On $s\ell_3$ KZ equations and $\mathcal{W}_3$ null-vector equations

Introduction. Many interesting 2d CFTs are based on affine Lie algebras and their cosets. For example, from the $\widehat{s\ell_2}$ algebra one can build the $SL(2,\mathbb{R})$ WZW model which is related to string theory in AdS_3, the Euclidean version the H_3^+ model, strings in the 2d BH. The simplest nonrational theory of this family is Liouville theory.

Families of non-rational 2d CFTs

Families of non-rational 2d CFTs
$s\ell_2$ and $s\ell_3$ families
The theories, their sym. alg., target space dim.

Recently there emerged a more precise meaning to this notion of a family of CFTs: a formula for arbitrary correlation functions of the H_3^+ model (and some of the others) in terms of certain correlation functions in Liouville theory [SR+Teschner]. Intuitively, the reason is: affine $s\ell_2$ representations are labelled by just one parameter (the spin), so even if a theory like the $SL(2,\mathbb{R})$ WZW model has a 3d target space, its dynamics are effectively 1d, due to the large symmetry of the theory.

Here I want to investigate whether the same might be true in the $s\ell_3$ family. The $SL(3,\mathbb{R})$ WZW model has a 8d target space, but we expect effectively 2d dynamics, since the Cartan subgroup is 2d. The simplest nonrational theory of the $s\ell_3$ family is indeed a theory of 2 interacting bosons, called conformal $s\ell_3$ Toda theory. Is there a hope to write correlation functions of the $SL(3,\mathbb{R})$ WZW model in terms of correlation functions of $s\ell_3$ Toda theory?

I will explore this question with the help of the $s\ell_3$ KZ equations, which all correlation functions (of primary fields) in theories with an $\widehat{s\ell_3}$ symmetry must obey. In the $s\ell_2$ case, the KZ equations are equivalent to certain second-order BPZ differential equations of Liouville theory; in the $s\ell_3$ case we thus expect the KZ equation to be related to certain third-order null-vector equations of the $s\ell_3$ Toda theory.

Symmetries and diff. equations

Symmetries and diff. equations
KZ-BPZ
$s\ell_3$ KZ
Gaudin
Sklyanin SOV

More precisely, the KZ equations involve Gaudin Hamiltonians, and in the $s\ell_2$ case the KZ-BPZ relation is found by using Sklyanin's separated variables in KZ equations. Similarly, I will write $s\ell_3$ KZ equations in terms of separated variables.

Conjectures and results. Consider an m-point function of affine primary fields in a theory with $\widehat{s\ell_N}$ symmetry. The theory is parametrized by the level $k > N$. The fields are parametrized by their position z on the Riemann sphere, the spin j with $N-1$ components and isospins x with $\frac{N(N-1)}{2}$ components.

We want to relate this to a correlation function in $s\ell_N$ conformal Toda theory which involves m corresponding fields with momenta $\alpha(j_i)$, plus $d = \frac{N(N-1)}{2}(m-2)$ degenerate fields satisfying order N differential equations. For instance, in the $s\ell_2$ case, second-order BPZ equations.

Correlation functions
Ω_m
$s\ell_2$ isospins
$\tilde{\Omega}_m$
relations

Correlation functions

The relation between x_i and y_a will be given by Sklyanin's SOV for the $s\ell_N$ Gaudin model. This is an integral transformation, which does not depend on the level k. Its kernel S is not known explicitly beyond the $s\ell_2$ case. The conjectured relation also involves a simples twist function Θ_m, with parameters λ, μ, ν to be determined as functions of the level k.

Status of our conjecture: compatible with KZ in $s\ell_2$, and $s\ell_3$ in the limit $k \to 3$. Proved in specific models in H_3^+-Liouville case. Sorry to disprove my own conjecture!

The conjecture
Θ_m
Conjecture
Status of conjecture

The conjecture

KZ equations in Sklyanin variables. Let me now explain how to construct the variables y_a, and how to write the KZ equations in terms of such variables. We introduce the differential operators D^a which appear in the definition of the fields and in the Gaudin Hamiltonians. We then build the operator-valued Lax matrix $L(u)$ where u is the spectral parameter. It satisfies a "linear" commutation relation.

From the Lax matrix, we should build the objects which define the separation of variables: two functions $A(u), B(u)$ and a characteristic equation. Sklyanin variables y_i are defined as the zeroes of $B(u)$, their conjugate momenta p_i as $p_i = A(y_i)$. For any given i, p_i, y_i and invariants built from $L(y_i)$ are related by the characteristic equation.

SOV in the Gaudin model
$J^a\Phi^j$, D^a, H_i
$s\ell_2$ example for D^a
$L(u)$
3 objects
$[y_i, y_j]$ etc

SOV in the Gaudin model

Let me review what these objects are in the $s\ell_2$ case and how they help rewrite the KZ equations. $A(u)$ and $B(u)$ are simply matrix elements of $L(u)$. The characteristic equation involves the Gaudin Hamiltonians. It is a kinematic identity. But now apply it to $S^{-1} \cdot \Omega_m$ so that $p_i = \frac{\partial}{\partial y_i}$, and inject the KZ equations. (If we were interested in diagonalizing Gaudin Hamiltonians we would have an eigenvalue E_ℓ instead of $S^{-1}\frac{\delta}{\delta z_\ell}S$, hence the term "separation of variables".) Then compute $S^{-1}\frac{\delta}{\delta z_\ell}S$, doable in $s\ell_2$. Resulting equations are equivalent to BPZ, modulo twist with right values of λ, μ, ν.

$s\ell_2$ KZ in Sklyanin variables
$A(u)$, $B(u)$, characteristic equation
Apply to $S^{-1}\Omega_m$, inject KZ
Compute $S^{-1}\frac{\delta}{\delta z_\ell}S$

$s\ell_2$ KZ in Sklyanin variables

We want to follow similar steps in the $s\ell_3$ case. We first have to derive the SOV, which is apparently not present in the literature. This is done by taking a limit of the related $s\ell_3$ Yangian model, where the SOV was derived by Sklyanin. Now the characteristic equation involves not only quadratic but also a cubic invariant built from the Lax matrix. The quadratic invariants $L^\beta_\alpha L^\alpha_\beta$ can be rewritten in terms of the Gaudin Hamiltonians, like in the $s\ell_2$ case.

The cubic invariant can be rewritten in terms of higher Gaudin Hamiltonians. In the CFT with $\widehat{s\ell_3}$ symmetry it is interpreted as an insertion of a field W which is a cubic invariant of the currents J, similar to the Sugawara construction of the stress-energy tensor T. Fields Φ^j are labelled by their spins j or equivalently by the $s\ell_3$ invariants Δ_j, q_j (eigenvalues of zero-modes of T, W).

SOV for $s\ell_3$ Gaudin

When applied to a correlation function $S^{-1}\Omega_m$, some things work like in $s\ell_2$: we still have $p_i = \frac{\partial}{\partial y_i}$, we can still use KZ equations to replace Gaudin Hamiltonians with z-derivatives. But the cubic term now gives rise to an insertion of W. We obtain $3m-6$ equations whereas we are really interested only in the KZ equations, because they are differential equations. We can get rid of the $2m$ non-differential terms with W_{-1}, W_{-2}, by taking appropriate linear combinations of the $3m-6$ equations. An equivalent way to do this is to work modulo terms of that type, an equivalence which we will denote as $\sim$. (It can be defined rigorously.)

$s\ell_3$ KZ in Sklyanin variables

$\mathcal{W}_3$ null-vector equations. *Let me explain the choice of the $\mathcal{W}_3$ degenerate field V_{α_d} in $\tilde{\Omega}_m$, which should reproduce similar differential equations. In the $s\ell_2$ case we had two choices for degenerate fields leading to second-order BPZ equations, but only one had the correct b-scaling. In the $s\ell_3$ case we are looking for a fully degenerate field with 3 independent null vectors at levels 1, 2, 3. There are 2 such fields with the correct b-scaling. They are related by the $s\ell_3$ Dynkin diagram automorphism. We therefore have a freedom to choose either field. This choice should however correspond to a choice which we made in the SOV for the Gaudin model: we decided that the Lax matrix lived in the fundamental representation, rather than the antifundamental. In our conventions this will correspond to the degenerate field $V_{-b^{-1}\omega_1}$.*

Choice of $\mathcal{W}_3$ degenerate field V_{α_d}

Now the equations for $\Theta_m\tilde{\Omega}_m$ follow from the choice of V_{α_d} as fully degenerate fields with 3 independent null vectors at levels 1, 2, 3. We also choose specific values for the parameters λ, μ, ν of Θ_m.

SOV for $s\ell_3$ Gaudin
$A(u)$, $B(u)$
Characteristic equation
T, W fields
Cubic term

$s\ell_3$ KZ in Sklyanin variables
Full equation
Neglect W-terms

Choice of $\mathcal{W}_3$ degenerate field V_{α_d}
$s\ell_2$ case
2 candidate fields in $s\ell_3$
Cartan matrix, bases
Correct field, relation with fundamental in $L(u)$
Relation of Φ^j and V_α

NVE for $\Theta_m\tilde{\Omega}_m$
The equation
D_1
D_2
Values of λ, μ, ν

NVE for $\Theta_m\tilde{\Omega}_m$

Let us finally compare this with the KZ equations in Sklyanin variables. These should agree according to the conjecture $\Theta_m\tilde{\Omega}_m = S^{-1}\Omega_m$. Some terms agree, some cannot be evaluated, and the D_1 term disagrees. The only cure seems to send k to 3 (critical level limit).

Comparison

Comparison
Rewriting of the two equations
Conjecture for D_2
Limit

A family of solvable non-rational CFTs. What happens if we modify the Liouville side in the H_3^+-Liouville relation by replacing $V_{-\frac{1}{2b}}$ with $V_{-\frac{r}{2b}}$? We do not get an m-point function in H_3^+, is it an m-point function in some new CFT?

I propose a Lagrangian for the new CFT in terms of the same bosonic fields $\phi, \beta, \bar{\beta}, \gamma, \bar{\gamma}$ which appear in the H_3^+ model. Let us study the symmetry algebra associated to this Lagrangian. This will help

1. show that it describes a CFT,
2. check that this CFT is solvable, i.e. that all correlation functions can be deduced from the primary field correlation functions for which we have an ansatz,
3. check that the correlation functions obey differential equations in the cases when it should, for instance $r = 2$.

Family of non-rational CFTs

Family of non-rational CFTs
The ansatz
The lagrangian
Need for symmetry algebra

We find a symmetry algebra generated by fields T, J^3, J^-. The stress-energy tensor satisfies Virasoro so the theory is conformal. The field J^3 is not quite a primary of dimension one, due to a central term in the TJ^3 OPE. In the case $r = 2$ we find a subsingular vector, i.e. a field R which vanishes provided J^- vanishes too. This leads to third-order differential equations associated to the zeroes of J^-. But $J^- = 0 \Leftrightarrow L_1^2 = 0$ so the zeroes of J^- are the Sklyanin variables. Moreover, the third-order differential equations do agree with what the third-order BPZ equations for $\tilde{\Omega}_m^{(2)}$.

Therefore, the theory with the Lagrangian $L^{(2)}$ and the proposed symmetry algebra has correlation functions which satisfy the right differential equations. This is strong evidence that these correlation functions are given in terms of Liouville correlation functions by our ansatz.

Symmetry algebra

Symmetry algebra
T, J^3, J^-
TJ^3 OPE
Case $r = 2$: R
Agreement

On $s\ell_3$ KZ equations and $\mathcal{W}_3$ null-vector equations

Sylvain Ribault (Montpellier)

Families of non-rational 2d CFTs

The $s\ell_2$ family

Alg.	Dim.	Theory
$\widehat{s\ell_2}$	3d	$SL(2,\mathbb{R})$ WZW $\sim$ strings in AdS_3
$\widehat{s\ell_2}$	3d	H_3^+ model
$\widehat{s\ell_2}/\widehat{u_1}$	2d	Strings in 2d black hole
$Vir = \mathcal{W}_2$	1d	Liouville theory

H_3^+-Liouville relation: [SR+Teschner 2005]

The $s\ell_3$ family

Alg.	Dim.	Theory
$\widehat{s\ell_3}$	8d	$SL(3,\mathbb{R})$ WZW
$\cdots$	$2 \leq d \leq 8$	$\cdots$
$\mathcal{W}_3$	2d	Conformal $s\ell_3$ Toda theory

Symmetries and differential equations

m-point function in theory with $\widehat{s\ell_N}$ symmetry

$$\Omega_m = \left\langle \Phi^{j_1}(x_1|z_1) \cdots \Phi^{j_m}(x_m|z_m) \right\rangle \tag{1}$$

obeys Knizhnik–Zamolodchikov equations

$$\left[(k-N) \frac{\delta}{\delta z_i} + H_i \right] \Omega_m = 0 \tag{2}$$

$H_1 \cdots H_m =$ Gaudin Hamiltonians (differential wrt x_i)

Gaudin model has *Sklyanin's separation of variables*

$$\widehat{s\ell_2} \text{ KZ} \quad \overset{\text{Sklyanin variables}}{\longrightarrow} \quad Vir \text{ BPZ} \tag{3}$$

$$\widehat{s\ell_3} \text{ KZ} \quad \overset{\text{Sklyanin variables}}{\longrightarrow} \quad \mathcal{W}_3 \text{ equations ?} \tag{4}$$

Correlation functions

$$\Omega_m = \left\langle \prod_{i=1}^{m} \Phi^{j_i}(x_i|z_i) \right\rangle \tag{5}$$

- $k > N$: level of $\widehat{s\ell_N}$ with $c = \frac{k(N^2-1)}{k-N}$
- $j \in \mathbb{C}^{N-1}$: spin of $\widehat{s\ell_N}$ representation
- $x \in \mathbb{C}^{\frac{N(N-1)}{2}}$: isospin variables
- $z \in \mathbb{C}$: position on Riemann sphere

$$\tilde{\Omega}_m = \left\langle \prod_{i=1}^{m} V_{\alpha(j_i)}(z_i) \prod_{a=1}^{\frac{N(N-1)}{2}(m-2)} V_{\alpha_d}(y_a) \right\rangle \tag{6}$$

- $b = \frac{1}{\sqrt{k-N}}$: parameter of $\mathcal{W}_N$ with $c = (N-1)[1 + N(N+1)(b+b^{-1})^2]$
- $\alpha(j) \in \mathbb{C}^{N-1}$: momentum of $\mathcal{W}_N$ representation
- $y \in \mathbb{C}$: position on Riemann sphere (Sklyanin variable)
- V_{α_d}: degenerate field $\rightarrow$ order N equation

The relation

Sklyanin's separation of variables = integral transformation

$$\Psi(x_1 \cdots x_m) = S \cdot \tilde{\Psi}(y_1 \cdots y_p)$$
$$= \int \prod_a dy_a \; S(x_1 \cdots x_m | y_1 \cdots y_p) \tilde{\Psi}(y_1 \cdots y_p) \quad (7)$$

S depends on $j_1 \cdots j_m$ and $z_1 \cdots z_m$ but not on k

$$\Theta_m = \prod_{a<b} (y_a - y_b)^\lambda \prod_{i,a} (y_a - z_i)^\mu \prod_{i<j} (z_i - z_j)^\nu \quad (8)$$

Conjectured relation: $\boxed{\Omega_m = S \cdot \Theta_m \tilde{\Omega}_m}$

Status of relation	$s\ell_2$ case	$s\ell_3$ case
KZ-Compatible	Yes [Stoyanovsky]	Only if $k = 3$
Proved in a model	Yes [SR+Teschner]	

Separation of Variables in the Gaudin model

$J^a(z)\Phi^j(x|w) = \frac{D^a\Phi^j(x|w)}{z-w} + \text{reg.}$ and $H_i = \sum_{\ell\neq i} \frac{D^a_{(i)}D^a_{(\ell)}}{z_i - z_\ell}$

where $[D^a_{(i)}, D^b_{(j)}] = \delta_{ij} f^{ab}_c D^c_{(i)}$ and $D^a_{(i)}D^a_{(i)} = C_2(j_i)$

($s\ell_2$ example $D^- = \frac{\partial}{\partial x},\; D^3 = x\frac{\partial}{\partial x} - j,\; D^+ = x^2\frac{\partial}{\partial x} - 2jx$)

Lax matrix $\boxed{L(u) = \sum_{i=1}^{m} \frac{t^a D^a_{(i)}}{u - z_i}}$ $\begin{cases} u = \text{spectral parameter} \\ L(u) \in Mat_{N\times N} \end{cases}$

$$[L^\gamma_\alpha(u), L^\epsilon_\beta(v)] = \frac{\delta^\epsilon_\alpha L^\gamma_\beta(u) - \delta^\gamma_\beta L^\epsilon_\alpha(u) - \delta^\epsilon_\alpha L^\gamma_\beta(v) + \delta^\gamma_\beta L^\epsilon_\alpha(v)}{u - v}$$

To be built from $L(u)$:

- A function $B(u)$ and its zeroes y_i (Sklyanin variables)
- A function $A(u)$ and $p_i = A(y_i)$ (momenta)
- A characteristic equation relating p_i, y_i and $L(y_i)$

such that $\quad [y_i, y_j] = 0 \quad , \quad [p_i, y_j] = \delta_{ij} \quad , \quad [p_i, p_j] = 0$

$s\ell_2$ KZ equations in Sklyanin variables

$$L(u) = \begin{pmatrix} L_1^1(u) & L_1^2(u) \\ L_2^1(u) & L_2^2(u) \end{pmatrix} \qquad \begin{cases} A(u) = L_1^1(u) \\ B(u) = L_1^2(u) \end{cases}$$

$$p_i^2 - \frac{1}{2}(L_\alpha^\beta L_\beta^\alpha)(y_i) = 0 \tag{9}$$

$$\Leftrightarrow \qquad p_i^2 - \sum_\ell \frac{1}{y_i - z_\ell}\left(H_\ell + \frac{1}{2}\frac{C_2(j_\ell)}{y_i - z_\ell}\right) = 0 \tag{10}$$

[Sklyanin] Then apply to $S^{-1}\Omega_m$ and inject KZ equations:

$$\left[\frac{\partial^2}{\partial y^2} + \sum_{\ell=1}^m \frac{k-2}{y-z_\ell}\left(S^{-1}\frac{\delta}{\delta z_\ell}S + \frac{\Delta_{j_\ell}}{y-z_\ell}\right)\right] S^{-1}\Omega_m = 0 \tag{11}$$

$$\Leftrightarrow \quad \left[\frac{1}{k-2}\frac{\partial^2}{\partial y^2} + \sum_{\ell=1}^m \frac{1}{y-z_\ell}\left(\frac{\partial}{\partial z_\ell} + \frac{\partial}{\partial y}\right) + \right.$$

$$\left. \sum_b \frac{1}{y-y_b}\left(\frac{\partial}{\partial y_b} - \frac{\partial}{\partial y}\right) + \sum_{\ell=1}^m \frac{\Delta_{j_\ell}}{(y-z_\ell)^2}\right] S^{-1}\Omega_m = 0 \tag{12}$$

$\Leftrightarrow$ BPZ for $\Theta_m\tilde{\Omega}_m$ if $\lambda = -\mu = \nu = \frac{1}{2b^2}$ and $V_{\alpha_d} = V_{-\frac{1}{2b}}$
[Stoyanovsky]

Separation of variables for the $s\ell_3$ Gaudin model

$$\begin{cases} A(u) = -L_1^1 + \frac{L_3^1 L_1^2}{L_3^2} \\ B(u) = L_2^1 L_3^2 L_3^2 - L_3^2 L_3^1 L_2^2 + L_3^1 L_3^2 L_1^1 - L_1^2 L_3^1 L_3^1 \end{cases}$$

Characteristic equation:

$$\begin{aligned} & p_i^3 - p_i \cdot \frac{1}{2}(L_\alpha^\beta L_\beta^\alpha)(y_i) \\ & + \frac{1}{4}(L_\alpha^\beta L_\beta^\alpha)'(y_i) + \frac{1}{6}\left(L_\alpha^\beta L_\beta^\gamma L_\gamma^\alpha + L_\beta^\alpha L_\gamma^\beta L_\alpha^\gamma\right)(y_i) = 0 \end{aligned} \quad (13)$$

Similarly to $T = -\frac{(J^a J^a)}{2(k-3)}$ let $W = \frac{\rho}{6} d_{abc}(J^a(J^b J^c))$

with $d_{abc} = \mathrm{Tr}\ (t^a t^b t^c + t^a t^c t^b)$ and $\rho = \frac{i}{(k-3)^{\frac{3}{2}}}$

Spin $j \Leftrightarrow (\Delta_j, q_j)$ with $W_0 \Phi^j(x|z) = q_j \Phi^j(x|z)$

$$\left[W(u) - \frac{\rho}{6}\left(L_\alpha^\beta L_\beta^\gamma L_\gamma^\alpha + L_\beta^\alpha L_\gamma^\beta L_\alpha^\gamma\right)(u)\right]\Omega_m = 0 \quad (14)$$

$s\ell_3$ KZ equations in Sklyanin variables

$3m-6$ equations for $\Omega_m = \left\langle \Phi^{j_1}(x_1|z_1)\cdots\Phi^{j_m}(x_m|z_m)\right\rangle$

$$\Bigg[\frac{\partial^3}{\partial y^3} + \frac{\partial}{\partial y}\cdot\sum_{i=1}^m \frac{k-3}{y-z_i}\left(S^{-1}\frac{\delta}{\delta z_i}S + \frac{\Delta_{j_i}}{y-z_i}\right)$$
$$+\sum_{i=1}^m \frac{k-3}{2(y-z_i)^2}\left(S^{-1}\frac{\delta}{\delta z_i}S + \frac{2\Delta_{j_i}}{y-z_i}\right)$$
$$-\frac{1}{\rho}\sum_{i=1}^m\left(\frac{S^{-1}W^{(i)}_{-2}S}{y-z_i} + \frac{S^{-1}W^{(i)}_{-1}S}{(y-z_i)^2} + \frac{q_{j_i}}{(y-z_i)^3}\right)\Bigg]$$
$$\cdot S^{-1}\Omega_m = 0 \quad (15)$$

$2m$ non-differential terms $\rightarrow$ $m-6$ differential equations:

$$\Bigg[\frac{\partial^3}{\partial y^3} + \frac{\partial}{\partial y}\cdot\sum_{i=1}^m \frac{k-3}{y-z_i}S^{-1}\frac{\delta}{\delta z_i}S + \sum_{i=1}^m \frac{(k-3)\Delta_{j_i}}{(y-z_i)^2}\frac{\partial}{\partial y}$$
$$-\sum_{i=1}^m \frac{\frac{1}{\rho}q_{j_i} + (k-3)\Delta_{j_i}}{(y-z_i)^3}\Bigg] S^{-1}\Omega_m \sim 0 \quad (16)$$

Choice of the $\mathcal{W}_3$ degenerate field V_{α_d}

$\underline{s\ell_2 \text{ case:}}$ 2nd order equation $\rightarrow$ level 2 null vector

$\rightarrow$ fields $V_{-\frac{1}{2}b^{-1}}, V_{-\frac{1}{2}b}$ $\rightarrow$ actually $V_{-\frac{1}{2}b^{-1}}$

$\underline{s\ell_3 \text{ case:}}$ 3rd order equation $\rightarrow$ levels 1, 2, 3 null vectors

$\rightarrow$ fields $V_{-b^{-1}\omega_1}, V_{-b^{-1}\omega_2}$ related by automorphism

Root space basis e_1, e_2 such that $\left(\begin{smallmatrix} (e_1,e_1) & (e_1,e_2) \\ (e_2,e_1) & (e_2,e_2) \end{smallmatrix}\right) = \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$

Dual basis ω_1, ω_2 such that $(e_i, \omega_j) = \delta_{ij}$

Choice $\boxed{V_{\alpha_d} = V_{-b^{-1}\omega_1}}$ $\Leftrightarrow$ Choice of fundamental in $L(u)$

Let $Q = (b + b^{-1})(e_1 + e_2)$ and $\Delta_\alpha = \frac{1}{2}(\alpha, 2Q - \alpha)$

and q_α = cubic

Relation $\Phi^j \leftrightarrow V_{\alpha(j)}$ with $\boxed{\alpha(j) = -bj + b^{-1}(e_1 + e_2)}$

$$\Rightarrow \begin{cases} \Delta_\alpha = \Delta_j + 2 + b^{-2} \\ q_\alpha = q_j \end{cases}$$

Null-vector equations for $\Theta_m\tilde{\Omega}_m$

$$\tilde{\Omega}_m = \left\langle \prod_{i=1}^m V_{\alpha(j_i)}(z_i) \prod_{a=1}^{3m-6} V_{\alpha_d}(y_a) \right\rangle$$

$$\Theta_m = \prod_{a<b}(y_a - y_b)^{-\frac{2}{3b^2}} \prod_{i,a}(y_a - z_i)^{\frac{1}{b^2}} \prod_{i<j}(z_i - z_j)^{-\frac{2}{b^2}}$$

$$\left[\frac{\partial^3}{\partial y^3} + \frac{1}{b^2}D_2 + \frac{1}{b^4}D_1 + \frac{1}{b^2}\sum_{i=1}^m \frac{\Delta_{j_i}}{(y-z_i)^2}\frac{\partial}{\partial y} + \sum_{i=1}^m \frac{\frac{i}{b^3}q_{j_i} - \frac{1}{b^2}\Delta_{j_i}}{(y-z_i)^3}\right]\Theta_m\tilde{\Omega}_m \sim 0 \quad (17)$$

$$D_1 = -\sum_i \frac{1}{(y-z_i)^2}\frac{\partial}{\partial y} + 3\sum_i \frac{1}{y-z_i}\sum_b \frac{1}{y-y_b}\left(\frac{\partial}{\partial y_b} - \frac{\partial}{\partial y}\right) + 2\left(\sum_i \frac{1}{y-z_i}\right)^2 \frac{\partial}{\partial y} - 2\sum_{b\neq c}\frac{1}{y-y_b}\frac{1}{y_b-y_c}\left(\frac{\partial}{\partial y_b} - \frac{\partial}{\partial y}\right) \quad (18)$$

$$D_2 = \sum_b \frac{1}{(y-y_b)^2}\frac{\partial}{\partial y} + \sum_i \frac{1}{y-z_i}\frac{\partial}{\partial y}\left(\frac{\partial}{\partial z_i} + 3\frac{\partial}{\partial y}\right) + \sum_b \frac{1}{y-y_b}\left(\frac{\partial}{\partial y_b} - \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial y_b} + 2\frac{\partial}{\partial y}\right) \quad (19)$$

Comparison

$$\left[\frac{\partial^3}{\partial y^3}+\frac{\partial}{\partial y}\cdot\sum_{i=1}^{m}\frac{k-3}{y-z_i}S^{-1}\frac{\delta}{\delta z_i}S+\sum_{i=1}^{m}\frac{(k-3)\Delta_{j_i}}{(y-z_i)^2}\frac{\partial}{\partial y}\right.$$
$$\left.-\sum_{i=1}^{m}\frac{\frac{1}{\rho}q_{j_i}+(k-3)\Delta_{j_i}}{(y-z_i)^3}\right]S^{-1}\Omega_m\sim 0 \quad (20)$$

$$\left[\frac{\partial^3}{\partial y^3}+\frac{1}{b^2}D_2+\frac{1}{b^4}D_1+\frac{1}{b^2}\sum_{i=1}^{m}\frac{\Delta_{j_i}}{(y-z_i)^2}\frac{\partial}{\partial y}\right.$$
$$\left.+\sum_{i=1}^{m}\frac{\frac{i}{b^3}q_{j_i}-\frac{1}{b^2}\Delta_{j_i}}{(y-z_i)^3}\right]\Theta_m\tilde{\Omega}_m\sim 0 \quad (21)$$

$\frac{1}{b^2}=k-3$

$D_2 \overset{?}{\sim} \frac{\partial}{\partial y}\cdot\sum_{i=1}^{m}\frac{1}{y-z_i}S^{-1}\frac{\delta}{\delta z_i}S$

Problem: D_1 term.

Only solution: $k\to 3 \Leftrightarrow b\to\infty$? (critical level)

A family of solvable non-rational CFTs

[SR, 2008]

H_3^+-Liouville: $S \cdot \Theta_m \tilde{\Omega}_m = \Omega_m$

$$\tilde{\Omega}_m^{(r)} \equiv \left\langle \prod_{i=1}^{m} V_{\alpha_i}(z_i) \prod_{a=1}^{m-2} V_{-\frac{r}{2b}}(y_a) \right\rangle \qquad (r \neq 1) \tag{22}$$

$$\boxed{S \cdot \Theta_m^{(r)} \tilde{\Omega}_m^{(r)} = \,?} \tag{23}$$

where S is the $s\ell_2$ Gaudin separation of variables

Answer: Lagrangian $L^{(r)} = \partial\phi\bar{\partial}\phi + \beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma} + (-\beta\bar{\beta})^r e^{2b\phi}$

with ϕ, β, γ bosons, $\Delta(\phi) = \Delta(\beta) = 0$, $\Delta(\gamma) = 1$

$$\text{Symmetry algebra} \rightarrow \begin{cases} \text{Differential equation if } r = 2? \\ \text{Are primaries enough?} \end{cases}$$

Symmetry algebra

From the Lagrangian $L^{(r)}$:

$$\begin{aligned} T &= -\beta\partial\gamma - (\partial\phi)^2 + (b + b^{-1}(1-r))\partial^2\phi && (24)\\ J^3 &= -\beta\gamma - rb^{-1}\partial\phi && (25)\\ J^- &= \beta && (26) \end{aligned}$$

$$T(z)J^3(w) = \frac{(1-r)(1-rb^{-2})}{(z-w)^3} + \frac{J^3(w)}{(z-w)^2} + \frac{\partial J^3(w)}{z-w} \quad (27)$$

<u>Case $r=2$:</u> subsingular vector $J^-(y)=0 \Rightarrow R(y)=0$ with

$$\begin{aligned} R =\ & \tfrac{3}{2}b^2(\partial J^-(J^3\partial J^3)) + \tfrac{1}{2}[b^2+1-2b^{-2}](\partial J^-\partial^2 J^3)\\ &+ \tfrac{1}{2}b^2(\partial J^-(J^3(J^3J^3))) + 2(\partial J^-(J^3T)) + [2b^{-2}+1](\partial J^-\partial T)\\ &- \tfrac{1}{2}(\partial^2 J^-(J^3J^3)) + [-1+b^{-2}](\partial^2 J^-\partial J^3) - 2b^{-2}(\partial^2 J^- T) \end{aligned} \quad (28)$$

$\rightarrow$ agrees with 3rd order BPZ equation for $V_{-\frac{1}{b}}(y)$

($J^-(y)=0 \Leftrightarrow L_1^2(y)=0$ from separation of variables)