RSOS paths, quasi-particles and fermionic characters

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(collaboration with Patrick Jacob)

Minimal model $M(p',p)$

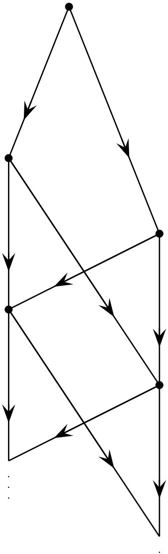
- Central charge
$$c = 1 - \frac{6(p-p')^2}{pp'}$$
(with p',p coprime and say $p > p'$)

- Conformal dimensions:
$$h_{r,s} = \frac{(pr - p's)^2 - (p-p')^2}{4pp'} = h_{p'-r,p-s}$$
$$1 \leq r \leq p'-1 \quad \text{and} \quad 1 \leq s \leq p-1$$

- Highest-weight modules are completely degenerate

Embedding pattern of singular vectors

$$(r,s) \sim (p'-r, p-s)$$



Character of the irreducible modules

- Character:

$$\chi_{r,s}^{(p',p)}(q) = \frac{1}{(q)_\infty} - \frac{q^{rs}}{(q)_\infty} - \frac{q^{(p'-r)(p-s)}}{(q)_\infty} + \frac{q^{rs+(p'+r)(p-s)}}{(q)_\infty} + \frac{q^{(p'-r)(p-s)+r(2p-s)}}{(q)_\infty} - \cdots$$

  where (Verma character)

$$\frac{1}{(q)_\infty} \equiv \frac{1}{\prod_{n\geq 1}(1-q^n)} = \sum_{n\geq 0} p(n)q^n$$

- This formula is thus an alternating sign expression ...
- obtained by representation theory...
- but not very physical !

Fermionic character formula

- Every character in minimal models has a representation in terms of a positive multiple-sum
- This is called a fermionic character
- It reflects the filling of the space of states with quasi-particles subject to restrictions and without singular vectors

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- Example: Ising vacuum:

$$\chi_{1,1}^{(3,4)}(q) = \sum_{m=0}^{\infty} \frac{q^{2m^2}}{(q)_{2m}}$$

where

$$(q)_m \equiv \prod_{i=1}^{m} (1 - q^i)$$

manifestly positive:

$$\frac{1}{(1-q^j)} = 1 + q^j + q^{2j} + \dots$$

The $M(2,2k+1)$ model [Feigin-Nakanashi-Ooguri '91]

$$\chi_{1,s}^{(2,2k+1)}(q) = \sum_{m_1,\cdots,m_{k-1}=0}^{\infty} \frac{q^{mBm+Cm}}{(q)_{m_1}\cdots(q)_{m_{k-1}}}$$

where

$$B_{ij} = \min(i,j) \qquad C_j = \max(j-s+1)$$

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Basis of states:

$$L_{-n_1}\cdots L_{-n_N}|h_{1,s}\rangle \qquad (n_i > 0)$$

with

$$n_i \geq n_{i+k-1}+2 \qquad \text{and} \qquad n_{p-s+1} \geq 2$$

(sort of generalized exclusion principle plus a boundary condition that selects the module)

Origin in CFT: These constraints come from the non-trivial vacuum singular vector

The $M(2,2k+1)$ model [Feigin-Nakanashi-Ooguri '91]

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where

$$B_{ij} = \min(i,j) \qquad C_j = \max(j-s+1)$$

Questions

- What is the CFT interpretation of the m_j?
- If a fermionic form is related to an integrable perturbation: this is the $\phi_{1,3}$ one: how does this enters in the structure?

The $M(3,p)$ models [Jacob, M '06; Feigin et al '06]

Extended algebra construction:

$$\phi_{2,1} \times \phi_{2,1} = \phi_{1,1} + \phi_{3,1} \ = \phi_{1,1}$$

e.g., with

$$\phi \equiv \phi_{2,1} \qquad \text{and} \qquad h \equiv h_{2,1} = \frac{p-2}{4}$$

$$\phi(z)\phi(w) = \frac{1}{(z-w)^{2h}} \left[I + (z-w)^2 \frac{2h}{c} T(w) + \cdots \right] S$$

and $S = (-1)^{p\mathscr{F}}$ where $\mathscr{F}$ counts the number of ϕ modes

Basis of states

Generalized commutation relations + singular vector of ϕ:

$$\phi_{-s_1}\phi_{-s_2}\cdots\phi_{-s_{N-1}}\phi_{-s_N}|\sigma_\ell\rangle\,,$$

with

$$s_i \geq s_{i+1} - \frac{p}{2} + 3\,, \qquad s_i \geq s_{i+2} + 1$$

and the boundary conditions:

$$s_{N-1} \geq -h + \frac{\ell}{2} + 1\,, \qquad s_N \geq h - \frac{\ell}{2}$$

where

$$s_{N-2i} \in \mathbb{Z} + h + \frac{\ell}{2} \qquad \text{and} \qquad s_{N-2i-1} \in \mathbb{Z} - h + \frac{\ell}{2}$$

The spectrum is fixed by associativity

The $M(3,p)$ character formula

$$\chi_{1,s}^{(3,p)}(q) = \sum_{m_1,m_2,\cdots m_k \geq 0} \frac{q^{mB'm+C'm}}{(q)_{m_1}\cdots(q)_{m_{k-1}}(q)_{2m_k}} ,$$

where k is defined via

$$p = 3k + 2 - \epsilon \qquad (\epsilon = 0,1)$$

with $1 \leq i,j \leq k-1$

$$B'_{ij} = \min(i,j) , \qquad B'_{jk} = B'_{kj} = \frac{j}{2} , \qquad B'_{kk} = \frac{k+\epsilon}{4} ,$$

and C' reads

$$C'_j = \max(j-s+1,0) , \qquad C'_k = \frac{k-\epsilon-s+1}{2} ,$$

The $M(3,p)$ character formula

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Questions

- What is the CFT interpretation of the m_j?
- If a fermionic form is related to an integrable perturbation: this is the $\phi_{1,3}$ one: how does this fit with a formulation in terms of the $\phi_{2,1}$ modes ?

On the CFT derivations of fermionic characters in $M(p',p)$ models

- The $M(2,p)$ and the $M(3,p)$ models are the only ones for which there is a 'complete' CFT derivation of the fermionic characters

- Generalization of the $M(3,p)$ case to $M(p',p)$:

  1- Replace $\phi_{2,1}$ by $\phi_{p'-1,1}$ [M,Ridout '07]

  2- Treat $\phi_{2,1}$ with its 2 channels
  [Feigen-Jimbo-Miwa-Mutkhin-Takayema '04,'06]

  - Monomial bases have been derived/conjectured for all models but the corresponding formula is not written

  - These bases can be reexpressed in terms of RSOS-type configuration sums

Origins of Fermionic forms

- Bases in affine Lie algebras (parafermions) [Lepowski-Primc '85]
- Bases for the $M(2,p)$ models [Feigin-Ooguri-Nakanishi '91]
- Dilogarithmic identities [Nahm et al '92; Kuniba-Nakanishi-Suzuki (generalized parafermions)]
- Many conjectured expressions for the minimal models [Kedem-Klassen-McCoy-Melzer '93]
- Counting of states in XXZ: truncation and q-deformation [Berkovich-McCoy-Schilling: '94-'95]
- Spinon bases for $\widehat{su}(2)_k$ [Bernard et al, Bouwknegt et al '94]
- Mathematical transformations of identities (Bayley and Burge transforms) [Foda-Quano, Berkovich-McCoy,...]

Origins of Fermionic forms: The RSOS side

- RSOS models:

  Andrews-Baxter-Forrester (1984) (unitary case)

  Forrester-Baxter (1985) (non-unitary case)

- Key observation [Date, Jimbo, Kuniba, Miwa, Okado]:

  1D configuration sums (obtained by CTM) in regime III
  (a lattice realization of the $\phi_{1,3}$ perturbation)
  are the $M(p',p)$ irreducible characters

- Configurations sums leads to fermionic character in a systematic way [Melzer, Warnaar, Foda, Welsh]

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- Configurations sums leads to fermionic character in a systematic way [Melzer, Warnaar, Foda, Welsh]

  ...but the derivation is not constructive and the underlying quasi-particle structure remains unclear (except in the unitary case)

RSOS(p',p) paths (regime-III)

States in the finitized $M(p',p)$ minimal models (with $p > p'$) are described by RSOS(p',p) configurations

Configurations

- Configuration = sequence of values of the height variables $\ell_i \in \{1,2,\cdots,p-1\}$ $(0 \leq i \leq L)$
- with the admissibility condition: $|\ell_i - \ell_{i+1}| = 1$
- and the boundary conditions: ℓ_0, ℓ_{L-1} and ℓ_L fixed

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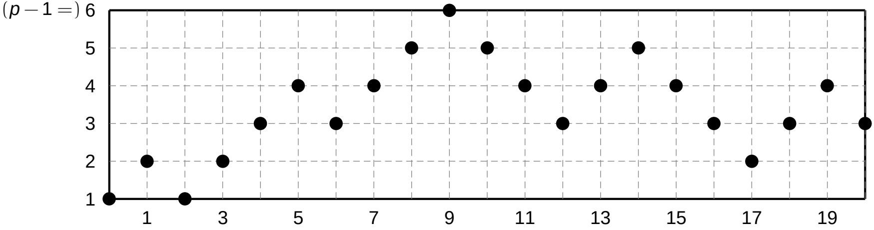
Paths

- A path is the contour of a configuration.
- Path = sequence of NE or SE edges joining the adjacent vertices (i, ℓ_i) and $(i+1, \ell_{i+1})$ of the configuration within the rectangle $1 \leq y \leq p-1$ and $0 \leq x \leq L$
- with ℓ_0 and ℓ_L fixed
- and fixed last edge: SE

A typical configuration for the $M(p',7)$ model: $\ell_0 = 1$, $\ell_{19} = 4, \ell_{20} = 3$



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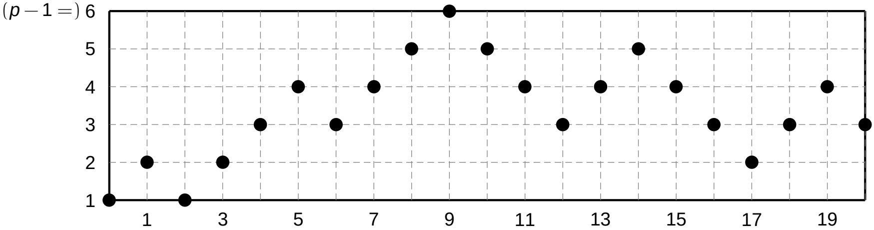
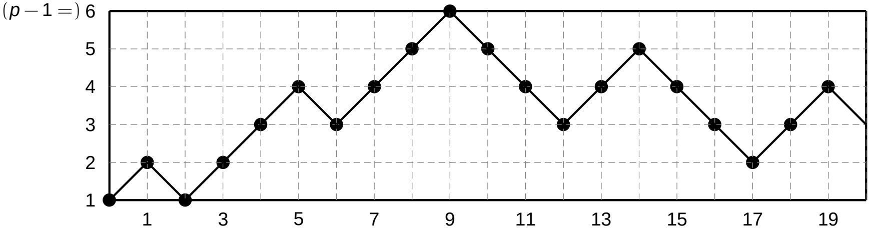
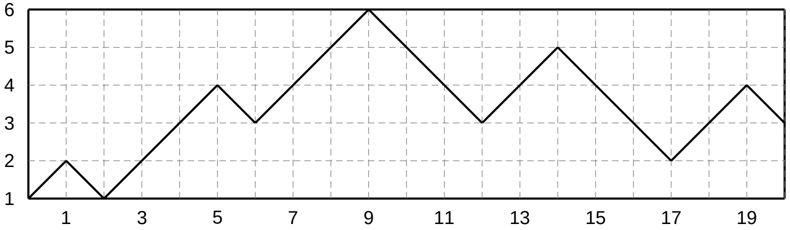


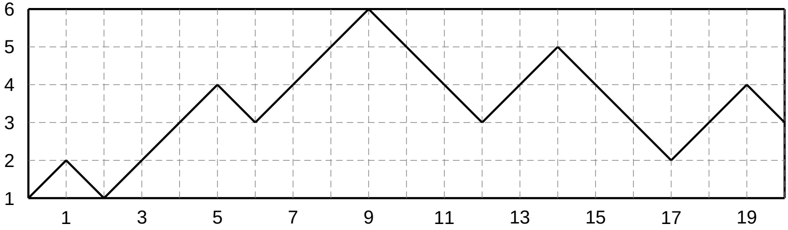
and the corresponding path (with $\ell_{20} = 3$)



A typical path for the $M(p',7)$ model: $\ell_0 = 1$ and $\ell_{20} = 3$ and final SE



A typical path for the $M(p', 7)$ model: $\ell_0 = 1$ and $\ell_{20} = 3$ and final SE

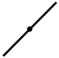
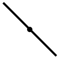


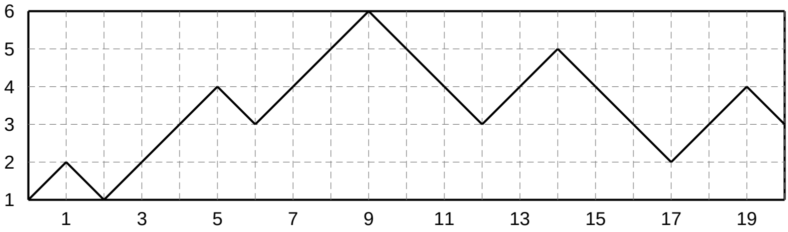
- But this corresponds to a state for which model ? (value of p'?)
- ...and to which module (r, s)?
- ...and what is its conformal dimension?

Weighting the path

The dependence of the path upon the parameter p' is via the weight function:

$$\tilde{w} = \sum_{i=1}^{L-1} \tilde{w}_i$$

	Vertex	$\tilde{w}_i$		Vertex	$\tilde{w}_i$
h	 i	$\frac{i}{2}$	$h+1$	 i	$-i\left\lfloor h\frac{(p-p')}{p}\right\rfloor$
h	 i	$\frac{i}{2}$	$h-1$	 i	$i\left\lfloor h\frac{(p-p')}{p}\right\rfloor$



The expressions of $\tilde{w}_i/i$ for the extrema

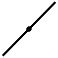
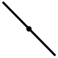
	$p'=2$		$p'=3$		$p'=6$	
h	max	min	max	min	max	min
6	-3	$-$	-2	$-$	0	$-$
5	-2	4	-2	3	0	0
4	-2	3	-1	2	0	0
3	-1	2	-1	2	0	0
2	0	2	0	1	0	0
1	$-$	1	$-$	1	$-$	0

The simplicity of unitary models

The weight function is not positive

Exception: the unitary models: $p' = p - 1$

$$\left\lfloor h\frac{(p-p')}{p}\right\rfloor = \left\lfloor \frac{h}{p}\right\rfloor = 0 \qquad \text{since} \quad h < p$$

Vertex	$\tilde{w}_i$	Vertex	$\tilde{w}_i$
h  i	$\frac{i}{2}$	$h+1$  i	0
h  i	$\frac{i}{2}$	$h-1$  i	0

Weight vs conformal dimension

- Classes of paths are specified by ℓ_0 and ℓ_L
- Ground-state path = unique path with minimal weight with ℓ_0, ℓ_L given
- This path represents a highest-weight state
- Let its weight be $\tilde{w}_{\mathrm{gs}}$
- The relative weight $\Delta\tilde{w} = \tilde{w} - \tilde{w}_{\mathrm{gs}}$ is the (relative) conformal dimension

Generating functions for paths

The GF is the q-enumeration of the paths

$$X^{(p',p)}_{\ell_0,\ell_L}(q) = \sum_{\substack{\text{paths with} \\ \ell_0 \text{ and } \ell_L \text{ fixed}}} q^{\Delta\tilde{w}}$$

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When $L \to \infty$: this is a character of $M(p',p)$:

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When $L \to \infty$: this is a character of $M(p',p)$:

But for which module?

Need to relate (r,s) to ℓ_0 and ℓ_L

A new weight function for the paths (FLPW)

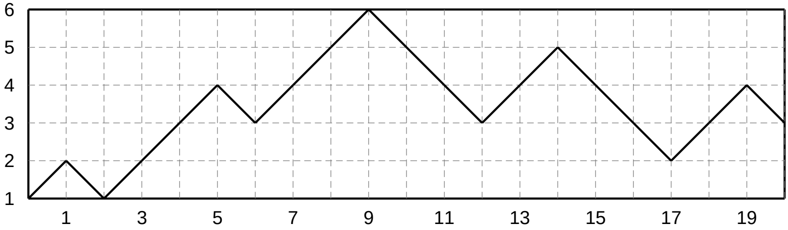
- Make the defining rectangle looks p'-dependent
- Color the $p'-1$ strips between the heights h and $h+1$ for which:

$$\left\lfloor \frac{hp'}{p} \right\rfloor = \left\lfloor \frac{(h+1)p'}{p} \right\rfloor - 1.$$

- Solutions:

$$h = h_r \equiv \left\lfloor \frac{rp}{p'} \right\rfloor \qquad \text{for} \qquad 1 \leq r \leq p'-1.$$

Our path for any $M(p',7)$ model



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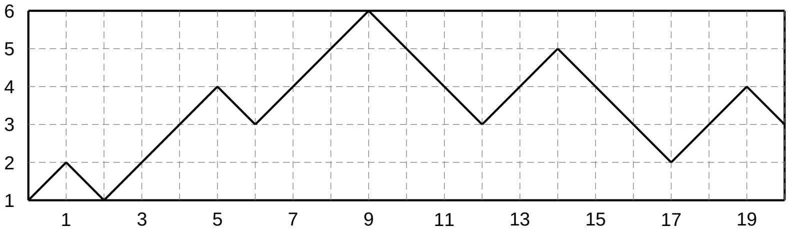
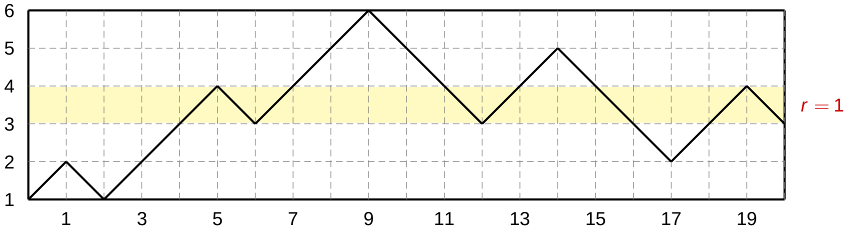


The same path for the $M(2,7)$ model.



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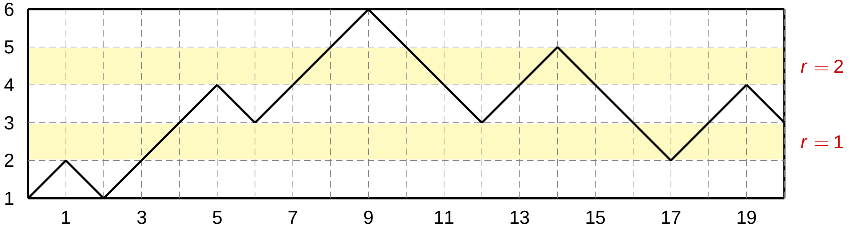

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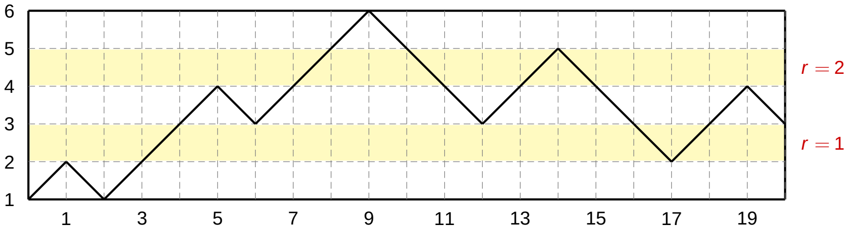
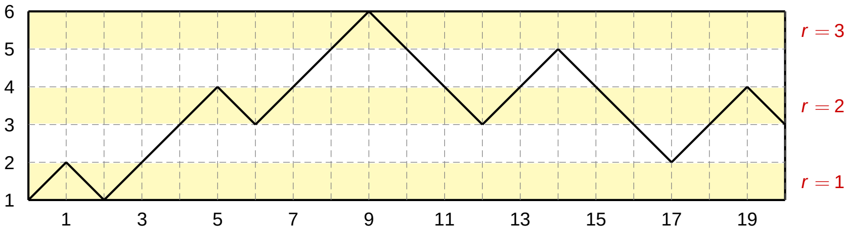
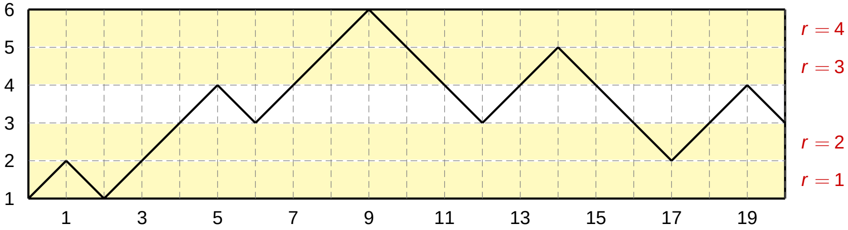


The same path for the $M(4,7)$ model.



The same path for the $M(5,7)$ model.



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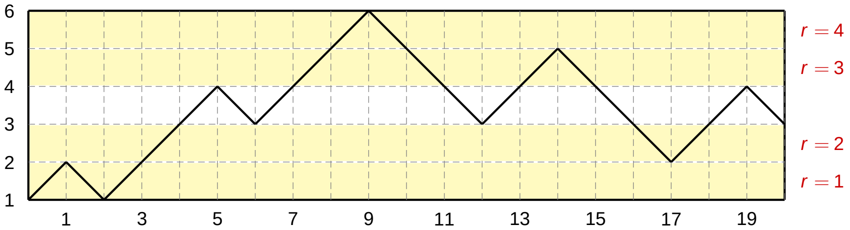
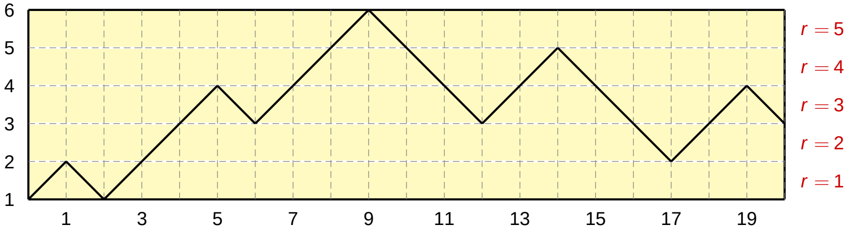


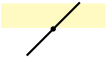
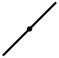
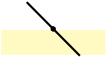
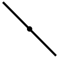
The same path for the $M(6,7)$ model.



Observations:

- The band structure is symmetric with respect to up-down reflection
- For unitary models, $p = p' + 1$, all the bands are colored
- For the $M(2,p)$ models, there is a single colored band
- Colored bands are isolated when $p \geq 2p' - 1$

New weight function for the paths: w (FLPW)

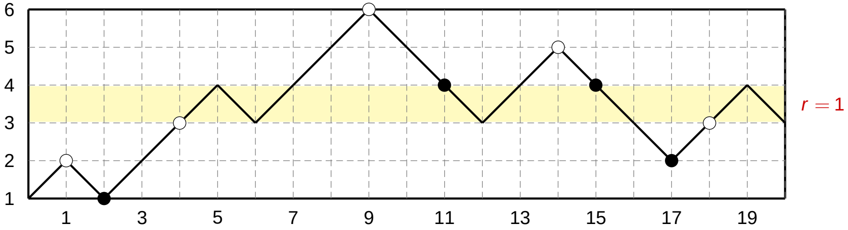
Vertex	Weight	Vertex	Weight
	0		u_i
	0		v_i
	u_i		0
	v_i		0

$$u_i = \frac{1}{2}(i - \ell_i + \ell_0)\,, \qquad v_i = \frac{1}{2}(i + \ell_i - \ell_0)$$

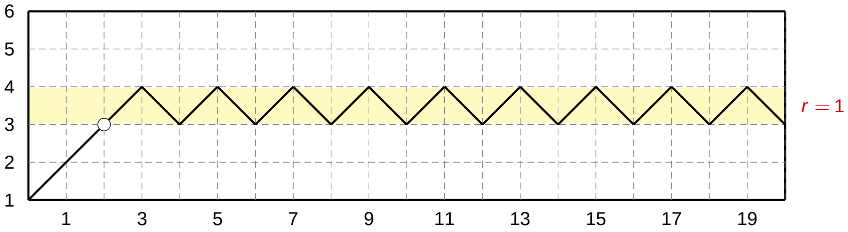
This is a positive definite weighting

Our $M(2,7)$ path with the "scoring vertices"

$\circ \leftrightarrow u_i \qquad \bullet \leftrightarrow v_i$



The ground-state path for the case $\ell_0 = 1$ and $\ell_L = 3$



A single scoring vertex:

$$u_i = \frac{1}{2}(i - \ell_i - \ell_0) \qquad \Rightarrow \qquad u_2 = \frac{1}{2}(2 - 3 - 1) = 0$$

The weight is absolute: $w = 0$

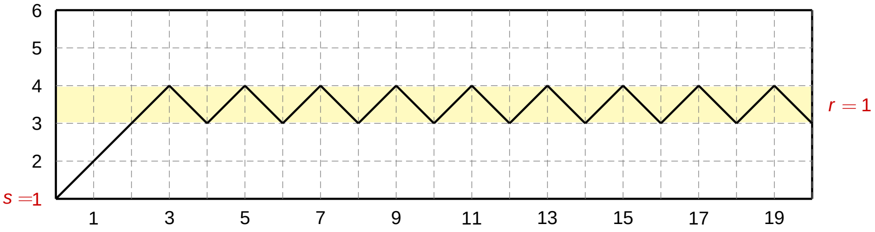
Module identification vs boundaries

- Tails in colored bands have weight $w = 0$
- Such tails are the proper ends for infinite paths
- Characterization of r, s:

$$\ell_0 = s \qquad \text{and} \qquad \ell_L = \left\lfloor \frac{rp}{p'} \right\rfloor$$

The modules are thus characterized by the boundary conditions

The path

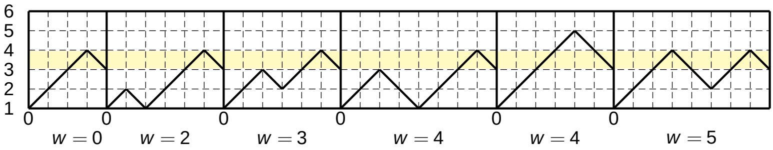


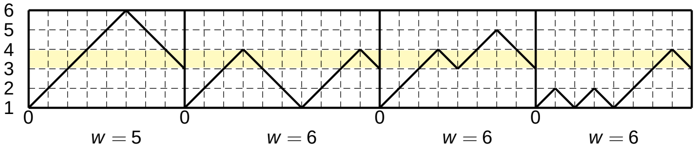
describes the state of lowest dimension in the vacuum module $|h_{1,1}\rangle = |0\rangle$ of $M(2,7)$

i.e.: it represents a finitized version of the vacuum state

All modules are covered by taking $1 \leq s \leq p-1 = 6$
(since $1 \leq r \leq p'-1 = 1$)

The first few sates in the $M(2,7)$ vacuum module:





These correspond to the first few terms in the character

$$\chi_{1,1}^{2,7}(q) = 1 + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + \cdots$$

Characters = GF for infinite paths

- GF for paths is

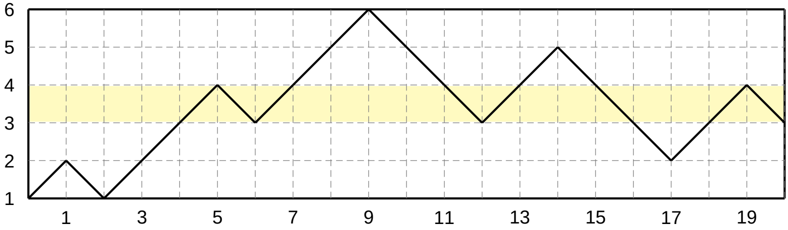
$$X^{(p',p)}_{\ell_0,\ell_L}(q) = \sum_{paths} q^w$$

- Set $\ell_0 = s$ and $\ell_L = \lfloor \frac{rp}{p'} \rfloor$
- GF = finitized version of the Virasoro characters $\chi^{(p',p)}_{r,s}(q)$
- The full Virasoro character is

$$\chi^{(p',p)}_{r,s}(q) = \lim_{L\to\infty} X^{(p',p)}_{s,\lfloor \frac{rp}{p'} \rfloor}(q)$$

Where are the quasi-particles in a RSOS$(2,p)$ path?

e.g., in the RSOS$(2,7)$ path?



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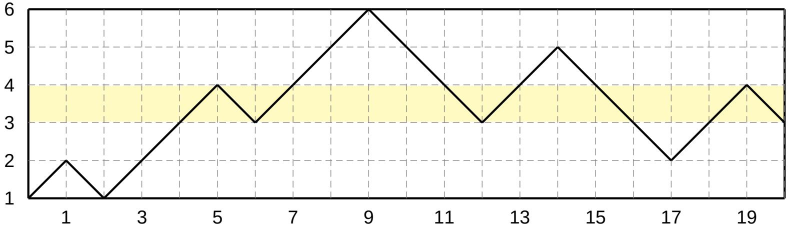
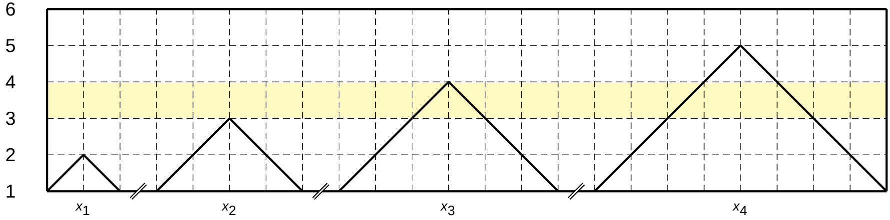


Idea: look at the “peaks in their whole”



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x_1
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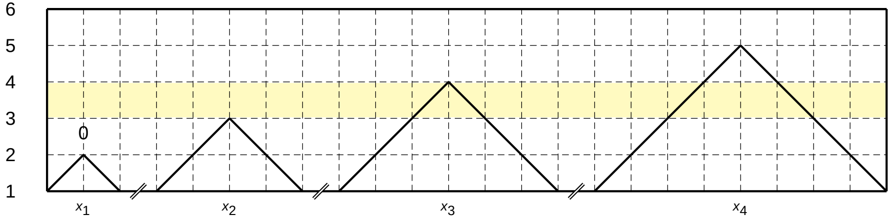
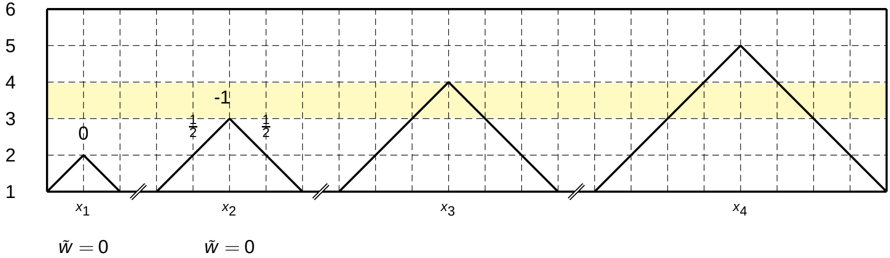
x_4



$\tilde{w} = 0$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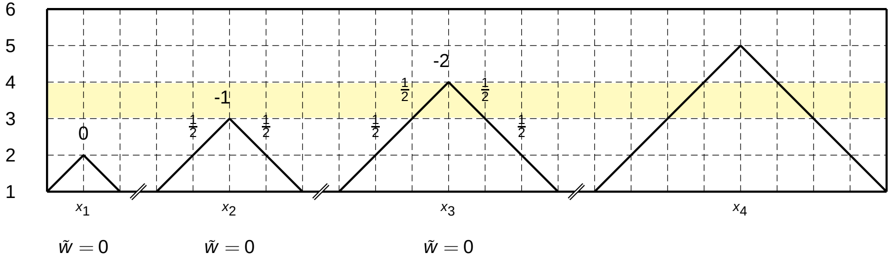
$\tilde{w} = 0$
$\tilde{w} = 0$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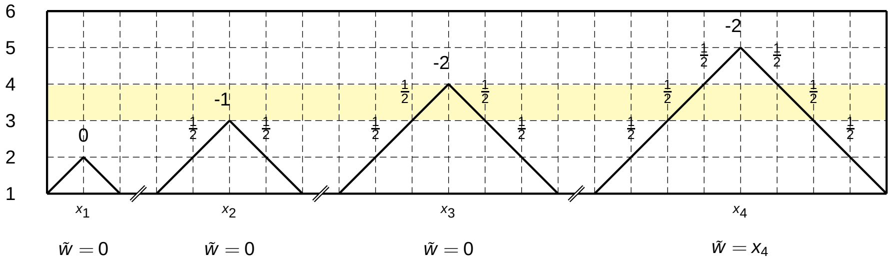

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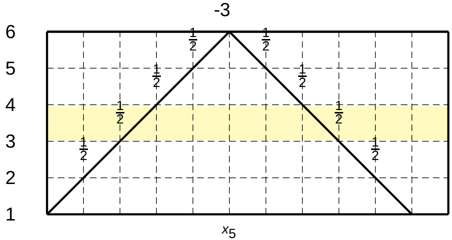
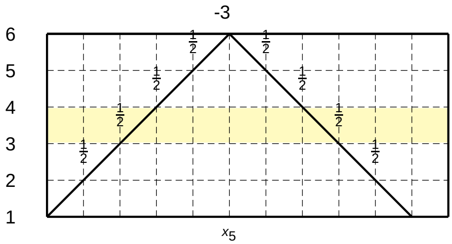
$\tilde{w} = 0$
$\tilde{w} = 0$
$\tilde{w} = 0$
$\tilde{w} = x_4$



$$\tilde{w} = x_5$$

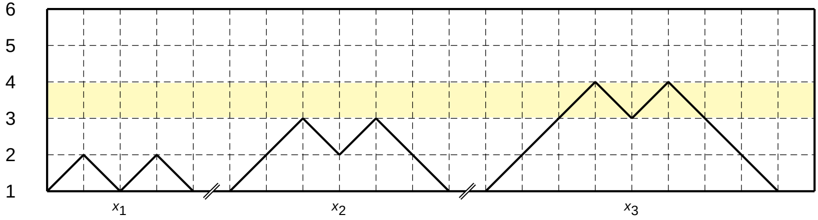


$\tilde{w} = x_5$

Observations:

- full peaks whose top is not above the colored band have $w = 0$
- full peaks whose top is above the colored band have

$$w = x\text{-position of the maximum}$$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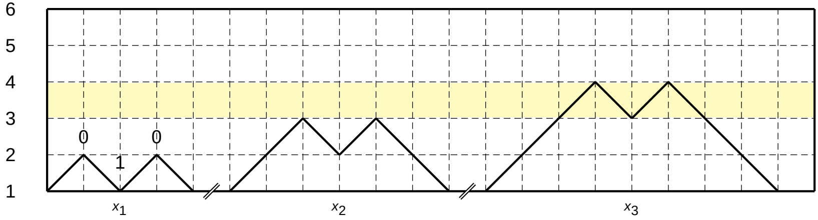
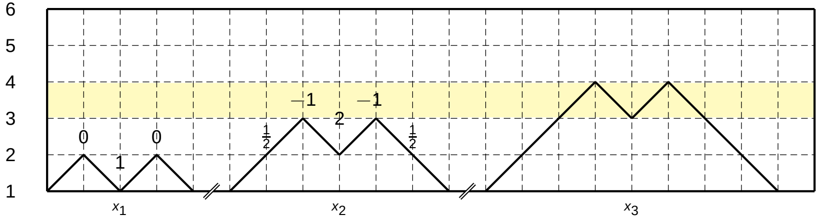
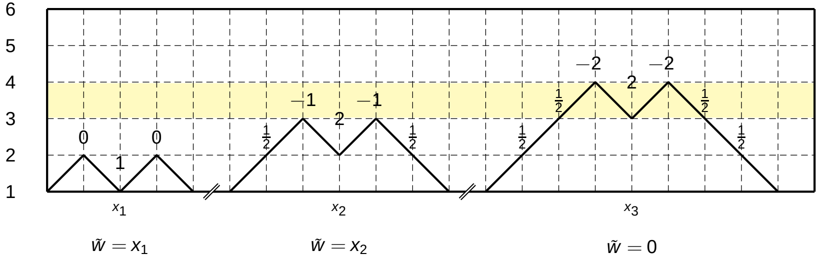


$\tilde{w} = x_1$



$\tilde{w} = x_1$ $\tilde{w} = x_2$



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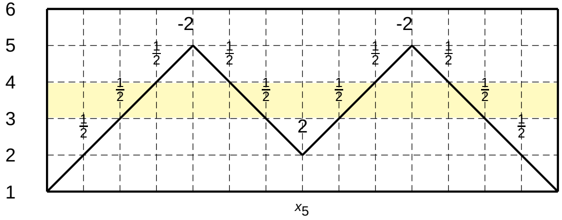
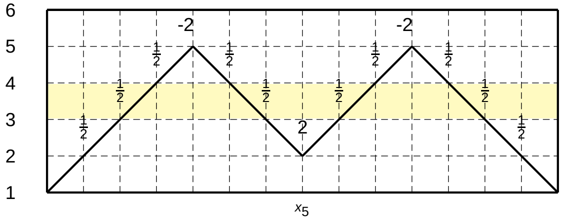
$\frac{1}{2}$
x_3
$\tilde{w} = x_1$
$\tilde{w} = x_2$
$\tilde{w} = 0$



$$\tilde{w} = 3x_5$$



$$\tilde{w} = 3x_5$$

Observations:

- valleys not below the colored band have zero weight
- valleys below the colored band have weight

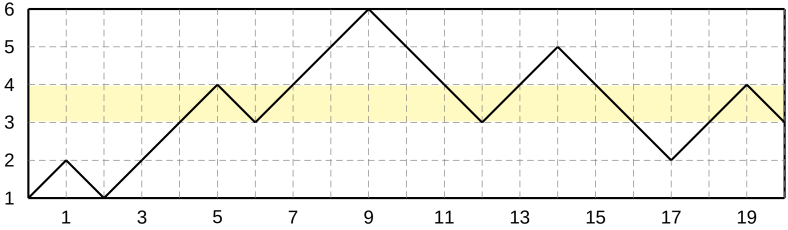
$$w = x\text{-position of the minimum}$$

- Above path: two peaks above and a valley below the colored band:

$$\tilde{w} = (x_5 - 3) + x_5 + (x_5 + 3) = 3x_5$$

Transformation of the RSOS$(2,p)$ paths

These observations suggest to transform the RSOS$(2,7)$ path



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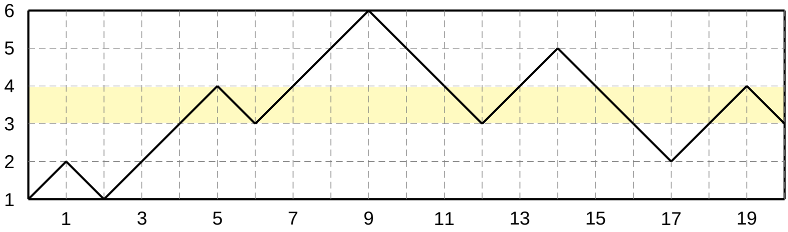
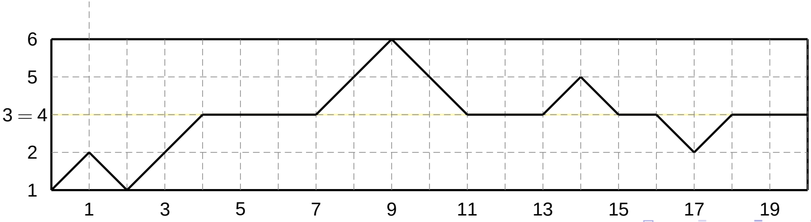
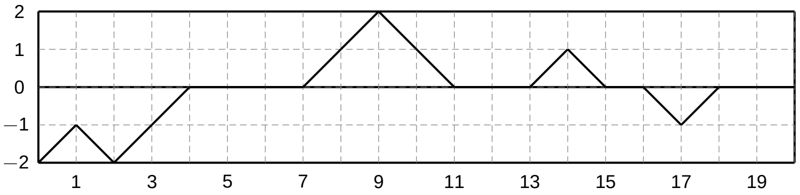


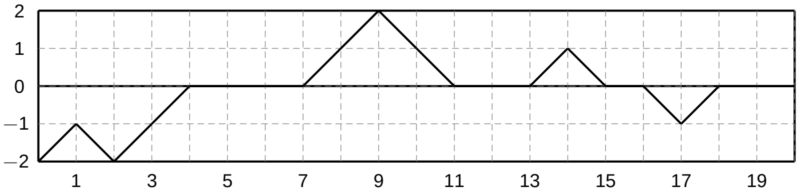
by flattening the colored band



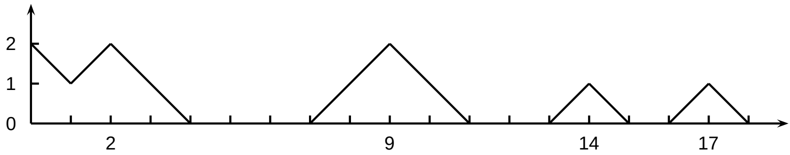
redefine the vertical axis



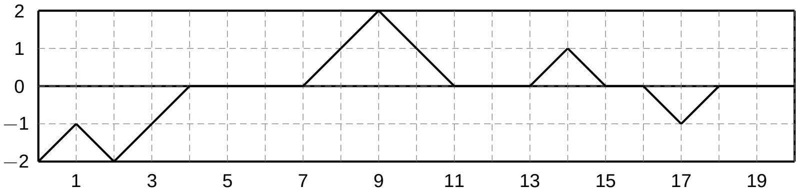
redefine the vertical axis



and fold the lower part onto the upper one



redefine the vertical axis



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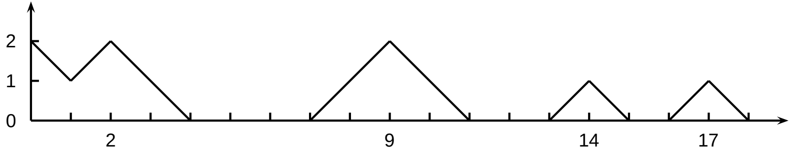
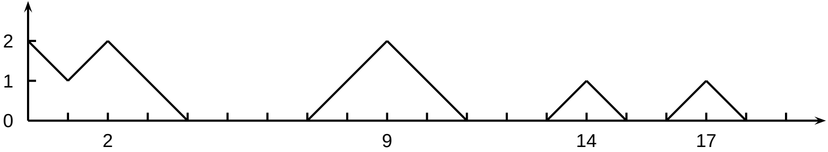


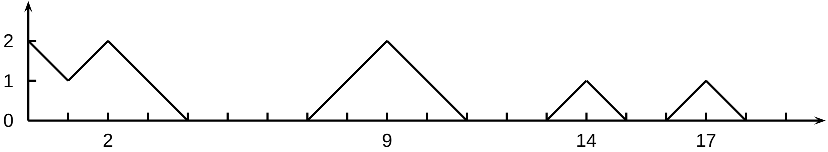
the weight is the x position of the peaks:

$$w = 2 + 9 + 14 + 17$$

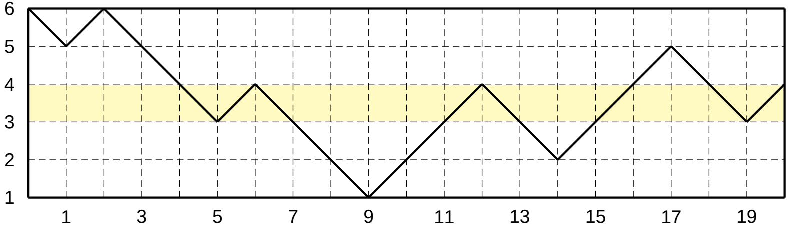
Is this 1-1?



Is this 1-1?



is also related to



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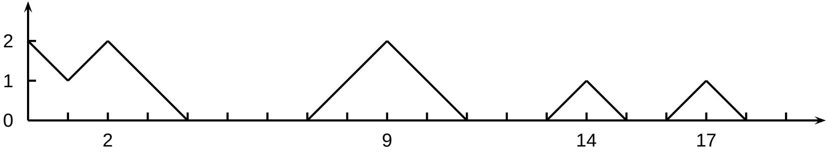

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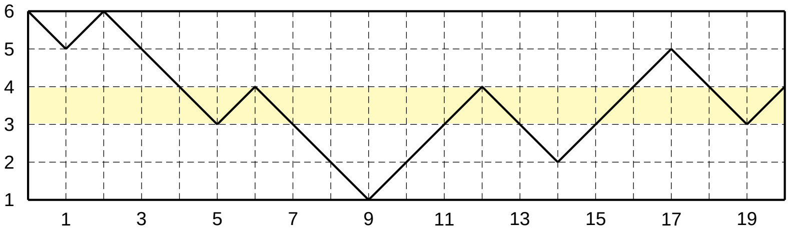


But this has a final NE edge: enforcing a final SE: 1-1 relation

Bressoud paths

These are integer lattice paths

- defined in the strip:

$$0 \le x \le \infty, \qquad 0 \le y \le k-1$$

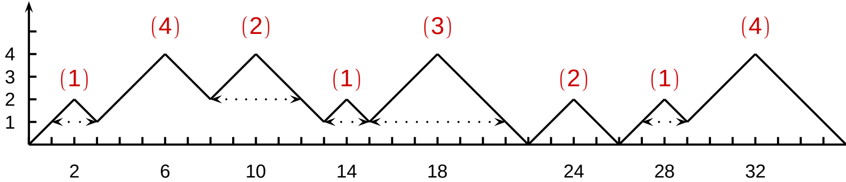
  (with $p = 2k+1$)

- composed of NE, SE and Horizontal edges (H iff $y = 0$)

- weight = x position of the peaks

Bressoud path = sequence of fermi-type charged particles

- An example of Bressoud path for $k = 5$ and initial point (0,0) as a sequence charged peaks (= particles)



- The charge content of the path is:

$$m_1 = 3, m_2 = 2, m_3 = 1, m_4 = 2$$

Bressoud paths $\approx$ 1D fermi-gas

Rules for constructing the generating function of all Bressoud paths with fixed boundaries (ex: $y_0 = 0$)

- For a fixed charge content (fixed $\{m_j\}$): determine the configuration of minimal weight (mwc)

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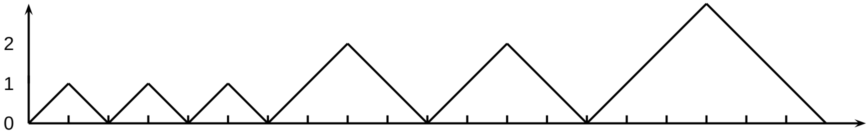
  Example: $m_1 = 3, m_2 = 2, m_3 = 1$:



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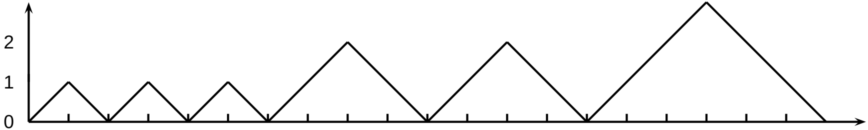
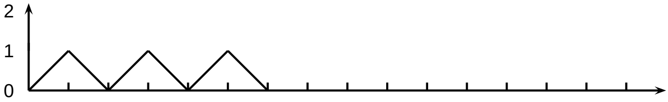


- Evaluate its weight: above $w_{\text{mwc}} = 1+3+5+8+12+17$

  In general

$$w_{\text{mwc}} = \sum_{i,j=1}^{k-1} \min(i,j)\, m_i\, m_j$$

- Move the particles (peaks) in all possible ways and q-count them
  Example: consider $m_1 = 3$



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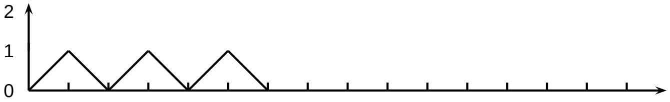
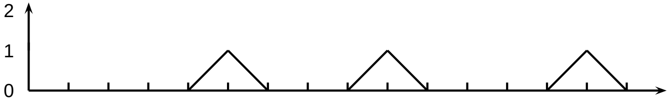


- Rule 1: Identical particles are impenetrable (hard-core repulsion):
  Example: move the rightmost by 9, the next by 6 and the third by 4



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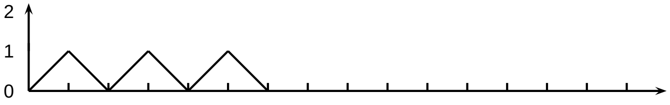

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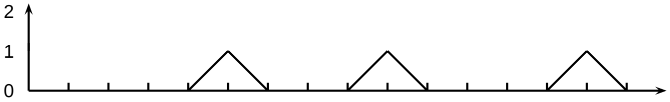
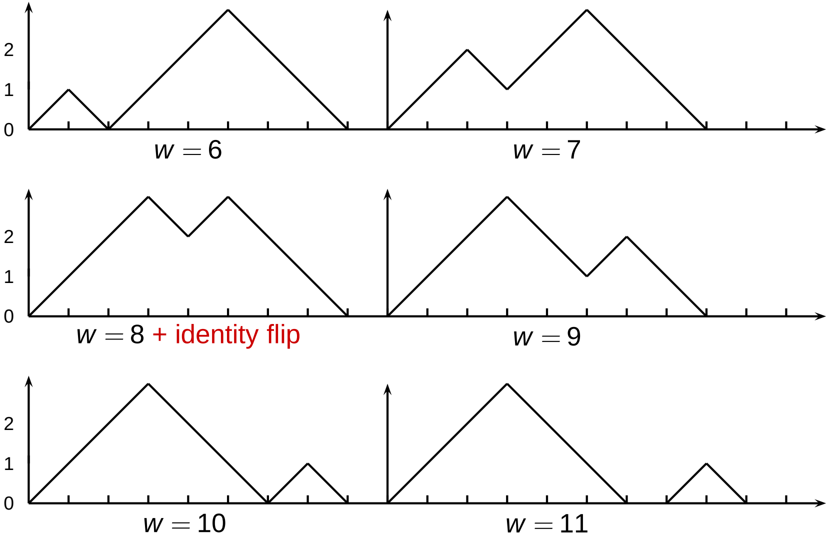


- Generating factor for these moves
  = the number of partitions with at most three parts:

$$\frac{1}{(1-q)(1-q^2)(1-q^3)} \equiv \frac{1}{(q)_3} \qquad \to \qquad \frac{1}{(q)_{m_1}}$$

- Rule 2: Particles of different charges can penetrate

  Consider the successive displacements of the peak 1 in :



- Every move of 1 unit increases the weight by 1 independently of the presence of higher charged particles

$$\text{i.e.} \quad \frac{1}{(q)_{m_1}} \quad \text{is generic}$$

- The same holds for the other particles:

$$\text{factor} \quad \frac{1}{(q)_{m_j}} \quad \text{for each type } 1 \leq j \leq k-1$$

- Generating functions for all paths with fixed charge content

$$G(\{m_j\}) = \frac{q^{w_{\mathrm{mwc}}}}{(q)_{m_1} \dots (q)_{m_{k-1}}}$$

with

$$w_{\mathrm{mwc}} = \sum_{i,j=1}^{k-1} \min(i,j)\, m_i\, m_j$$

- Full generating function:

$$G = \sum_{m_1,\cdots,m_{k-1}} G(\{m_j\})$$

i.e.

$$G = \chi_{1,1}^{(2,2k+1)} = \sum_{m_1,\cdots,m_{k-1}=0}^{\infty} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_1+\cdots+N_{k-1}}}{(q)_{m_1}\cdots(q)_{m_{k-1}}}$$

with N_j defined as

$$N_j = m_j + \cdots + m_{k-1}$$

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with N_j defined as

$$N_j = m_j + \cdots + m_{k-1}$$

- This is the fermionic character of the $M(2,2k+1)$ vacuum module (FNO)
- derived directly from the RSOS$(2,2k+1)$ paths (using the Fermi-gas method of Warnaar)

From RSOS(p',p) to generalized Bressoud paths

- Restriction to $p \geq 2p'-1$: isolated colored bands
- Flatten all colored bands and fold the part below the first band
- Result: generalized Bressoud paths defined in

$$0 \leq x \leq L \qquad 0 \leq y \leq p-p'-\left\lfloor \frac{p}{p'} \right\rfloor$$

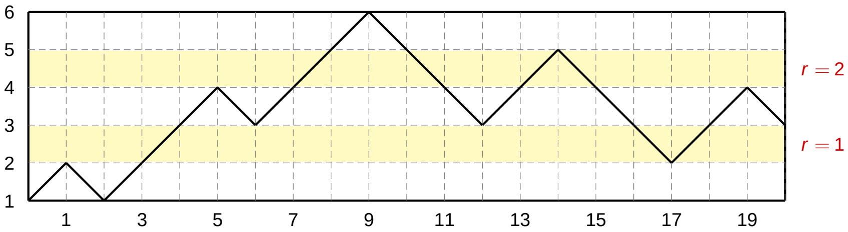
- ...with H edges allowed at height

$$y(r) = \left\lfloor \frac{rp}{p'} \right\rfloor - \left\lfloor \frac{p}{p'} \right\rfloor - r + 1$$

(with a condition relating the parity of successive H edges and the change of direction of the path)

- ...and weight = (half) x position of the (half) peaks

Our $M(3,7)$ path



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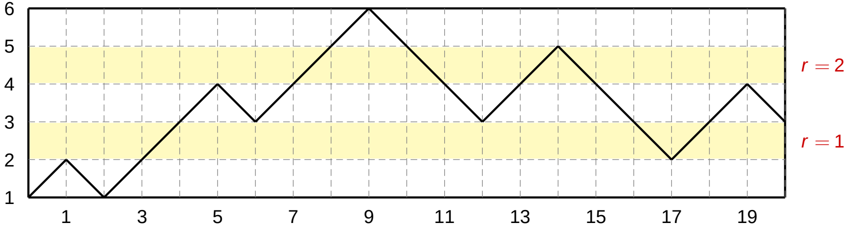
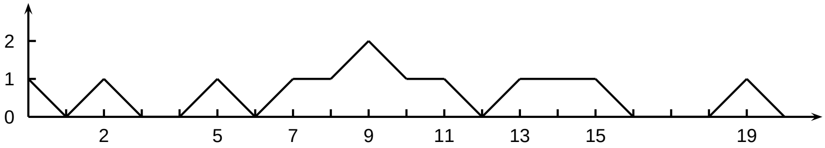
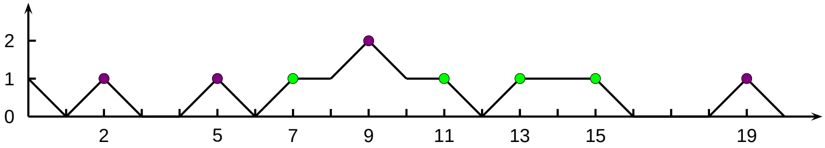


is transformed into

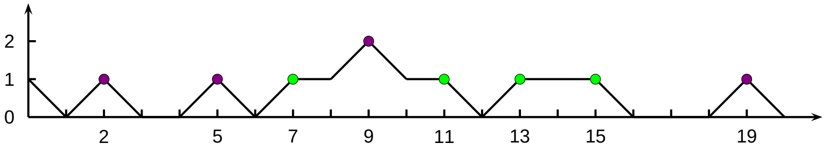


with H edges allowed at $y = 0, 1$ but not $y = 2$

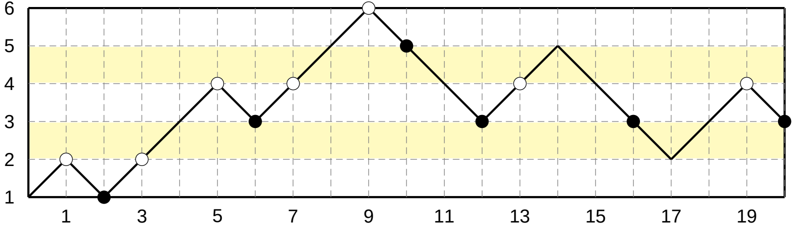
Weight of the path



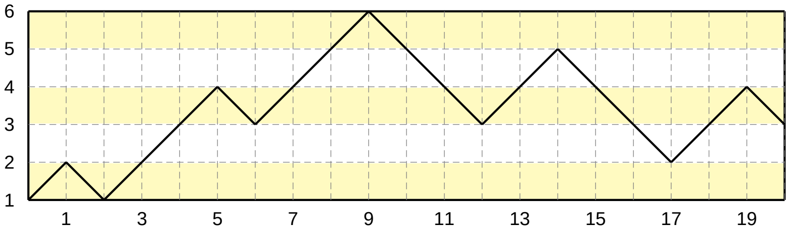
$$w = 2 + 5 + 9 + 19 + \frac{1}{2}\left(7 + 11 + 13 + 15\right)$$



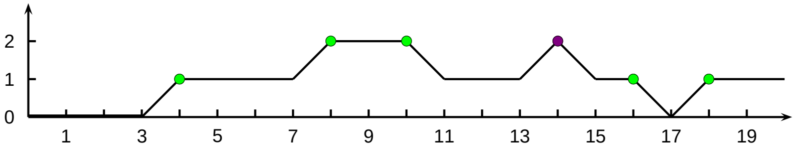
These contributing vertices are not the “scoring vertices”



Similary, our $M(4,7)$ path



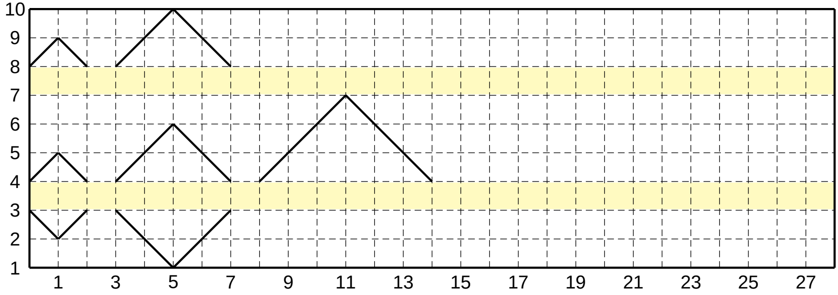
is transformed into:



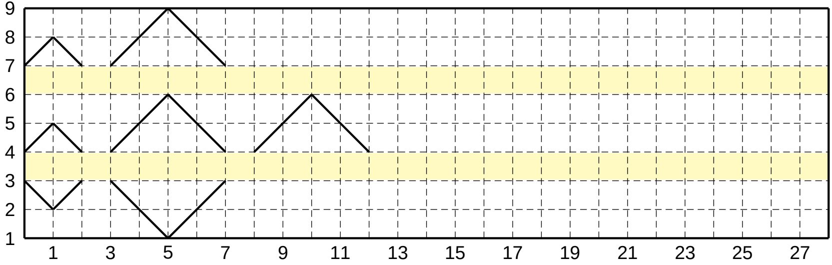
where H edges are allowed at $y = 0, 1, 2$ and
$w = 14 + \frac{1}{2}(4 + 8 + 10 + 16 + 18) - (w_{\text{gs}} = 1)$

Fermi-gas analysis of the $B(3,p)$ paths

$M(3,11)$ (case $p = 3k+2$): 3 particles



$M(3,10)$ (case $p = 3k+1$): 3 particles



Direct Fermi-gas analysis:

$$\chi_{1,1}^{(3,p)}(q) = \sum_{m_1, m_2, \cdots m_k \geq 0} \frac{q^{mBm + Cm - \epsilon m_k^2}}{(q)_{m_1} \cdots (q^{1+\epsilon}; q^{1+\epsilon})_{m_{k-1}} (q)_{2m_k}} ,$$

where k and $\epsilon = 0, 1$ are defined by

$$p = 3k + 2 - \epsilon$$

and

$$(a)_n \equiv (a;q)_n = \prod_{i=0}^{n-1} (1 - aq^i)$$

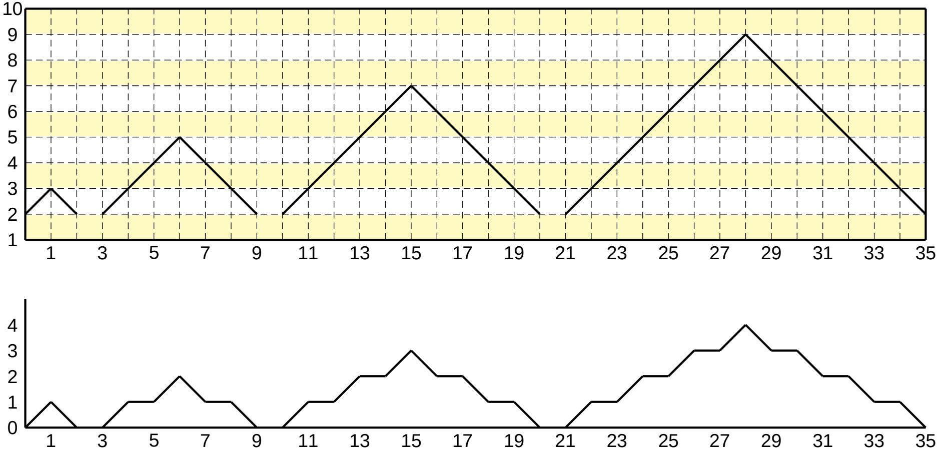
with

$$B_{ij} = \min(i,j) , \qquad C_j = j.$$

New expression when $\epsilon = 1$

Fermi-gas analysis of the $\mathrm{B}(k+2, 2k+3)$ paths

$M(6, 11)$: 4 $(= k)$ particles



Character resulting from the direct Fermi-gas analysis

$$\chi_{1,1}^{(k+2,2k+3)}(q) = \sum_{m_1,\cdots,m_k} \frac{q^{mBm+Cm}}{(q)_{p_0}} \prod_{i=1}^{k-1} \begin{bmatrix} m_i + p_i \\ m_i \end{bmatrix},$$

where

$$B_{i,j} = B_{j,i} \quad B_{i,j} = (2i-1)j \quad \text{if} \quad i \leq j \quad \text{and} \quad C_j = j$$

and

$$\begin{bmatrix} a \\ b \end{bmatrix}_q = \begin{cases} \frac{(q)_a}{(q)_{a-b}(q)_b} & \text{if} \quad 0 \leq b \leq a, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$p_j = 2m_{j+2} + 4m_{j+2} + \cdots + 2(k-j+1)m_k$$

so that

$$p_0 = \text{ number of half peaks}$$

Conclusion

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- Can this be lifted to a CFT interpretation?