EIKONAL METHODS IN THE ADS/CFT CORRESPONDENCE

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GGI Workshop, May 2007

(hep-th 0611122, 0611123 and to appear)

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Introduction

- String theory in AdS_{d+1} has two dimensionless parameters

$G\ell^{1-d}$	quantum gravity effects
$\alpha'\ell^{-2}$	string effects

- Usual attention on $\alpha'\ell^{-2}$ corrections

- In **flat space** perturbative expansion in G for 4–point function under control in some kinematical regimes

    1. Eikonal (small angle $|t| \ll s$) scattering with spin–j exchange
    2. Graviton ($j = 2$) dominates

- Can we apply eikonal methods to physics in AdS ?

- Can we include string corrections ?

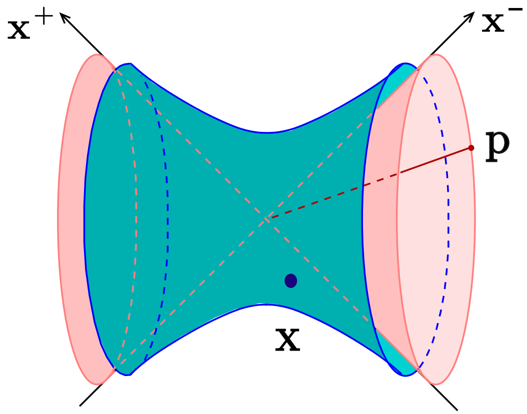
- What is the relation with the perturbative treatment of high energy scattering in YM theories ?

- Information about CFT_d 4–point functions for specific cross–ratios **to all orders in** N^{-1}

Notation and Preliminaries

- AdS$_{d+1}$ given by

$$\mathbf{x} = \left(x^+, x^-, x\right) \in \mathbb{M}^2 \times \mathbb{M}^d$$
$$\mathbf{x}^2 = -x^+x^- + x^2 = -\ell^2$$



- Boundary points given by rays on the light cone $(\lambda > 0)$

$$\mathbf{p}^2 = 0 \qquad\qquad \mathbf{p} \sim \lambda \mathbf{p}$$

- Scalar amplitudes

$$A\left(\mathbf{p}_i\right)$$

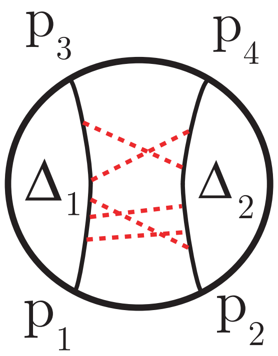
depend on invariants

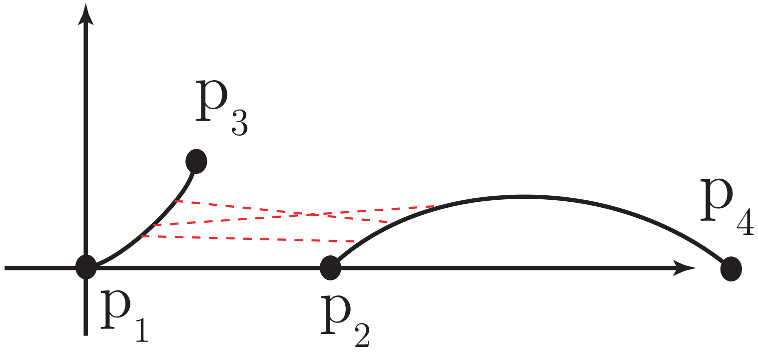
$$\mathbf{p}_{ij} = -2\mathbf{p}_i \cdot \mathbf{p}_j$$

and are homogeneous

$$A\left(\cdots, \lambda \mathbf{p}_i, \cdots\right) = \lambda^{-\Delta_i} A\left(\cdots, \mathbf{p}_i, \cdots\right)$$

with Δ_i the conformal dimension of the i–th scalar





- Four point function

$$A\left(\mathbf{p}_1, \cdots, \mathbf{p}_4\right) = \frac{1}{\mathbf{p}_{13}^{\Delta_1} \mathbf{p}_{24}^{\Delta_2}} A\left(z, \bar{z}\right)$$

with cross ratios defined by

$$\begin{aligned} z\bar{z} &= \frac{\mathbf{p}_{13}\mathbf{p}_{24}}{\mathbf{p}_{12}\mathbf{p}_{34}} \\ \left(1 - z\right)\left(1 - \bar{z}\right) &= \frac{\mathbf{p}_{14}\mathbf{p}_{23}}{\mathbf{p}_{12}\mathbf{p}_{34}} \end{aligned}$$

Impact Parameter Representation

- T and S–channels

$$z, \bar{z} \;\; \longrightarrow \;\; 0 \qquad\qquad \text{T–channel}$$

$$z, \bar{z} \;\; \longrightarrow \;\; \infty \qquad\qquad \text{S–channel}$$

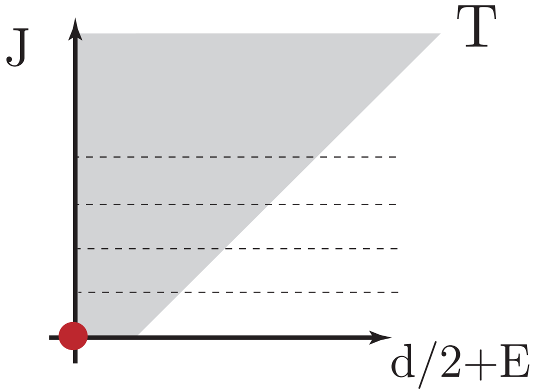
Two scattering angles $z, \bar{z}$ analogous to $-t/s$

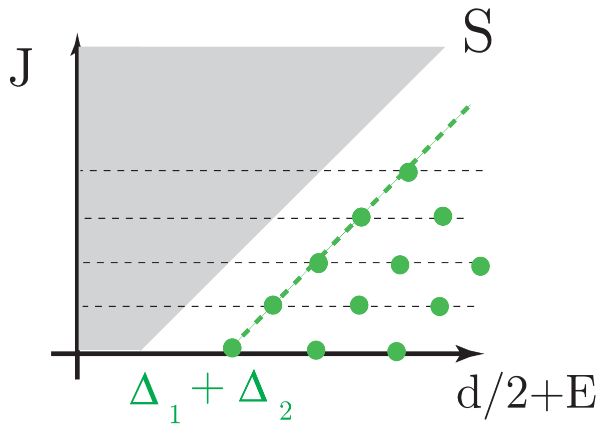
- Partial waves in T and S–channels for the exchange of a primary of dimension and spin

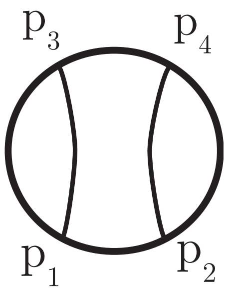
$$\frac{d}{2} + E \qquad\qquad\qquad J$$

- Contribution to $\mathcal{A}(z,\bar{z})$ in T and S–channels

$\mathcal{T}_{E,J}(z,\bar{z})$ $\mathcal{S}_{E,J}(z,\bar{z})$







- Free AdS propagation

$$\mathcal{A}_0 = 1 = \mathcal{T}_{-\frac{d}{2},0} = \sum_{E,J \text{ in } \Gamma} \mathcal{S}_{E,J}$$

with sum on lattice Γ

$$\frac{d}{2} + E = \Delta_1 + \Delta_2 + J + 2n$$

of double–trace composites $\mathcal{O}_1 \partial^{J+2n} \mathcal{O}_2$

- Limit $z, \bar{z} \longrightarrow 0$ of $\mathcal{T}_{E,J}$ given by OPE

$$\mathcal{T}_{E,J} \sim (z\bar{z})^{(d+2E)/4} \quad (\bar{z}/z)^{\pm J/2}$$

- For $\mathcal{S}_{E,J}$ must consider the limit

$$z, \bar{z} \longrightarrow 0 \qquad\qquad E, J \longrightarrow \infty$$

with

$$z, \bar{z} \sim E^{-2}, J^{-2}$$

- External points

$$\mathbf{p}_1 = (0, 1, 0) \qquad \mathbf{p}_2 = (-1, 1, p)$$
$$\mathbf{p}_3 = \left(r^2, -1, rq\right) \qquad \mathbf{p}_4 = (1, 0, 0)$$

with $q, p \in \mathsf{H}_{d-1} \subset \mathbb{M}^d$. The cross ratios are

$$z\bar{z} = r^2 \qquad \sqrt{\bar{z}/z} + \sqrt{z/\bar{z}} = 2p \cdot q$$

- Consider a sum

$$\sum_{E,J \text{ in } \Gamma} \sigma_{E,J} \, \mathcal{S}_{E,J}$$

Defining

$$\begin{aligned} h + \bar{h} &= \frac{d}{2} + E \\ h - \bar{h} &= J \end{aligned}$$

and rewriting

$$\begin{aligned} \sigma_{E,J} &\sim \int \frac{d\omega}{2\pi i} \, \sigma_\omega \left(x, y\right) (h\bar{h})^{\omega - 1} \\ -2x \cdot y &= \bar{h}/h + h/\bar{h} \end{aligned}$$

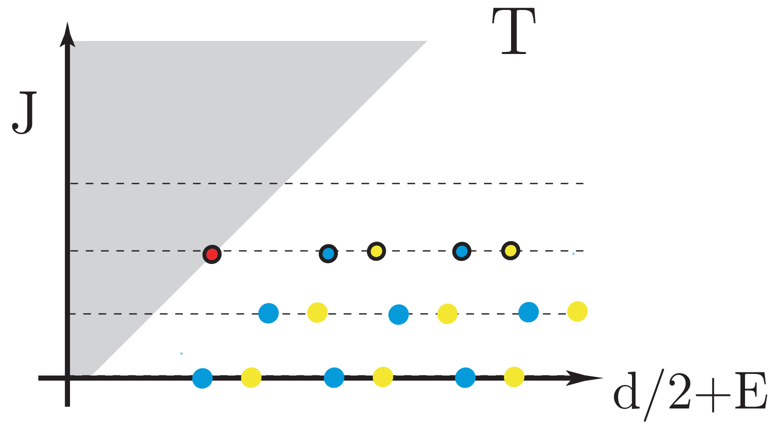
with $x, y \in \mathsf{H}_{d-1}$ we have the impact parameter representation

$$\int \frac{d\omega}{2\pi i} \, r^{1-\omega} \int_{\mathsf{H}_{d-1}} \widetilde{dx}\widetilde{dy} \, \frac{\sigma_\omega \left(x, y\right)}{(-2q \cdot x)^{2\Delta_1 + \omega - 1} (-2p \cdot y)^{2\Delta_2 + \omega - 1}}$$

Tree Level Exchanges

- Exchange of a spin j particle in the bulk. The T−channel decomposition contains only partial waves $\mathcal{T}_{E,J}$ with $J \leq j$

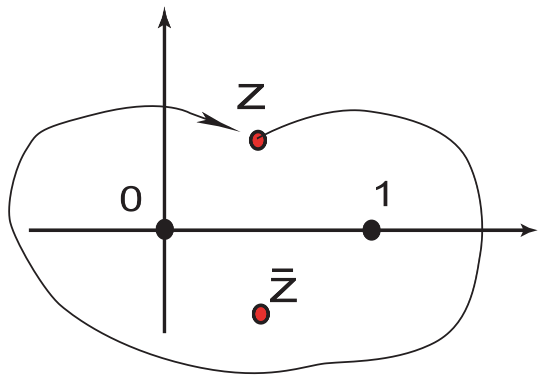
$$\mathcal{A}_1 = \sum_{J \leq j} \mu_{E,J} \; \mathcal{T}_{E,J}$$



- In the S–channel it corresponds to moving the free lattice

$$\mathcal{A}_1 = \sum_{E,J \text{ in } \Gamma} \rho_{E,J}\, \mathcal{S}_{E,J} + 2 \sum_{E,J \text{ in } \Gamma} \sigma_{E,J}\, \frac{\partial}{\partial E} \mathcal{S}_{E,J}$$

- Consider the Lorentzian amplitude $\widehat{\mathcal{A}}$ obtained by rotating z around $0, 1, \bar{z}$ clockwise



- In the limit $z, \bar{z} \to 0$ we have

$$\widehat{\mathcal{T}}_{E,J} \sim r^{1-J} \; \Pi_E \, (p, q)$$

with

$$\Pi_E \text{ propagator in transverse space } H_{d-1}$$

Inversion of roles

$$\frac{d}{2} + E \leftrightarrow 1 - J$$

The discontinuity dominates the $z, \bar{z} \to 0$ limit for positive energy partial waves

- In the S–channel we have

$$\widehat{\mathcal{S}}_{E,J} = \mathcal{S}_{E,J} \qquad\qquad \frac{\partial}{\partial E}\widehat{\mathcal{S}}_{E,J} = \frac{\partial}{\partial E}\mathcal{S}_{E,J} + 2\pi i \; \mathcal{S}_{E,J}$$

- Dominant contribution for $\widehat{\mathcal{A}}_1$ is

$$\widehat{\mathcal{A}}_1 \sim 4\pi i \sum_{E,J} \sigma_{E,J}\ \mathcal{S}_{E,J}$$

given by an impact parameter representation with

$$\sigma_\omega(x,y) \sim \frac{1}{\omega - j}\ \sigma(x,y)$$

$$\widehat{\mathcal{A}}_1 \sim r^{1-j} \int_{\mathsf{H}_{d-1}} \widetilde{dx}\widetilde{dy}\ \frac{\sigma(x,y)}{(-2q\cdot x)^{2\Delta_1+j-1}(-2p\cdot y)^{2\Delta_2+j-1}}$$

- The decomposition of $\widehat{\mathcal{A}}_1$ in terms of $r^{1-j}\ \Pi_E$ determines the coefficients $\mu_{E,J}$ for $J=j$

A Sample Interaction

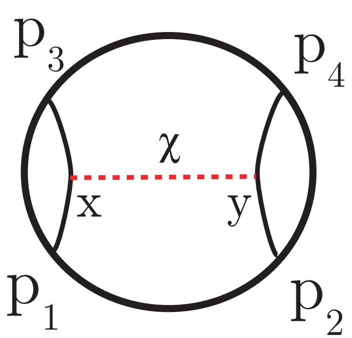
- Denote with $\mathbf{\Pi}_\chi$ the propagator in AdS$_{d+1}$

- Bulk to boundary propagator

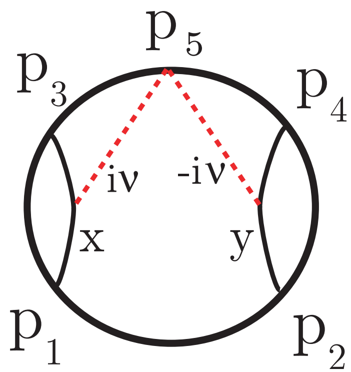
$$\frac{1}{(-2\mathbf{x}\cdot\mathbf{p})^{\Delta}}$$

- Basic scalar interaction

$$\int_{\mathrm{AdS}_{d+1}} \widetilde{d\mathbf{x}}\widetilde{d\mathbf{y}}\ \frac{1}{(-2\mathbf{x}\cdot\mathbf{p}_1)^{\Delta_1}(-2\mathbf{x}\cdot\mathbf{p}_3)^{\Delta_1}}\ \mathbf{\Pi}_\chi(\mathbf{x},\mathbf{y})\ \cdot$$
$$\cdot\frac{1}{(-2\mathbf{y}\cdot\mathbf{p}_2)^{\Delta_2}(-2\mathbf{y}\cdot\mathbf{p}_4)^{\Delta_2}}$$



$$= \int \frac{d\nu}{\nu^2+\chi^2}\int_{\partial \mathrm{AdS}_{d+1}} \widetilde{d\mathbf{p}_5}$$



- Representation of the scalar propagator in the form

$$\mathbf{\Pi}_\chi = \int d\nu \, \frac{\mathbf{\Omega}_{i\nu}}{\nu^2+\chi^2}$$

with

$$\mathbf{\Omega}_{i\nu} = -\frac{\nu}{2\pi i}\left(\mathbf{\Pi}_{i\nu} - \mathbf{\Pi}_{-i\nu}\right)$$

regular eigenfunctions of the Laplacian. Gives a complete radial Fourier transform with dimensions

$$\frac{d}{2} + i\nu$$

- Representation of $\mathbf{\Omega}_{i\nu}$ as

$$\mathbf{\Omega}_{i\nu} \sim \int_{\partial \mathsf{AdS}_{d+1}} \widetilde{d\mathbf{p}_5} \; \frac{1}{(-2\mathbf{x}\cdot\mathbf{p}_5)^{\frac{d}{2}+i\nu}} \frac{1}{(-2\mathbf{y}\cdot\mathbf{p}_5)^{\frac{d}{2}-i\nu}}$$

- Three point coupling

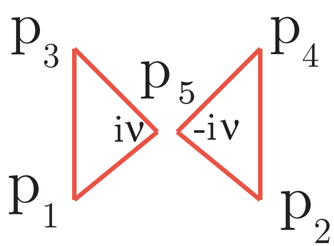
$$\int_{\mathsf{AdS}_{d+1}} \frac{\widetilde{d\mathbf{x}}}{(-2\mathbf{x}\cdot\mathbf{p}_1)^{\Delta_1}\,(-2\mathbf{x}\cdot\mathbf{p}_3)^{\Delta_1}\,(-2\mathbf{x}\cdot\mathbf{p}_5)^{\frac{d}{2}+i\nu}}$$

determined by conformal invariance

$$\frac{1}{\mathbf{p}_{13}^{\Delta_1}} \left(\frac{\mathbf{p}_{13}}{\mathbf{p}_{15}\mathbf{p}_{35}} \right)^{\frac{d+2i\nu}{4}} a_1(\nu)$$

- Finally integrate over $\int_{\partial \mathrm{AdS}_{d+1}} \widetilde{d\mathbf{p}_5}$, using the representation of conformal partial waves of spin 0

$$\int_{\partial \mathrm{AdS}_{d+1}} \widetilde{d\mathbf{p}_5}$$



$=$

$$= \int_{\partial \mathrm{H}_{d+1}} \widetilde{d\mathbf{p}_5} \left(\frac{\mathbf{p}_{13}}{\mathbf{p}_{15}\mathbf{p}_{35}}\right)^{\frac{d+2i\nu}{4}} \left(\frac{\mathbf{p}_{24}}{\mathbf{p}_{25}\mathbf{p}_{45}}\right)^{\frac{d-2i\nu}{4}} \sim \mathcal{T}_{i\nu,0} + \mathcal{T}_{-i\nu,0}$$

- We are left with a single integral

$$\int d\nu \; \frac{1}{\nu^2 + \chi^2} \left[a_1(\nu)\, a_2(-\nu) + a_1(-\nu)\, a_2(\nu)\right] \; \mathcal{T}_{i\nu,0}$$

Close the ν–contour in the lower half complex plane $\mathrm{Im}\, \nu \leq 0$

- In general, for spin j exchange, get partial waves $\mathcal{T}_{E,J}$ with

$$J \leq j$$

Eikonal Limit

- Basic replacement

$$\mathcal{T}_{i\nu,\ j} \rightarrow r^{1-j}\ \Pi_{i\nu}(p,q)$$

and get (for $j = 0$)

$$\widehat{\mathcal{A}}_1 \sim r \int d\nu\ \frac{1}{\nu^2 + \chi^2} a_1(\nu)\, a_2(-\nu)\, \Omega_{i\nu}(p,q)$$

Transverse radial Fourier transform of the impact parameter representation with $\sigma(x,y) \sim \Pi_\chi(x,y)$

- Holds for spin j minimal interaction and we get

$$\sigma_\omega(x,y) \sim \frac{1}{\omega - j}\ \Pi_\chi(x,y)$$

- Phase shift $\sigma_{E,J}$ exponentiates for the full $\hat{\mathcal{A}} = \hat{\mathcal{A}}_0 + \hat{\mathcal{A}}_1 + \cdots$

$$\hat{\mathcal{A}} \sim \int_{\mathsf{M}} \frac{dx}{|x|^{d-2\Delta_1}} \frac{dy}{|y|^{d-2\Delta_2}}\ \ e^{2q\cdot x + 2p\cdot y}\ e^{-2\pi i \sigma_{E,J}}$$

with

$$\begin{aligned} x^2 y^2 &= r^2\ h\bar{h} \\ -2x\cdot y &= r\ (h^2 + \bar{h}^2) \end{aligned}$$

- The phase shift $\sigma_{E,J}$ for large E, J is determined by the tree–level interaction

String Corrections at Strong Coupling λ

- For pure graviton exchange $\chi = \frac{d}{2}$ and

$$\sigma_\omega(x,y) \sim \int \frac{d\nu}{\omega - 2} \, \frac{1}{\nu^2 + (d/2)^2} \, \Omega_{i\nu}(x,y)$$

- String corrections

$$\frac{1}{\sqrt{\lambda}} = \frac{\alpha'}{\ell^2}$$

- To use flat space results for small strings define

$$-\ell^2 \, t(\nu) \simeq \left(\frac{d}{2}\right)^2 + \nu^2 + \cdots$$

$$j_0(\nu) \simeq 2 + \frac{\alpha'}{2} t(\nu) \simeq 2 - \frac{d^2}{8\sqrt{\lambda}} - \frac{1}{2\sqrt{\lambda}} \nu^2 - \cdots$$

and

$$F\left(\alpha' t\right) \simeq \frac{\Gamma\left(-\alpha' t/4\right)}{\Gamma\left(1 + \alpha' t/4\right)} e^{-i\pi\alpha' t/4}$$

- Include string corrections from leading Regge trajectory of the graviton

$$\sigma_\omega(x, y) \sim \int \frac{d\nu}{\omega - j_0(\nu)} F\left(\alpha' t(\nu)\right) \; \Omega_{i\nu}(x, y)$$

Comments:

- Intercept at vanishing ν at

$$2 - \frac{d^2}{8\sqrt{\lambda}} - \cdots$$

- For large ν we expect corrections to the t–ν relation. Long rotating strings have a linear energy–spin relation

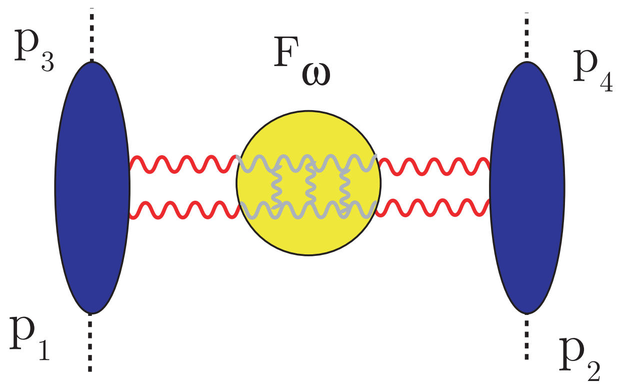
$$t\,(\nu) \sim \nu$$

Weak Coupling and the BFKL equation

- From now on $d = 4$
- In non–abelian YM theories, pomeron exchange dominates high energy scattering. It is a trajectory with the quantum numbers of the vacuum, like the graviton exchange. We look at conformal theories (or at scattering well above the confinement scale)
- At energy s we have the following amplitude

$$\int \frac{d\omega}{2\pi i} s^{\omega} \int d^2x_1 \cdots d^2x_4 \; a_1(x_1, x_3) \, F_\omega(x_1, \cdots, x_4) \, a_2(x_2, x_4)$$

with a_1, a_2 impact factors dependent on the interaction vertex and on the momentum transfer



- Pomeron propagator F_ω is conformal in $2d$ transverse space

$$F_\omega \sim \sum_{n\geq 0} \int \frac{d\nu}{\omega - j_n(\nu)} \frac{1}{\nu^2 + (n-1)^2} \mathcal{T}^{\perp}_{i\nu,n}$$

with $\mathcal{T}^{\perp}_{i\nu,n}$ the transverse $2d$ partial waves and

$$j_n(\nu) = 1 - \frac{\lambda}{2\pi^2} \mathrm{Re}\left(\Psi\left(\frac{1+i\nu+n}{2}\right) - \Psi(1)\right) + \cdots$$

- $\lambda \to 0$ corresponds to two gluons in a color singlet state of spin 1

$$\frac{1}{\omega - 1} \sum_{n \geq 0} \int \frac{d\nu}{\nu^2 + (n-1)^2} \mathcal{T}^{\perp}_{i\nu,n}$$

- The $n = 0$ term corresponds to graviton exchange, with

$$j_0(\nu) \sim 1 + \frac{\ln 2}{\pi^2}\lambda - \frac{7\zeta(3)}{8\pi^2}\nu^2 + \cdots$$

Intercept raising from 1 to 2 from weak to strong coupling

- At $\lambda \to 0$ we have

$$\frac{1}{\omega - 1} \int \frac{d\nu}{\nu^2 + 1} \mathcal{T}^{\perp}_{i\nu,0}$$

with a pole at $i\nu = 1$. In AdS_5 the dimension is

$$\frac{d}{2} + i\nu = 3$$

Massless spin 1 effective interaction. Corresponds to the HS gauge multiplet interaction

Conclusions

- We have a consistent picture of high energy interactions in AdS and in the dual CFT.

More to be done

1. Express impact factors in CFT language so as to match the transverse BFKL propagator at weak λ to the large λ string propagator

2. Relate the AdS eikonal results to the literature on multi–pomeron exchange at ultra–high energies

3. Understand pomeron exchange directly in AdS. This requires non–local interactions of higher spin fields. Technology ready to tackle the simplest HS theories of Vasiliev