New results on block entanglement in 1D systems

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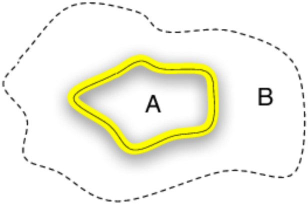
Florence September 2008

With J. Cardy, M, Campostrini & B. Nienhuis, A. Lefevre

Entanglement: what is it?

Quantum system in a pure state $|\Psi\rangle$
The density matrix is $\rho = |\Psi\rangle\langle\Psi|$
$(\mathrm{Tr}\rho^n = 1)$ $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$



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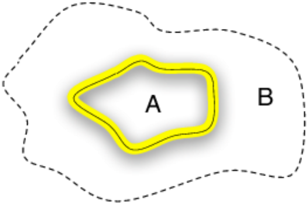

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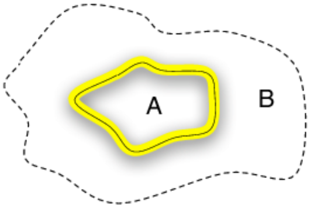

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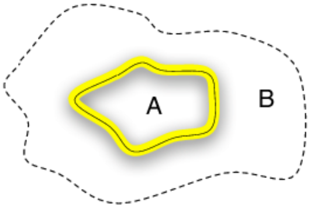

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A natural measure is the *entanglement entropy*

$$S_A = -\sum_n c_n^2 \log c_n^2 = S_B$$

$S_A = 0$ when $|\Psi\rangle$ is unentangled and its maximal $= \log \dim \mathcal{H}_{\min A,B}$

$|\Psi\rangle$ is the ground state of a local Hamiltonian H
Is entanglement special?

Entanglement meets cond-mat (and QFT)

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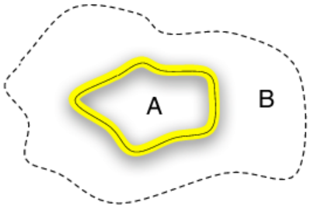
Is entanglement special?

Yes, if A is a large compact spatial subset



- How does S_A depend on the size of A?
- What about the shape of A?
- Is there any universality?

Area law and criticality

- Area Law: $S_A \propto \mathcal{A}$ [Non extensive]

Srednicki '93

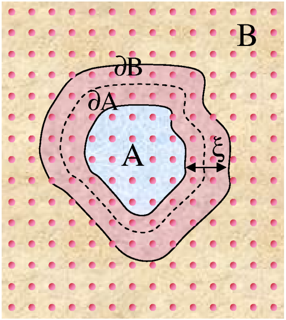
$\downarrow$

(lots of works)

$\downarrow$

Wolf et al '07

Only in gapped systems



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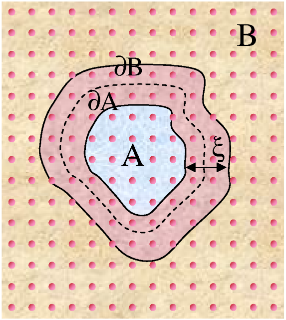

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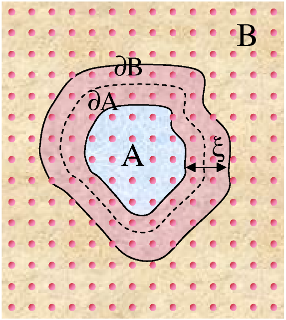

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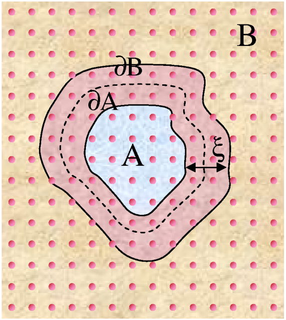

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- Extensive reviews by Amico et al., Eisert et al. [RMP]

Entanglement and CFT (with J. Cardy)

Replica trick: $$S_A = -\mathrm{Tr}\rho_A \log \rho_A = -\lim_{n\to 1}\frac{\partial}{\partial n}\mathrm{Tr}\rho_A{}^n$$

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In CFT, $\mathrm{Tr}\rho_A^n$ transforms like the correlation function of m (# of points between A & B) primary fields with scaling dimension

$$\Delta_\Phi = \tfrac{c}{24}\left(n - \tfrac{1}{n}\right) \Rightarrow \quad \mathrm{Tr}\,\rho_A^n = c_n \left(\frac{\ell}{a}\right)^{-\frac{c}{6}\left(n-\frac{1}{n}\right)} \quad \Rightarrow \quad S_A = \frac{c}{3}\ln\frac{\ell}{a} + c_1'$$

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Finite temperature

$$S_A = \frac{c}{3}\log\left(\frac{\beta}{\pi a}\sinh\frac{\pi\ell}{\beta}\right) + c_1' \simeq \begin{cases} \dfrac{\pi c}{3}\dfrac{\ell}{\beta}\,, & \ell \gg \beta \qquad \text{classical extensive} \\[2ex] \dfrac{c}{3}\log\dfrac{\ell}{a}\,, & \ell \ll \beta \qquad T = 0 \ \text{non}-\text{extensive} \end{cases}$$

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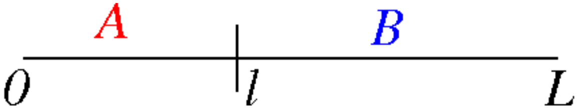
Finite size

$$S_A = \frac{c}{3} \log\left(\frac{L}{\pi a} \sin \frac{\pi \ell}{L}\right) + c_1'$$ Symmetric $\ell \to L - \ell$. Maximal for $\ell = L/2$

Open systems



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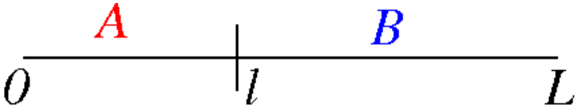

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and finite size

$$S_A(L) = \frac{c}{6}\log\left(\frac{2L}{\pi a}\sin\frac{\pi\ell}{L}\right) + \tilde{c}'_1$$

$\tilde{c}'_1 - c'_1/2 = \ln g$ boundary entropy
[Affleck, Ludwig]

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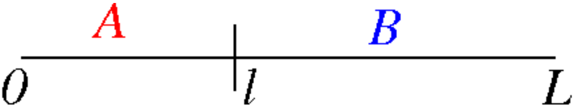

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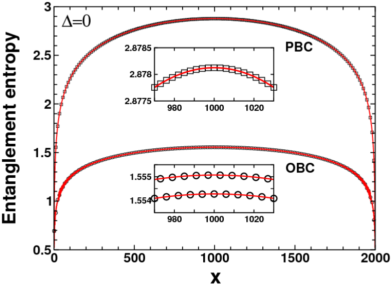

[From Laflorencie et al '06]

Developments

Since the early papers in 2003 about 1000 papers on the subject!

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- Effective way of detecting and characterizing quantum criticality
- In random (no conformal invariance!) quantum spin chains $S_A \propto \log \ell$

  Rafael and Moore, Laflorencie, Santachiara...

  It is related to the number of broken singlets. Is it true for clean chains? NO

  Alet et al, Jacobsen and Saleur

- Topological entanglement entropy

$$S_A = \alpha L - \gamma, \quad \gamma \,\text{is the topological charge}$$

Kitaev and Preskill, Levin and Wen, Fradkin and Moore, Schoutens et al., Furukawa and Misguich, Li and Haldane...

- New numerical methods based on entanglement to simulate $d > 1$

  Vidal, Latorre, Cirac, Hastings ....

- Time dependence and DMRG-like simulability of non-equilibrium

  PC and JC, Vidal, Schollwoeck, Kollath, Eisert, Cirac, Hastings, Peschel ....

- Holography: $S_A =$ length of the geodesic in the AdS bulk

  Ryu and Takayanagi...

- Too many more, sorry if YOUR name is not here!

Universal finite size scaling in Heisenberg chains

Joint work with B. Nienhuis and M. Campostrini

$$H = -\sum_{j=1}^{L} [\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y - \Delta \sigma_j^z \sigma_{j+1}^z]$$

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with periodic boundary conditions

1. $-1 \leq \Delta \leq 1$: gapless
2. $\Delta = 0$: free fermions
3. $\Delta = -1/2$ with L odd: magic
   - Doubly degenerate ground state with no FS for the energy $E_0 = -3/2L$ exactly Baxter
   - The components of the ground-state wavefunction (suitable normalized) are integer numbers related to the combinatorics of Alternating Sign Matrices, Plane partitions etc Razumov-Stroganov
     Correlations are simple functions (rational/factorial) of L

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What about the reduced density matrix?

The magic of $\Delta = 1/2$

ρ_A can be written in terms of integers numbers (obvious)
Method:

- Getting the GS for a sequence of L
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$$\rho_1(L) = \begin{bmatrix} (L+1)/2L & 0 \\ 0 & (L-1)/2L \end{bmatrix}$$

$$\rho_2(L) = \frac{1}{2^4 L^2} \begin{bmatrix} 2((L+2)^2-1) & 0 & 0 & 0 \\ 0 & 6L^2-6 & 5L^2+3 & 0 \\ 0 & 5L^2+3 & 6L^2-6 & 0 \\ 0 & 0 & 0 & 2((L-2)^2-1) \end{bmatrix}$$

We worked out the analytic expression for any L up to $\ell = 6$

For $L \to \infty$ reduces to Sato and Shiroishi

The denominators of $\rho_n(L)$ are: $2^{n^2} L^n \prod_{k=1}^{[n/2]} (L^2 - 4k^2)^{n-2k}$

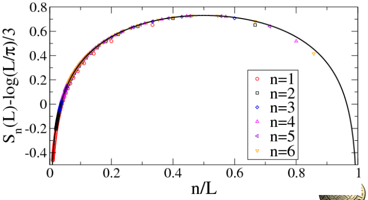
The magic of $\Delta = 1/2$: combinatorics?

- $\rho_n(L)[1,1]$ is the emptiness formation $\Rightarrow$ follows the ASM sequence
- Few other elements of $\rho_n(L)$ can be derived trough recursion relations. We were not able to recognize the others
- The eigenvalues are not simple for general L, also in the TD limit
- In the TD limit

  $$S_1 = \ln 2, S_2 = 0.95075, S_3 = 1.09287, S_4 = 1.19076, S_5 = 1.26588, S_6 = 1.32701$$

  also from Sato Shiroishi. Growing like $1/3 \log \ell$. OK
- For finite L

$$S_n(L) = \frac{1}{3} \log \frac{L}{\pi} \sin \frac{\pi n}{L} + 0.730503 + O(1/L^2)$$



The magic of $\Delta = 1/2$: combinatorics? II

- $L = \infty$, $\mathrm{Tr}\rho_n^2 = r_n/2^{2n^2} (\propto n^{-c/4}$ CFT)

$$r_1 = 2, \quad r_2 = 130, \quad r_3 = 107468, \quad r_4 = 1796678230$$
$$r_5 = 413605561988912, \quad r_6 = 17682565053024999353800$$

Grows too quickly to be guessed

- numerically:

$$R_1 = 0.5, \quad R_2 = 0.5078, \quad R_3 = 0.4099, \quad R_4 = 0.4183, \quad R_5 = 0.3673, \quad R_6 = 0.3744$$

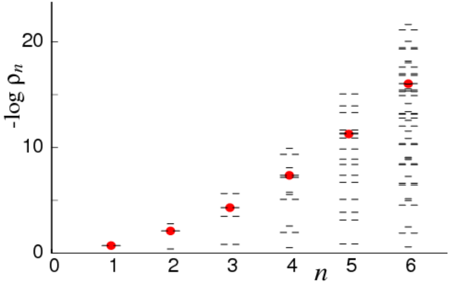
It alternates!!

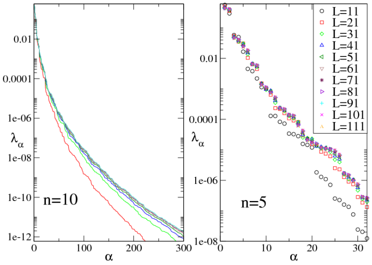
- A finite size sequence $Q_n = \mathrm{Tr}\rho_n^2 (L = (n \pm 1)/2)$

$$Q_1 = \frac{5}{9}, \quad Q_2 = \frac{327}{625}, \quad Q_3 = \frac{11393}{24696}, \quad Q_4 = \frac{3865135}{8732691}, \quad Q_5 = \frac{135038791915}{326039858001}$$

Grows slower, but still too quick to be guessed!

The spectrum of the reduced density matrix





- The pattern repeats at the top and bottom of the spectrum
- The smallest eigenvalues seems to scale like e^{-an^2}
- This is true for the "all up" eigenvalue ($\rho[1,1] =$EFP) that at 2/3 (3/4) of the spectrum for n odd (even).

Effective Hamiltonian for the subsystem

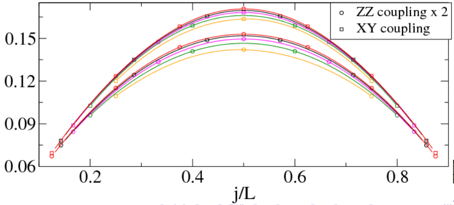
$$\rho_n(L) = e^{-\hat{H}_n}$$

See also Li & Haldane

Properties of $\hat{H}_n$:

- A nearest neighbor hopping term ($J_j S_j^+ S_{j+1}^-$ +h.c)
- A diagonal interaction term ($J_j^z S_j^z S_{j+1}^z$)
- in approximately the same ratio as in the original H ($\sim 1/2$)
- Other terms (multiple hops, far hops, multispin) are at least one (typically two) order of magnitude smaller
- The couplings in $\hat{H}$ depends on the position quadratically

$$J_j^z(n) \simeq A\frac{j(n-j)}{n}$$



A surprise

GS doubly degenerate at $L \Rightarrow$ symmetrized density matrix

$$\rho_A^s = \frac{1}{2}(|\Psi_0^+\rangle\langle\Psi_0^+| + |\Psi_0^-\rangle\langle\Psi_0^-|),$$

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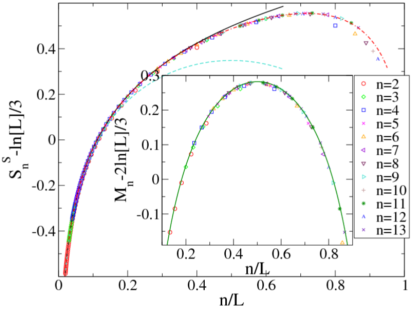

Entanglement is measured by

$$M_n = S_n + S_{L-n} - S_L\,,$$

restoring the symmetry (by definition), and roughly a parabola (obvious)

Larger n for asymptotic scaling: DMRG

$$S_\alpha \equiv \frac{1}{1-\alpha} \mathrm{Tr} \rho_A^\alpha = \frac{c}{6}(1+\alpha^{-1}) \ln \frac{L}{\pi} \sin \frac{\pi n}{L} + c'_\alpha$$

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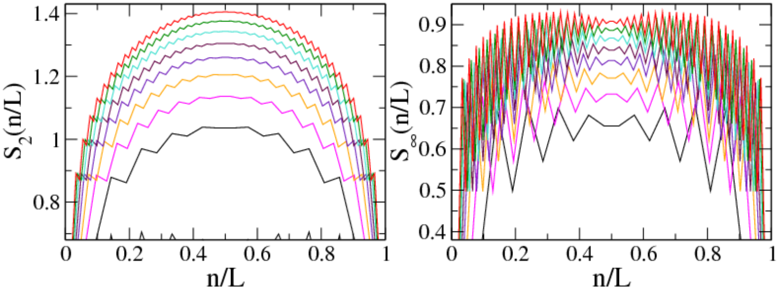

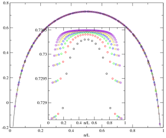

Exact result for XX

TD limit

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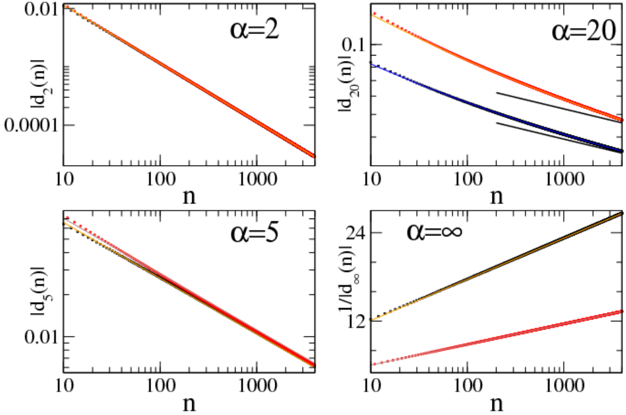
$$d_\alpha(n) \equiv S_\alpha(n) - S_\alpha^{\rm CFT}(n) = n^{-p_\alpha} f_\alpha^{\pm}$$

Using the exact knowledge of c_α

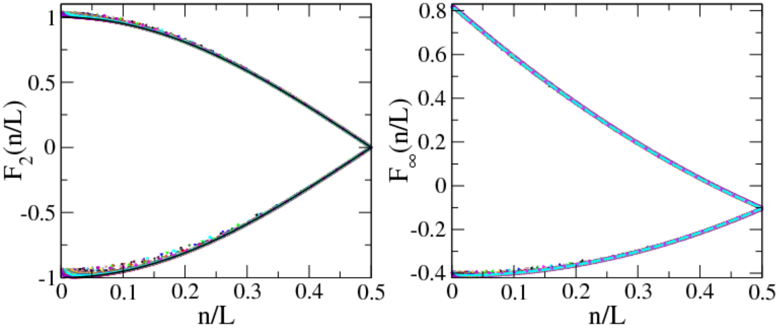


$\Rightarrow p_\alpha = 2/\alpha!!$

Exact result for XX [Finite Size]

$$S_\alpha(n,L) = S_\alpha^{\mathrm{CFT}}(n,L) + \left[\frac{L}{\pi}\sin\frac{\pi n}{L}\right]^{-p_\alpha} F_\alpha^{\pm,\pm}(n/L)$$

All n for several odd L from 17 to 4623 [$\sim 10^5$ points]:



$F_2^{\pm,-}(x) \propto \cos\pi x$ and $F_2^{\pm,+}(x)$ x independent

The finite size ansatz for any Δ

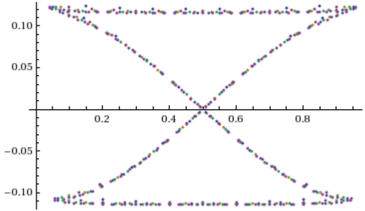
$$S_\alpha(n,L) = S_\alpha^{\rm CFT}(n,L) + \left[\frac{L}{\pi}\sin\frac{\pi n}{L}\right]^{-p_\alpha} F_\alpha^{\pm,\pm}(n/L)$$

p_α and $F_\alpha(x)$ are universal. They are not due to irrelevant operators, but are characteristic of the fixed point.

Similar the the "Friedel" oscillations with OBC for S_A [Laflorencie et al], but here is PBC

The analytic derivation remains an open problem!

For $\Delta = -1/2$, $\alpha = 2$



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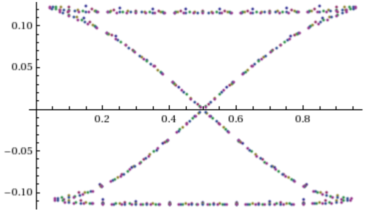

- Similar plots for any Δ, odd and even L, any α
- F_α depends on the parity of L, p_α does not
- $F_\alpha(x)$ has no symmetry, but for $\alpha = 2$ looks perfectly antisymmetric
- $p_\alpha = 2K/\alpha$!!! why?

The density of eigenvalues (with A. Lefevre)

$\mathrm{Tr}\rho_A^\alpha = \sum_i \lambda_i^\alpha = c_\alpha \ell^{-\frac{c}{6}(\alpha - \frac{1}{\alpha})} = c_\alpha e^{-b(\alpha-\frac{1}{\alpha})}$ gives more info than S_A

E.G.: Maximum eigenvalue $\lambda_M = e^{-b}$ Peschel, Orus et al.

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Also the full distribution: $P(\lambda) = \sum_i \delta(\lambda - \lambda_i)$

$$\sum_i \mathcal{L}^{-1}_{\alpha\to t}(\lambda_i^\alpha) = \sum_i \delta(t + \log\lambda_i) \to \int d\lambda P(\lambda)\delta(t+\log\lambda) = P(e^{-t})$$

Ignoring α-dependence of c_α we get

$$P(\lambda) = \delta(\lambda_M - \lambda) + \theta(\lambda_M - \lambda)\frac{b}{\lambda}\sqrt{\frac{1}{b\log\frac{\lambda_M}{\lambda}}}I_1\left(2\sqrt{b\log\frac{\lambda_M}{\lambda}}\right)$$

$P(\lambda)$ starts from λ_M with a delta peak

The density of eigenvalues: simple consequences

- $\#$ eigenvalues larger than λ:

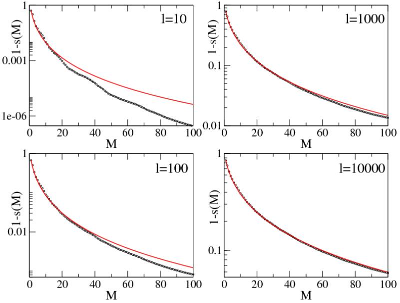
$$n(\lambda) = \int_{\lambda}^{\lambda_{\max}} d\lambda P(\lambda) = I_0(2\sqrt{b \ln(\lambda_M/\lambda)})\,.$$

- Normalization: $\sum \lambda_i = 1 \Rightarrow \int \lambda P(\lambda) d\lambda = 1$
- Entanglement entropy: $S = -\int_0^{\lambda_M} \lambda \ln \lambda P(\lambda) d\lambda = -2 \ln \lambda_M$
- Majorization:

$$s(M) \equiv \sum_{i=1}^{M} \lambda_i \to \lambda_M \left[1 + \int_0^{I_0^{-1}(M)} dy e^{-y^2/4b} I_1(y) \right]$$

at fixed M, is a monotonous function of λ_M (that is a monotonous function of ℓ). agrees Orus et al.

The density of eigenvalues: Check in the XX chain



Deviations from CFT at $M \simeq \ln \ell$ [lattice effects]

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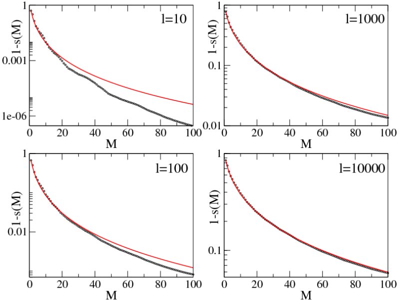
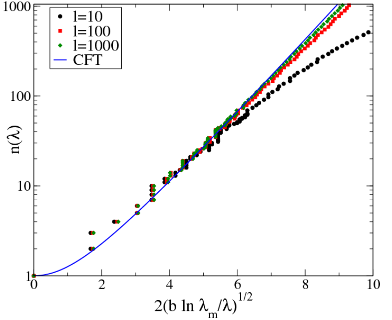


Deviations from CFT at $M \simeq \ln \ell$ [lattice effects]

Scaling variable $x = 2\sqrt{b \ln(\lambda_M/\lambda)}$

$n(\lambda) = I_0(x)$ model indipendent !!.



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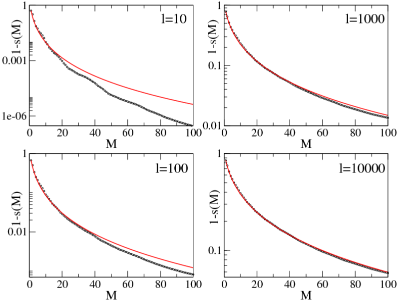

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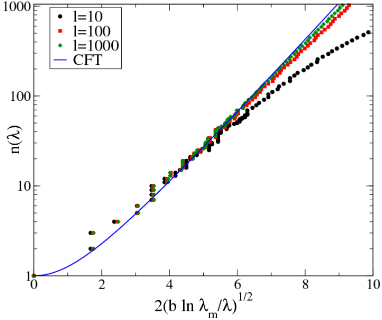

The degeneracies of the eigenvalues are not reproduced, but we observe

$$b \ln \frac{\lambda_\mu}{\lambda_\nu} \simeq k \Rightarrow \qquad \frac{\lambda_\nu}{\lambda_\mu} \simeq e^{-\frac{6k}{\ln \ell/a}}$$

"entanglement gap", related to the scaling of the eigenvalues of the corner transfer matrix Peschel & Truong '87

Entanglement has still a lot to teach us even in 1D!

Thank you