Hidden Grassmann structure in the XXZ model

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We present an exponential formula for the correlation functions of the XXZ model. The formula is given in terms of a kind of monodromy operators in the sense of the quantum inverse scattering method. The opertors satisfy anti-commutation relations, and acting on the space of quasi-local operators as annihilation operators. Construction of creation operators is also given.

Hamiltonian of the XXZ model is given by

$$H_{XXZ} = \tfrac{1}{2} \sum_{k=-\infty}^{\infty} (\sigma_k^1 \sigma_{k+1}^1 + \sigma_k^2 \sigma_{k+1}^2 + \Delta \sigma_k^3 \sigma_{k+1}^3),$$

where

$$\Delta = \tfrac{q+q^{-1}}{2}, \quad q = e^{\pi i \nu}.$$

Let $|\mathrm{vac}\rangle$ be the lowest eigenvector of the Hamiltonian. Our goal is to compute the correlation functions.

$$\langle \mathcal{O} \rangle = \frac{\langle \mathrm{vac}| q^{\alpha \sum_{j=-\infty}^{0} \sigma_j^3} \mathcal{O} |\mathrm{vac}\rangle}{\langle \mathrm{vac}| q^{\alpha \sum_{j=-\infty}^{0} \sigma_j^3} |\mathrm{vac}\rangle}.$$

Integrability of the Hamiltonian is based on the Yang-Baxter equation.

$$R(\zeta)=\begin{pmatrix} q\zeta-q^{-1}\zeta^{-1} & & & \\ & \zeta-\zeta^{-1} & q-q^{-1} & \\ & q-q^{-1} & \zeta-\zeta^{-1} & \\ & & & q\zeta-q^{-1}\zeta^{-1} \end{pmatrix},$$

$$R_{1,2}(\zeta_1/\zeta_2)R_{1,3}(\zeta_1/\zeta_3)R_{2,3}(\zeta_2/\zeta_3)$$
$$=R_{2,3}(\zeta_2/\zeta_3)R_{1,3}(\zeta_1/\zeta_3)R_{1,2}(\zeta_1/\zeta_2) \text{ on } (\mathbb{C}^2)_1\otimes(\mathbb{C}^2)_2\otimes(\mathbb{C}^2)_3.$$

Transfer matrices commute.

$$T_a(\zeta)=R_{a,m}(\zeta/\xi_m)\cdots R_{a,1}(\zeta/\xi_1)\in \mathrm{End}\left((\mathbb{C}^2)_a\otimes(\mathbb{C}^2)^{\otimes m}_{[1,m]}\right),$$

$[t(\zeta_1),t(\zeta_2)]=0$ where $t(\zeta)=\mathrm{Tr}_aT_a(\zeta)$

YBE is a consequence of $U_q(\widehat{\mathfrak{sl}}_2)$ symmetry.

$U_q(\mathfrak{sl}_2) : q^h e q^{-h} = q^2 e, q^h f q^{-h} = q^{-2} f, [e, f] = \dfrac{q^h - q^{-h}}{q - q^{-1}}$

$U_q(\widehat{\mathfrak{sl}}_2) : e_0, f_0, h_0, e_1, f_1, h_1$ level 0 : $h_0 + h_1 = 0$

Δ gives tensor product representation $\Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i,$

$\Delta(e_i) = e_i \otimes 1 + q^{h_i} \otimes e_i,\ \Delta(f_i) = e_i \otimes q^{-h_i} + 1 \otimes f_i.$

universal R matrix intertwines two representations

$U^+ = \langle e_0, e_1, q^{\pm h_1} \rangle,\ U^- = \langle f_0, f_1, q^{\pm h_1} \rangle,\ \mathcal{R} \in U^+ \otimes U^-$

$\mathcal{R}\Delta(x) = P\big(\Delta(x)\big)\mathcal{R},\ P(x \otimes y) = y \otimes x$

$\mathcal{R}_{1,2}\mathcal{R}_{1,3}\mathcal{R}_{2,3} = \mathcal{R}_{2,3}\mathcal{R}_{1,3}\mathcal{R}_{1,2}$

Commuting transfer matrix, which acts on **quantum** space, is obtained by taking trace over **auxiliary** space.

$$t(\zeta)_{\mathrm{quan}} = \mathrm{Tr}_{\mathrm{aux}}\left(\pi_{\mathrm{aux}}(\zeta) \otimes \pi_{\mathrm{quan}}\right)\mathcal{R}.$$

We replace the auxiliary space $\mathrm{End}(\mathbb{C}^2)$ by the q oscillator algebra Osc, which is generated by $\mathbf{a}, \mathbf{a}^*, q^{\pm D}$:

$q^D\mathbf{a}^*q^{-D} = q\mathbf{a},\ q^D\mathbf{a}q^{-D} = q^{-1}\mathbf{a},\ \mathbf{a}\mathbf{a}^* = 1-q^{2D+2},\ \mathbf{a}^*\mathbf{a} = 1-q^{2D}.$

Up to a scalar multiple, $\mathcal{R}$ is represented in $Osc\otimes\mathrm{End}(\mathbb{C}^2)$ by

$$L(\zeta) = \begin{pmatrix} q^{-D} - \zeta^2 q^{D+2} & -\zeta\mathbf{a}q^D \\ -\zeta\mathbf{a}^*q^D & q^D \end{pmatrix}$$

In the construction of L operator we use the algebra homomorphism $U_q^+ \to Osc$,

$$e_0 \mapsto \frac{\zeta}{q-q^{-1}}\mathbf{a},\ e_1 \mapsto \frac{\zeta}{q-q^{-1}}\mathbf{a}^*,\ t_1 = t_0^{-1} \mapsto q^D.$$

To obtain commuting transfer matrix we use the α-**trace** $\mathrm{Tr}_A^\alpha : Osc \to \mathbb{C}(q^\alpha)$:

$$\mathrm{Tr}_A^\alpha q^{mD} \stackrel{\text{def}}{=} \mathrm{Tr}_A q^{2\alpha D} q^{mD} = \frac{1}{1-q^{2\alpha+m}} \quad (m \in \mathbb{Z}).$$

Transfer matrix is α-**twisted** accordingly.

$$T_A^{(\alpha)}(\zeta) = q^{2\alpha D_A} L_{A,m}(\zeta/\xi_m)\cdots L_{A,1}(\zeta/\xi_1) \in Osc_A \otimes \mathrm{End}\left((\mathbb{C}^2)^{\otimes m}_{[1,m]}\right)$$

$[Q^{(\alpha)}(\zeta_1), Q^{(\alpha)}(\zeta_2)] = 0$ where $Q^{(\alpha)}(\zeta) = \mathrm{Tr}_A^\alpha T_A^{(\alpha)}(\zeta)$

We have a triangular decomposition of **fusion** $\{a, A\}$:

$$
\begin{aligned}
L_{\{a,A\}}(\zeta) &\overset{\text{def}}{=} F_{a,A}^{-1} R_{a,j}(\zeta) L_{A,j}(\zeta) F_{a,A} \\
&= \begin{pmatrix} * \, L_{A,j}(q\zeta) q^{-\sigma_j^3/2} & 0 \\ C_{A,j}(\zeta) & * \, L_{A,j}(q^{-1}\zeta) q^{\sigma_j^3/2} \end{pmatrix}
\end{aligned}
$$

Baxter's TQ relation is obtained from this decomposition. (BLZ construction)

$$
t^{(\alpha)}(\zeta) Q^{(\alpha)}(\zeta) = * \, Q^{(\alpha)}(q^{-1}\zeta) + * \, Q^{(\alpha)}(q\zeta)
$$

Here $*$ means irrelevant scalar factors.

For the formulas of the correlation functions, we use **adjoint** version, and take the **off diagonal** part.

We denote $\mathbb{T}_*(\zeta,\alpha)(X) = T_*(\zeta) q^{\alpha H_*} X T_*(\zeta)^{-1}$.

Here $* = a$ or A, $H_* = \sigma_a^3$ or $2D_A$, and $X \in \mathrm{End}\left((\mathbb{C}^2)^{\otimes m}_{[1,m]}\right)$ is a local operator.

We define two operators.

$$\mathbf{t}^*(\zeta,\alpha)(X) = \mathrm{Tr}_a\{\mathbb{T}_a(\zeta,\alpha)(X)\},$$

$$\mathbf{k}(\zeta,\alpha)(X) = \mathrm{Tr}_{A,a}\{\sigma^+ \mathbb{T}_a(\zeta,\alpha)\mathbb{T}_A(\zeta,\alpha)\zeta^{\alpha-\mathbb{S}}(q^{-2S}X)\}$$

Here $S = \sum_{j=1}^m \sigma_j^3$ is the total spin operator and $\mathbb{S}$ is its adjoint.

Correlation functions are given in the **exponential form**:

$$\langle \mathcal{O} \rangle = \mathbf{tr}^{\alpha} \left(e^{\mathbf{\Omega}} q^{2\alpha S(0)} \mathcal{O} \right).$$

Here, $\mathcal{O}$ is a local operator. It is multiplied by the **primary field**,

$$q^{2\alpha S(0)} = \cdots \otimes q^{\alpha\sigma^3} \otimes q^{\alpha\sigma^3} \otimes 1 \otimes 1 \otimes \cdots .$$

$$\mathbf{tr}^{\alpha}(X) = \cdots \mathrm{tr}_1^{\alpha} \mathrm{tr}_2^{\alpha} \mathrm{tr}_3^{\alpha} \cdots (X),$$

$$\mathrm{tr}^{\alpha}(x) = \frac{\mathrm{tr}(q^{-\frac{1}{2}\alpha\sigma^3} x)}{\mathrm{tr}(q^{-\frac{1}{2}\alpha\sigma^3})},$$

is used to ensure the **reduction relation**,

$$\mathrm{tr}^{\alpha} q^{\alpha\sigma^3} = \mathrm{tr}^{\alpha} 1 = 1.$$

The exponent $\mathbf{\Omega}$ is a **nilpotent** operator acting on the space of quasi-local operators. In particular, we have $\mathbf{\Omega}(q^{2\alpha S(0)}) = 0$. It is given in terms of **Grassmann** operators $\mathbf{b}$ and $\mathbf{c}$:

$$\mathbf{\Omega} = \mathrm{res}_{\zeta_1^2=1}\mathrm{res}_{\zeta_1^2=1}\omega(\zeta_1/\zeta_2)\mathbf{b}(\zeta_1)\mathbf{c}(\zeta_2)\frac{d\zeta_1^2}{\zeta_1^2}\frac{d\zeta_2^2}{\zeta_2^2}$$

$$\omega(\zeta,\alpha) = \omega_0(\zeta,\alpha) + \int_{-i\infty}^{i\infty}\zeta^{\alpha+u}\frac{\sin\frac{\pi}{2}(u-\nu(u+\alpha))}{\sin\frac{\pi}{2}u\cos\frac{\pi\nu}{2}(u+\alpha)}du$$

$$\omega_0(\zeta,\alpha) = -\left(\frac{1-y}{1+y}\right)^2\Delta_\zeta(\psi(\zeta,\alpha))$$

$$y = q^\alpha, \quad \psi(\zeta,\alpha) = \frac{1}{2}\frac{\zeta^2+1}{\zeta^2-1}\zeta^\alpha,$$

$$\Delta_\zeta(F(\zeta)) = F(q\zeta) - F(q^{-1}\zeta)$$

The following are the basic properties of $\mathbf{b}(\zeta)$ and $\mathbf{c}(\zeta)$.

(i) Grassmann

$$[\mathbf{b}(\zeta_1), \mathbf{b}(\zeta_2)]_+ = [\mathbf{c}(\zeta_1), \mathbf{c}(\zeta_2)]_+ = [\mathbf{b}(\zeta_1), \mathbf{c}(\zeta_2)]_+ = 0$$

(ii) Singular expansion

$$\begin{aligned}\mathbf{k}(\zeta,\alpha) &= \bar{\mathbf{c}}(\zeta, \alpha) + \mathbf{c}(q\zeta, \alpha) + \mathbf{c}(q^{-1}\zeta, \alpha) \\ &\quad + \mathbf{f}(q\zeta, \alpha) - \mathbf{f}(q^{-1}\zeta, \alpha),\end{aligned}$$

$$\mathbf{b}(\zeta) = \zeta^{-\alpha}\left(\mathbf{b}_0 + \sum_{p=1}^{\infty}(\zeta^2 - 1)^{-p}\mathbf{b}_p\right)$$

$$\mathbf{c}(\zeta) = \zeta^{\alpha}\left(\mathbf{c}_0 + \sum_{p=1}^{\infty}(\zeta^2 - 1)^{-p}\mathbf{c}_p\right)$$

(iii) Reduction

b (resp., **c**) send quasi-local operators of twist α and spin s to those of twist $\alpha+1$ (resp., $\alpha-1$) and spin $s-1$ (resp., $s+1$).

(iv) Annihilation ($\mathbf{x}=\mathbf{b}$ or $\mathbf{c}$)

$$\mathrm{supp}X\subset[1,n]\Rightarrow\begin{cases}\mathrm{supp}\,\mathbf{x}_p(X)\subset[1,m] & {\scriptstyle 1\le p\le m-1}\\ \mathrm{supp}\,\mathbf{x}_n(X)\subset[1,m-1] & {\scriptstyle p=m}\\ \mathbf{x}_p(X)=0 & {\scriptstyle p>m}\end{cases}$$

(v) Large kernel created by local integrals

$$[\mathbf{t}^*(\zeta_1),\mathbf{t}^*(\zeta_2)]=[\mathbf{c}(\zeta_1),\mathbf{t}^*(\zeta_2)]=[\mathbf{b}(\zeta_1),\mathbf{t}^*(\zeta_2)]=0$$

We can construct creation operators $\mathbf{b}^*, \mathbf{c}^*$ conjugate to $\mathbf{b}, \mathbf{c}$.

$$\mathbf{b}^*(\zeta,\alpha) = \mathbf{f}(q\zeta,\alpha) + \mathbf{f}(q^{-1}\zeta,\alpha) - \mathbf{t}^*(\zeta,\alpha)\mathbf{f}(\zeta,\alpha),$$

$$\mathbf{b}^*(\zeta) = \zeta^{\alpha-2}\sum_{p=1}^{\infty}(\zeta^2-1)^{p-1}\mathbf{b}^*_p$$

$$\mathbf{c}^*(\zeta) = \zeta^{-\alpha-2}\sum_{p=1}^{\infty}(\zeta^2-1)^{p-1}\mathbf{c}^*_p$$

CAR holds for $\{\mathbf{b}^*_p, \mathbf{c}^*_p, \mathbf{b}_p, \mathbf{c}_p\}_{p\geq 1}$

$\mathbf{b}^*_p, \mathbf{c}^*_p$ also commute with $\mathbf{t}^*$

$$\mathrm{supp} X \subset [1,n] \Rightarrow \mathrm{supp}\,\mathbf{b}^*_p(X) \subset [1,n+p]$$

In terms of basis created by $\mathbf{t}^*, \mathbf{b}^*.\mathbf{c}^*$, the correlation functions are given in the form of determinants.

$$\mathbf{tr}^\alpha \mathbf{t}^*(\zeta)(X) = 2\mathbf{tr}^\alpha(X),$$

$$\mathbf{tr}^\alpha \mathbf{b}^*(\zeta)(X) = \mathrm{res}_{\xi^2=1}\omega_0(\zeta/\xi,\alpha)\mathbf{tr}^\alpha \mathbf{c}(\xi)(X)\frac{d\xi^2}{\xi^2}.$$

In other words, the functional $v^{(\alpha)}$ given by

$$v^{(\alpha)}(X) = \mathbf{tr}^\alpha\left(e^{\mathbf{\Omega}_0}X\right),$$

$$\mathbf{\Omega}_0 = \mathrm{res}_{\zeta_1^2=1}\mathrm{res}_{\zeta_1^2=1}\omega_0(\zeta_1/\zeta_2,\alpha)\mathbf{b}(\zeta_1)\mathbf{c}(\zeta_2)\frac{d\zeta_1^2}{\zeta_1^2}\frac{d\zeta_2^2}{\zeta_2^2}$$

serves as the dual vacuum:

$$v^{(\alpha)}(\mathbf{t}^*(\zeta)X) = 2v^{(\alpha)}(X),\ v^{(\alpha)}(\mathbf{b}^*(\zeta)X) = v^{(\alpha)}(\mathbf{c}^*(\zeta)X) = 0.$$