Integrability in AdS/CFT: open problems

D. Serban,
IPhT Saclay

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Summary

- the AdS/CFT corespondence
- arguments for integrability
- conjectured Bethe Ansatz equations
- spin chain vs. sigma model features
- connection with the Hubbard model
- TBA and finite size effects
- integrability and the amplitudes

AdS/CFT correspondence

$\mathcal{N}$=4 gauge theory: superconformal symmetry PSL(2,2|4)

conformal group SO(4,2)≅SU(2,2)

R-group SO(6)≅SU(4)

Field content
SU(N) matrices: $A_\mu, \Phi_I \ (I = 1, ..., 6), \ \Psi_\alpha(\alpha = 1, ..., 4)$ and derivatives

$\updownarrow$

[Maldacena 97]
[Witten 98]
[Gubser, Klebanov, Polyakov 98]

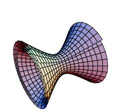
Type IIB string theory on AdS_5 x S^5: sigma model on PSL(2,2|4)/SO(4,1)xSO(5)

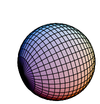
[Metsaev, Tseytlin 98]

AdS/CFT correspondence

‘t Hooft coupling $g^2 = \frac{g_{\rm YM}^2 N}{16\,\pi^2}$	String tension $T = 2g$
Number of colors N planar limit strong coupling	String coupling $g_{\rm s} = \frac{g}{N}$ free strings classical strings
Local operators $\mathrm{Tr}\,(\Phi_{I_1}\Phi_{I_2}\ldots\Phi_{I_L})$ Scaling dimension $\Delta(g)$ *R*-charges	String states Energy of the string E Angular momenta J_a

E(g), S₁, S₂, J₁, J₂, J₃





[Lipatov, 98]

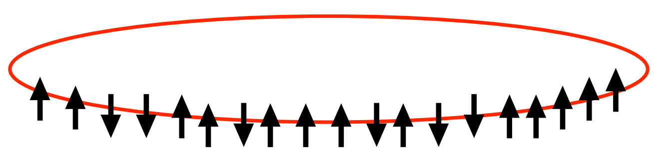
Integrability

One loop dilatation operator = integrable spin chain

[Minahan, Zarembo, 02]

$$Z = \Phi_1 + i\Phi_2$$
$$W = \Phi_3 + i\Phi_4$$

$$\mathrm{tr}\, ZZZWWZZZWWWZWZZZZ\ldots$$



$$\hat{D}_1 = 2\sum_{l=1}^{L}(1 - P_{l,l+1})$$

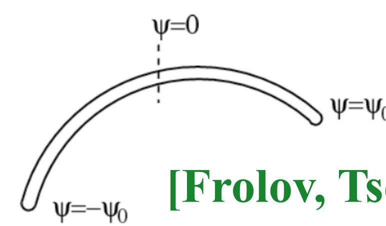
solution in terms of Bethe Ansatz equations

String sigma model is classically integrable

[Bena, Polchinski, Roiban, 02]

$$I = -\frac{\sqrt{\lambda}}{4\pi}\int d\tau d\sigma \left[G^{(AdS_5)}_{mn}\,\partial_a X^m\,\partial^a X^n + G^{(S^5)}_{mn}\,\partial_a Y^m\,\partial^a Y^n\right]$$
$$+ \text{ fermions}$$

string solution, *e.g.*



[Frolov, Tseytlin, 02]

[Kazakov, Marshakov, Minahan, Zarembo, 04]

solution of the classical sigma model in terms of an algebraic curve

Integrability

extends to the whole PSL(2,2|4) group

[Beisert, Staudacher 03]

survives at higher loops

[Beisert, Kristjansen, Staudacher 03]
[Beisert 03-04]

[conjecture]

There exists a model which is integrable for any value of the coupling constant g

?

spin chain at $g \to 0$

perturbative N=4 SYM

sigma model at $g \to \infty$

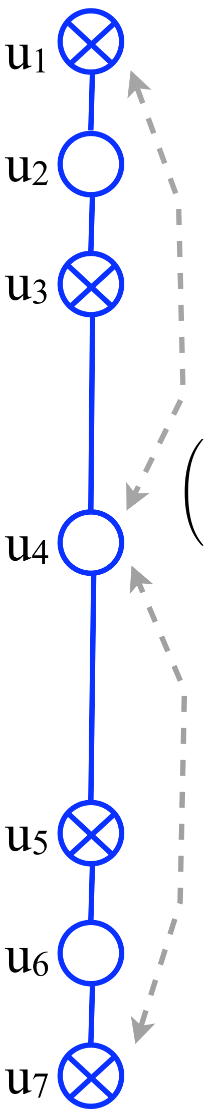
perturbative string theory on AdS_5 x S^5

The all-loop Bethe Ansatz equations

psu(2,2|4)

[Beisert, Staudacher, 05]
[Beisert, 05]

[Arutynov, Frolov, Zamaklar, 06]



$$1 = \prod_{j=1}^{K_2} \frac{u_{1,k}-u_{2,j}+\frac{i}{2}}{u_{1,k}-u_{2,j}-\frac{i}{2}} \prod_{j=1}^{K_4} \frac{1-1/x_{1,k}x_{4,j}^+}{1-1/x_{1,k}x_{4,j}^-},$$

$$1 = \prod_{j\neq k}^{K_2} \frac{u_{2,k}-u_{2,j}-i}{u_{2,k}-u_{2,j}+i} \prod_{j=1}^{K_3} \frac{u_{2,k}-u_{3,j}+\frac{i}{2}}{u_{2,k}-u_{3,j}-\frac{i}{2}} \prod_{j=1}^{K_1} \frac{u_{2,k}-u_{1,j}+\frac{i}{2}}{u_{2,k}-u_{1,j}-\frac{i}{2}},$$

$$1 = \prod_{j=1}^{K_2} \frac{u_{3,k}-u_{2,j}+\frac{i}{2}}{u_{3,k}-u_{2,j}-\frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{3,k}-x_{4,j}^+}{x_{3,k}-x_{4,j}^-},$$

$$\left(\frac{x_{4,k}^+}{x_{4,k}^-}\right)^L = \prod_{j\neq k}^{K_4} \frac{u_{4,k}-u_{4,j}+i}{u_{4,k}-u_{4,j}-i}\, \sigma^2(x_{4,k},x_{4,j})$$

$$\times \prod_{j=1}^{K_1} \frac{1-1/x_{4,k}^-x_{1,j}}{1-1/x_{4,k}^+x_{1,j}} \prod_{j=1}^{K_3} \frac{x_{4,k}^- - x_{3,j}}{x_{4,k}^+ - x_{3,j}} \prod_{j=1}^{K_5} \frac{x_{4,k}^- - x_{5,j}}{x_{4,k}^+ - x_{5,j}} \prod_{j=1}^{K_7} \frac{1-1/x_{4,k}^-x_{7,j}}{1-1/x_{4,k}^+x_{7,j}},$$

Dressing factor

[Janik'06;
Beisert-Hernandez-Lopez'06;
Beisert-Eden-Staudacher'06]

$$1 = \prod_{j=1}^{K_6} \frac{u_{5,k}-u_{6,j}+\frac{i}{2}}{u_{5,k}-u_{6,j}-\frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{5,k}-x_{4,j}^+}{x_{5,k}-x_{4,j}^-},$$

$$1 = \prod_{j\neq k}^{K_6} \frac{u_{6,k}-u_{6,j}-i}{u_{6,k}-u_{6,j}+i} \prod_{j=1}^{K_5} \frac{u_{6,k}-u_{5,j}+\frac{i}{2}}{u_{6,k}-u_{5,j}-\frac{i}{2}} \prod_{j=1}^{K_7} \frac{u_{6,k}-u_{7,j}+\frac{i}{2}}{u_{6,k}-u_{7,j}-\frac{i}{2}},$$

$$1 = \prod_{j=1}^{K_6} \frac{u_{7,k}-u_{6,j}+\frac{i}{2}}{u_{7,k}-u_{6,j}-\frac{i}{2}} \prod_{j=1}^{K_4} \frac{1-1/x_{7,k}x_{4,j}^+}{1-1/x_{7,k}x_{4,j}^-}.$$

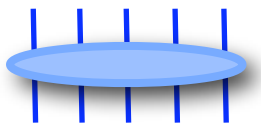
$$\boxed{x+\frac{1}{x}=\frac{u}{g}}$$

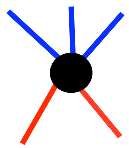
$$x^\pm + \frac{1}{x^\pm} = \frac{1}{g}\left(u \pm \frac{i}{2}\right)$$

magnon symmetry: centrally extended [su(2|2)]^2

Particular features of the N=4 SYM integrable model

spin chain

- one-parametrer integrable super-spin chain
- long-range interaction (~ Inozemtsev or Hubbard at half filling) g^{2n} 

- length-changing interaction g^3 
- BAE only asymptotic (L $\rightarrow \infty$)

sigma model

- crossing-like symmetry (particle/antiparticle) $\rightarrow$ **dressing phase**

$$E(p) = \pm\sqrt{1+16g^2\sin^2 p/2} - 1$$

- not (exactly) relativistically invariant
- scattering matrix does not depend on rapidity difference

Connection(s) with the Hubbard model

2 seemingly unrelated connections with the 1d Hubbard model

- su(2) sector reproducible from the Hubbard model at half filling (except for the dressing phase) **[Rej, Serban, Staudacher, 06]**

$$e^{ip_k L} = \prod_{\substack{j=1 \\ j\neq k}}^{M} \frac{u_k - u_j + i}{u_k - u_j - i}\,, \qquad u(p) = \frac{1}{2}\cot\frac{p}{2}\sqrt{1+16g^2\sin^2\frac{p}{2}}\,,$$

[Beisert, Dippel, Staudacher, 04]

$$E(p) = \sqrt{1+16g^2\sin^2\frac{p}{2}} - 1$$

- Beisert's su(2|2) symmetric **S-matrix** $\sim$ Hubbard Shastry's **R-matrix**

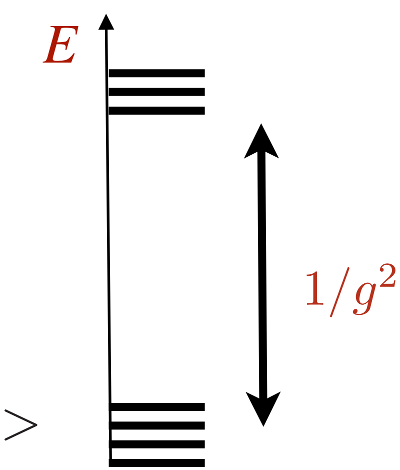
$\Rightarrow$ hidden supersymmetry in the Hubbard model **[Beisert, 06]**

The BDS model from Hubbard at half-filling

- itinerant fermions with onsite repulsion (***U=1/g***):

$$H = \frac{1}{2\,g}\sum_{i=1}^{L}\sum_{\sigma=\uparrow,\downarrow}\left(e^{i\phi_\sigma}\;c^\dagger_{i,\sigma}c_{i+1,\sigma} + e^{-i\phi_\sigma}\;c^\dagger_{i+1,\sigma}c_{i,\sigma}\right) - \frac{1}{2g^2}\sum_{i=1}^{L}c^\dagger_{i,\uparrow}c_{i,\uparrow}c^\dagger_{i,\downarrow}c_{i,\downarrow} \qquad \phi = \frac{\pi(L+1)}{2L}$$

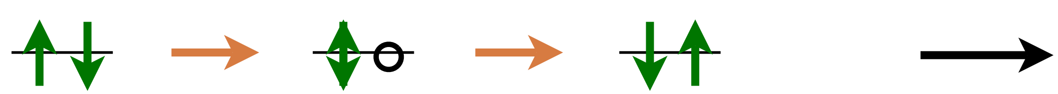
- Hilbert space: 4 states per site: $\circ \quad \uparrow \quad \downarrow \quad \downarrow\uparrow$



- at half filling $N=L$ and $g \to 0$ the spin states decouple:

singly-occupied states $|\uparrow\downarrow\uparrow\uparrow \dots \uparrow>$

- fluctuations $\sim$ g^2



XXX model

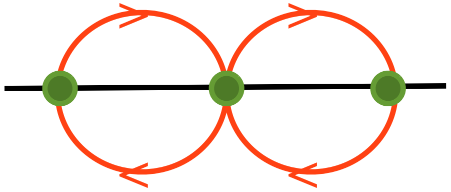
spin permutation

The BDS model from Hubbard at half-filling

higher orders: itinerant fermion making 2n step walks on the lattice

$$
\begin{aligned}
h_2 &= (1 - P_{i,i+1})\,, \\
h_4 &= -2(1 - P_{i,i+1}) + \frac{1}{2}(1 - P_{i,i+2})\,, \\
h_6 &= \frac{15}{2}(1 - P_{i,i+1}) - 3(1 - P_{i,i+2}) + \frac{1}{2}(1 - P_{i,i+3}) \\
&\quad -\frac{1}{2}(1 - P_{i,i+3})(1 - P_{i+1,i+2}) \\
&\quad +\frac{1}{2}(1 - P_{i,i+2})(1 - P_{i+1,i+3})\,.
\end{aligned}
$$

g^4



- coincides with the dilatation operator up to three loops
- corrects a result from **[Klein, Seitz, 73]**

The BDS model from Hubbard at half-filling

Lieb-Wu equations (half filling): [Lieb, Wu, 68]

$$e^{i\widetilde{q}_n L} = \prod_{j=1}^{M} \frac{u_j - 2g\sin(\widetilde{q}_n + \phi) - i/2}{u_j - 2g\sin(\widetilde{q}_n + \phi) + i/2}\,, \qquad n = 1, \ldots, L$$

$$\prod_{n=1}^{L} \frac{u_k - 2g\sin(\widetilde{q}_n + \phi) + i/2}{u_k - 2g\sin(\widetilde{q}_n + \phi) - i/2} = \prod_{\substack{j=1 \\ j\neq k}}^{M} \frac{u_k - u_j + i}{u_k - u_j - i}\,, \quad k = 1, \ldots, M$$

L large, M small

Shiba (particle/hole) transformation:

$$H(g;\phi,\phi) \to -H(-g;\pi-\phi,\phi) - \frac{M}{2g^2}$$

$$\uparrow \;\leftrightarrow\; \circ$$

$$\downarrow \;\leftrightarrow\; \updownarrow$$

Dual Lieb-Wu equations:

$$e^{iq_n L} = \prod_{j=1}^{M} \frac{u_j - 2g\sin(q_n - \phi) - i/2}{u_j - 2g\sin(q_n - \phi) + i/2}\,, \qquad n = 1, \ldots, 2M$$

$$\prod_{n=1}^{2M} \frac{u_k - 2g\sin(q_n - \phi) + i/2}{u_k - 2g\sin(q_n - \phi) - i/2} = -\prod_{\substack{j=1 \\ j\neq k}}^{M} \frac{u_k - u_j + i}{u_k - u_j - i}\,, \quad k = 1, \ldots, M$$

The BDS model from Hubbard at half-filling

Bound-state solutions (strings): [Takahashi, 72] $L \to \infty$

complex momenta: $$q_1 - \phi = \frac{\pi}{2} + \frac{p}{2} + i\,\beta\,, \qquad q_2 - \phi = \frac{\pi}{2} + \frac{p}{2} - i\,\beta$$

$$u \pm i/2 = 2g\ \cos\left(\frac{p}{2} \mp i\beta\right)$$

$2g\sin(q_2 - \phi)$

u

$2g\sin(q_1 - \phi)$

$$u(p) = \frac{1}{2}\cot\frac{p}{2}\,\sqrt{1 + 16g^2\sin^2\frac{p}{2}}$$

Lieb-Wu eq. $\rightarrow$ BDS eq.

$$e^{ip_k L} = \prod_{\substack{j=1\\ j\neq k}}^{M} \frac{u_k - u_j + i}{u_k - u_j - i}\,,$$

$$E(p) = \frac{1}{2g^2}\left(\sqrt{1 + 16g^2\sin^2\frac{p}{2}} - 1\right)$$

But the Hubbard construction:

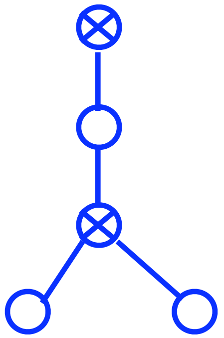
- does not extends to other sectors than su(2), *e.g.* su(1|1)

- does not take into account the dressing phase

  - is it possible to define an all-loop hamiltonian?

  - how to take into account the fine-size effects?

Remark: the su(2|2) symmetric S-matrix is also an essential ingredient of the new **AdS_4 x CP^3** duality **[Aharony, Bergman, Jafferis Maldacena, 08]** cf **[Gromov, Vieira 08]**



OSp(2,2|6)

TBA program

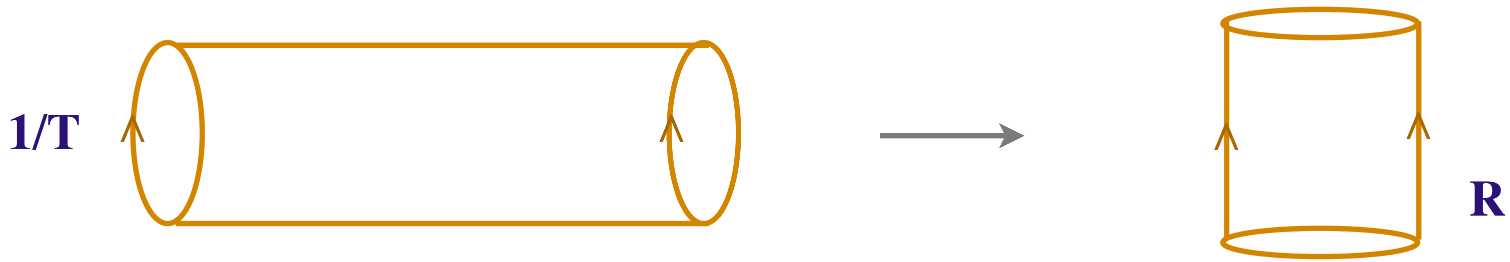
Use the field-theoretical methods to compute finite-size corrections:

[Ambjorn, Janik, Kristjansen 05]

- Lüscher terms **[Janik, Lukowski 07,...]**

- put the theory on the cylinder and make a "double Wick rotation" **$1/T \rightarrow R$**
  [Arutynov, Frolov 07; Bajnok, Janik,08]

- **difficulty**: the rotated theory is not equivalent to the original one ("mirror theory")



simplest wrapping correction: the four loop L=4 (Konishi operator)

[Fiamberti, Santambrogio, Sieg, Zanon ,08]: perturbative computation in N=4 SYM

=

[Bajnok, Janik,08]: from TBA

The origin of integrability?

There is more in N=4 SYM than the dilatation operator...

- the multigluon amplitudes have a particular structure at higher loops - >
  BDS conjecture **[Bern, Dixon, Smirnov 05]** (fails for n>5)

- this structure was checked at strong coupling for 4 (and many) gluons
  [Alday, Maldacena 07]

- dual superconformal symmetry **[Drummond, Henn, Korchemsky, Sokatchev, 07-08]**
  (and duality between multigluon amplitudes and the Wilson loops with lightlike cusps)

Is there any connection between this structure and the integrability?

[Berkovits, Maldacena, 08]
[Beisert, Ricci, Tseytlin, Wolf, 08]

Integrable open spin chain for gluon amplitudes [Lipatov, 08]

Conclusion

- the AdS/CFT correspondence provides an unusual integrable structure
- it puts together many known integrable models into a highly symmetric structure
- the complete definition still not under control (what is the hamiltonian?)
- what are the consequences of integrability on the overall structure of N=4 SYM?
- which are the other (integrable) dualities?