of the chain is **protected** by additional topological symmetry.

Local perturbations do not gap the system.

Is this special to $SU(2)_3$?

Topological stability for all k

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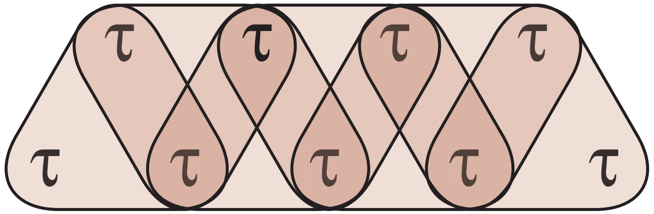
The predictions have been checked numerically, and both the AFM and FM spin-1/2 chains are stable!

Majumdar-Ghosh chain

Consider a (competing) three anyon-fusion term
➠ neither translational nor topological symmetry are broken

SU(2)$_3$ anyons



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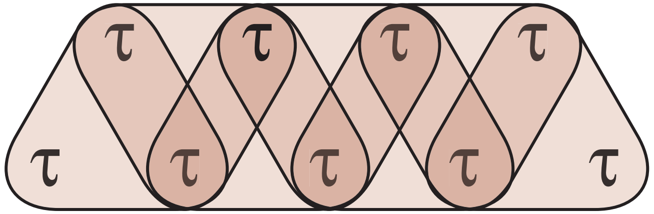

SU(2) spins

$$
\begin{aligned}
H_{\mathrm{MG}} &= J \sum_i \vec{T}^{\,2}_{i-1,i,i+1} \\
&= J \sum_i \vec{S}_i \vec{S}_{i+1} + \frac{J}{2} \sum_i \vec{S}_i \vec{S}_{i+2}
\end{aligned}
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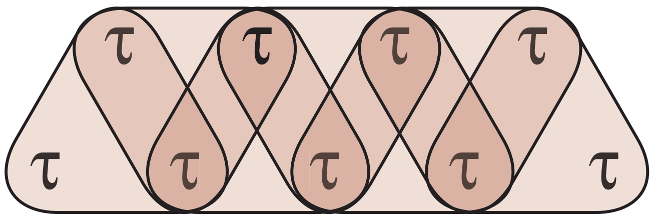

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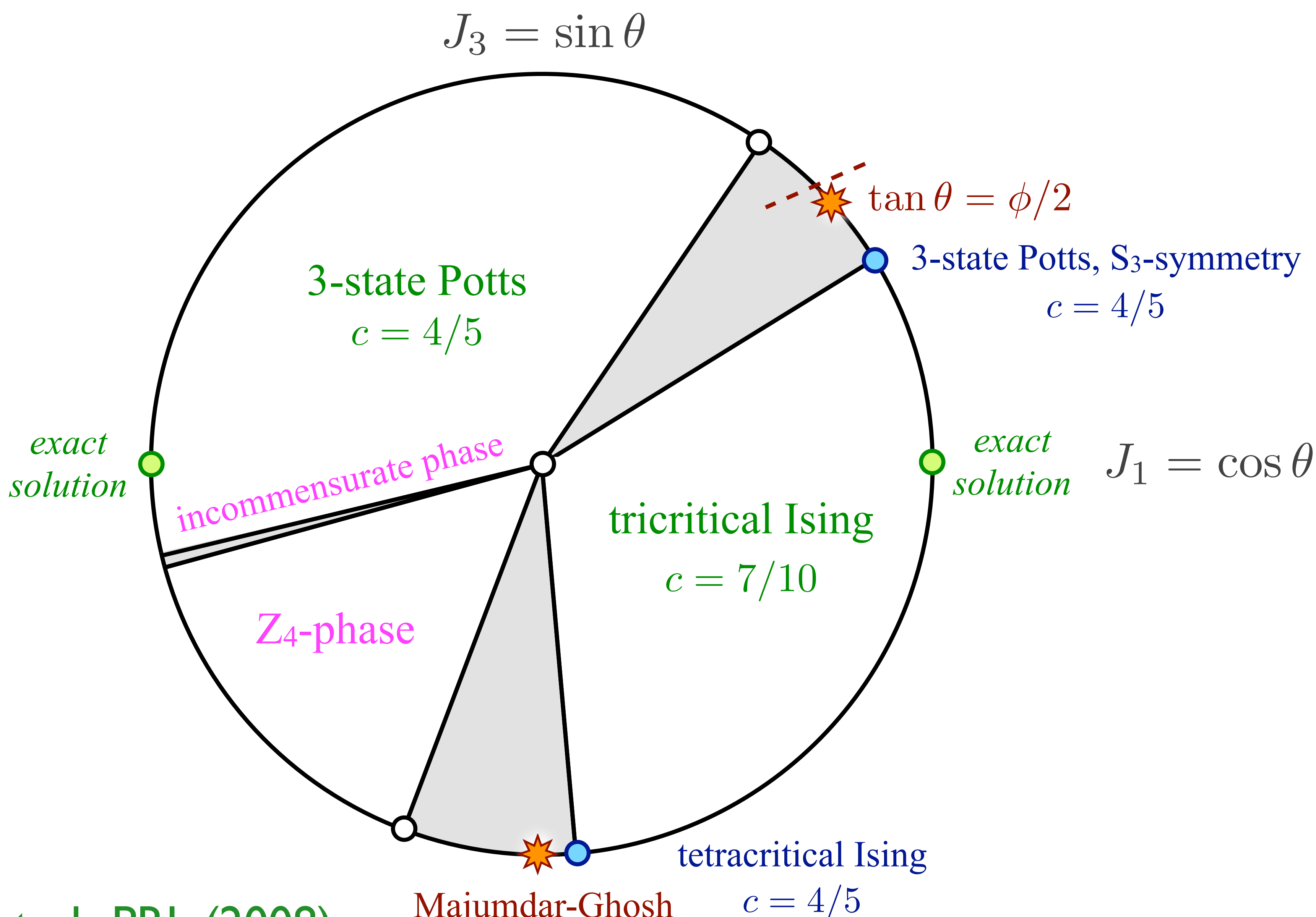
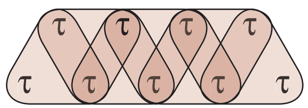
\end{aligned}
$$

$$
\begin{aligned}
H_i \quad = \quad & \mathcal{P}_{\tau 1 \tau 1} + \mathcal{P}_{1\tau 1\tau} + \mathcal{P}_{\tau\tau\tau 1} + \mathcal{P}_{1\tau\tau\tau} + 2\phi^{-2}\mathcal{P}_{\tau\tau\tau\tau} + \\
& \phi^{-1}\left(\mathcal{P}_{\tau 1\tau\tau} + \mathcal{P}_{\tau\tau 1\tau}\right) - \phi^{-2}\Big(|\tau\tau 1\tau\rangle\langle\tau 1\tau\tau| + \text{h.c.}\Big) + \\
& \phi^{-5/2}\left(|\tau 1\tau\tau\rangle\langle\tau\tau\tau\tau| + |\tau\tau 1\tau\rangle\langle\tau\tau\tau\tau| + \text{h.c.}\right)
\end{aligned}
$$

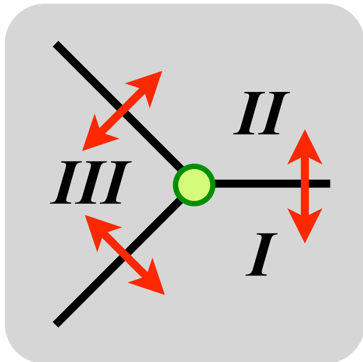
Phase diagram



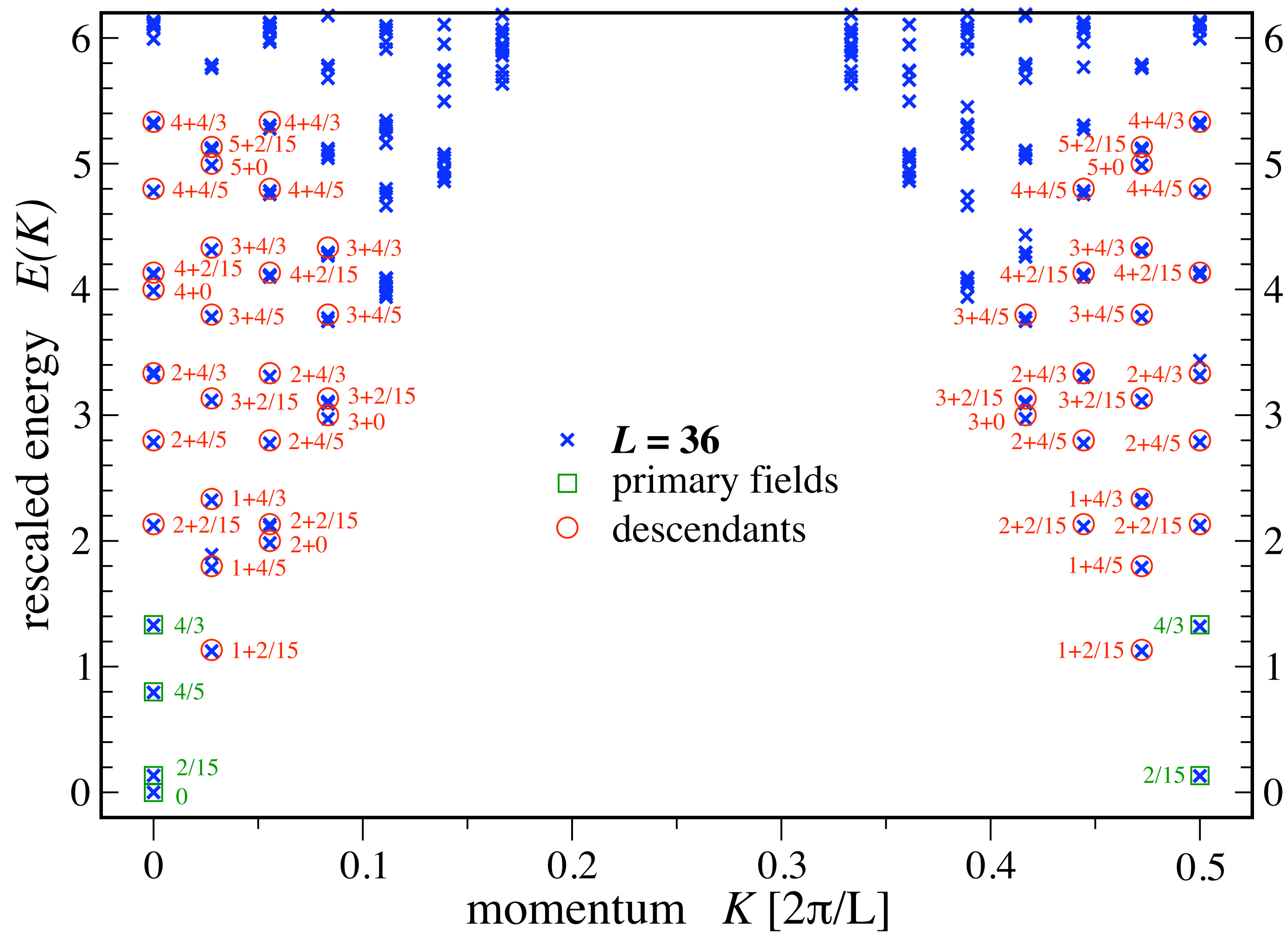
S. Trebst et. al., PRL (2008)



S_3-symmetric point



$\theta = 0.176\pi$



Conformal field theory: parafermions with central charge $c = 4/5$.

$\mathrm{su}(2)_5$ spin-1 chain

Take the even sector of $\mathrm{su}(2)_5$: $1 \quad \alpha \quad \beta$

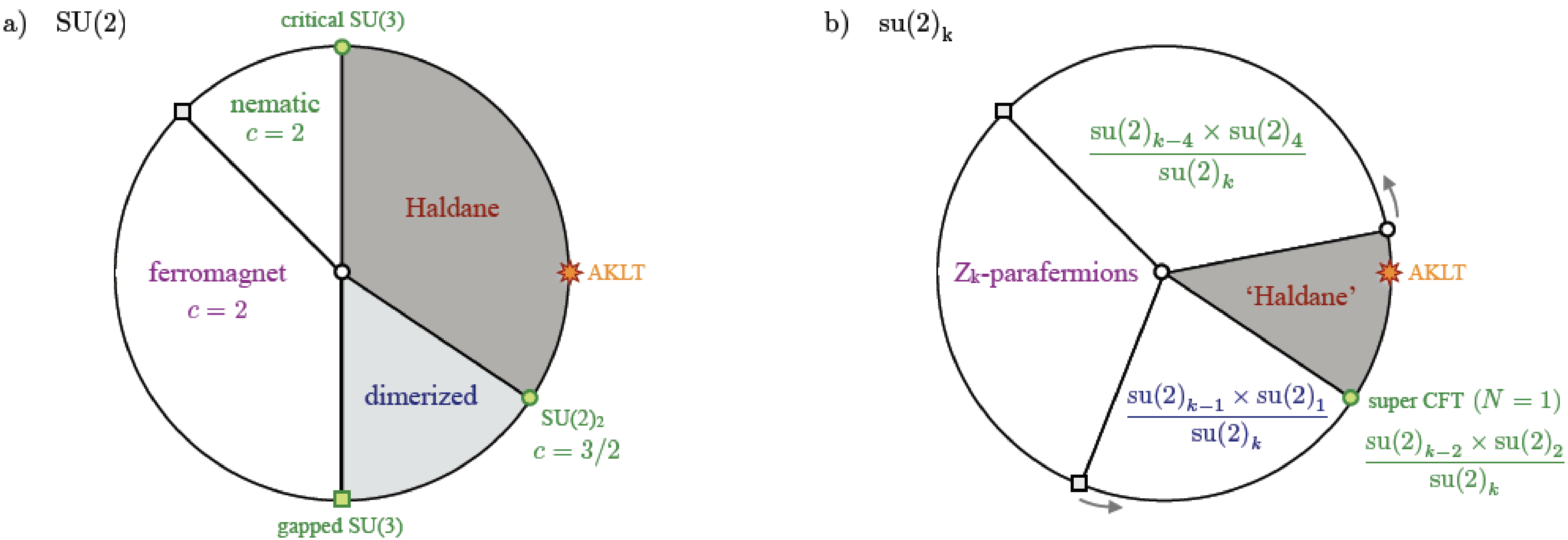
$$\alpha\times\alpha = 1+\beta \quad \alpha\times\beta = \alpha+\beta \quad \beta\times\beta = 1+\alpha+\beta$$

Use the β particles as the building block of the chain, and define the hamiltonian:

$$\mathrm{H} = \sin\theta P_\beta - \cos\theta P_\alpha$$

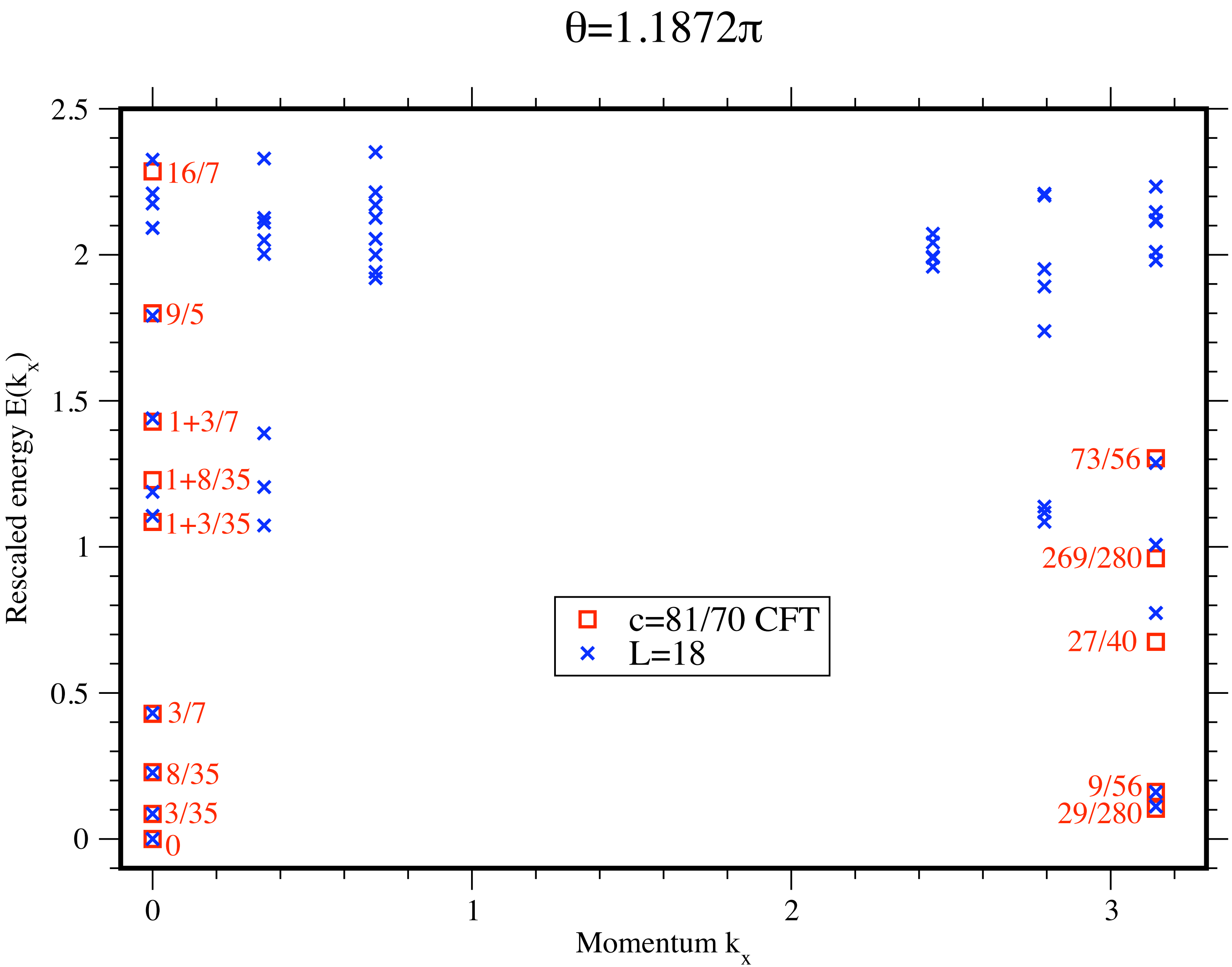
P_β , P_α project on the β , α channel.

$\mathrm{su}(2)_5$ spin-1 chain phase diagram



C. Gils et. al., in preparation

c=80/71 super cft spectrum



Conclusions

- Topological quantum spin chains have been studied analytically and numerically
- They support different phases: (topologically) protected criticality, gapped phases etc.
- Many aspects remain open:
  - 2-d systems
  - different types of disorder (some progress)
  - higher genus surfaces

Nordita program

August 17 - September 11, 2009:

Quantum Hall Physics - Novel systems and applications