Teleportation, Majorana zero modes and long distance entanglement

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Idea and Question

Is it possible to construct a quantum state characterized by:

- a wave function peaked at two different spatially separated locations;
- such that "interacting" with system at 0 something happens at L?

If exist then one should have entanglement and a sort of teleportation. We shall see that:

- it is possible only when in a system emerge Majorana fermions interacting with pertinent (soliton-antisoliton) background

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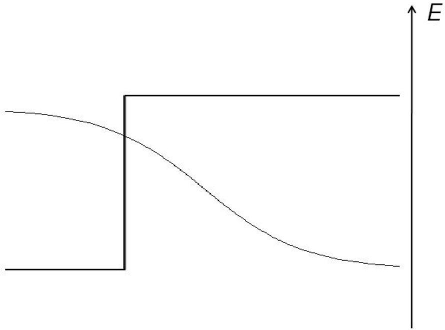
Degeneracy, tunneling . . .

Majorana fermions induce exotic entanglement in quantum states
Let us imagine to induce teleportation by quantum tunneling . . .



tail of wavefunction too small to be of any practical use

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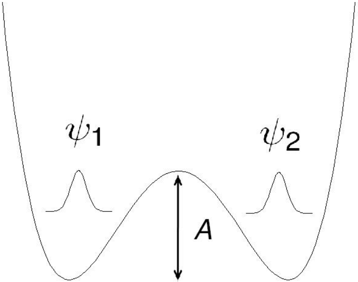

localization of minima well separated and large barrier $\rightarrow$ semiclassical method

scenario in which the wavefunction has well separated peaks (spatially separated) with eventually a forbidden region in between

Degeneracy, tunneling . . .

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$$\psi_s = \psi_1 + \psi_2$$

May I use ψ_s for exotic entanglement? i.e.: may I interact with the system in the vicinity of point 1 and see something of the other hand?

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NO WAY!

Degeneracy, tunneling ...

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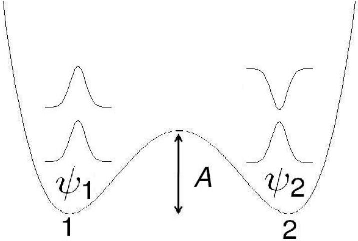

On top of $\psi_S = \psi_1 + \psi_2$ there is $\psi_A = \psi_1 - \psi_2$ and the two states on split by $\Delta E = f(A)$ and $\Delta E \to 0$ as $A \to a$.

$\Longrightarrow$ thus the two states are almost degenerate.

Degeneracy forbids our dream of teleportation?

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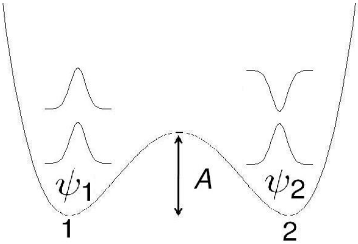

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Since

$$\frac{1}{\sqrt{2}}(\psi_S + \psi_A) = \sqrt{2}\psi_1(x)$$

i.e.: when I interact with the system near 1 I see ψ_S and ψ_A and therefore I am populating a state which is linear combination of the 2; for instance ψ_1.

Furthermore:

$$\frac{1}{\sqrt{2}}(\psi_S - \psi_A) = \sqrt{2}\psi_2(x)$$

ψ_1 and ψ_2 are not stationary states

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they mix . . . very slowly too!

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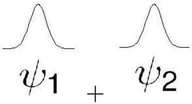

which is confined away from other states in the spectrum

Schr*ö*dinger equation? $\rightarrow$ NO!

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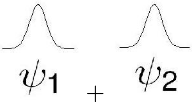

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Dirac like equation

Simple one dimensional model

$$[i\gamma^{\mu}\partial_{\mu} + \phi(x)]\,\psi(x,t) = 0$$

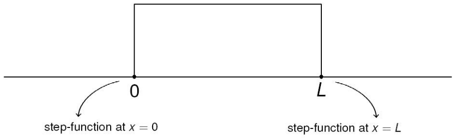
$$\{\gamma^{\mu},\gamma^{\nu}\} = 2g^{\mu\nu} \quad , \quad g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

<u>soliton-antisoliton</u>

$\theta = 0$ as the vacuum

Dirac like equation

soliton-antisoliton



$$\phi(x) \begin{cases} \phi_0 & x < 0 \ \text{ or } \ x > L \\ -\phi_0 & 0 < x < L \end{cases}$$

If $\psi(x,t) = \psi_E(x) e^{-iEt}$

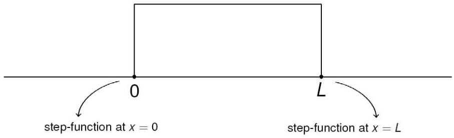
$$i \begin{pmatrix} 0 & \frac{d}{dx} + \phi(x) \\ \frac{d}{dx} - \phi(x) & 0 \end{pmatrix} \begin{pmatrix} u_E(x) \\ v_E(x) \end{pmatrix} = E \begin{pmatrix} u_E(x) \\ v_E(x) \end{pmatrix} \tag{1}$$

$\psi_{-E}(x) = \psi_E^*(x)$ particle-hole symmetry of Dirac equation

Equation (1) has exactly two bound states.

Dirac like equation

<u>soliton-antisoliton</u>



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- These states have energies well separated from the rest of the spectrum (continuum starts at $E = \phi_0$)
- The energies are exponentially close to 0 as $L \to \infty$
- Each wavefunction has two peaks: one at $x = 0$ and one at $x = L$
- Second quantized Dirac:

$$\Psi(x,t) = \psi_+(x)e^{-iE_+t}\mathbf{b} + \psi_+^*(x)e^{iE_+t}\mathbf{b}^\dagger$$

since $\psi_-(x) = \psi_+^*(x)$ and $E_- = -E_+$

- When L is large one can consider quasi-stationary states:

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with support near $x = 0$

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$$\psi_L(x) = \frac{1}{\sqrt{2}i}\left(e^{iE_0 t}\psi_+ - e^{-iE_0 t}\psi_-\right) =$$
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$$\Psi(x,t) = \psi_0(x,t)\frac{1}{\sqrt{2}}\left(\mathbf{a}+\mathbf{b}^\dagger\right) + \psi_L(x,t)\frac{1}{\sqrt{2}i}\left(-\mathbf{a}+\mathbf{b}^\dagger\right) + \dots$$

$$\alpha = \tfrac{1}{\sqrt{2}}\left(\mathbf{a}+\mathbf{b}^\dagger\right) \qquad \alpha^\dagger = \tfrac{1}{\sqrt{2}}\left(\mathbf{a}^\dagger+\mathbf{b}\right)$$
$$\beta = \tfrac{1}{\sqrt{2}i}\left(\mathbf{a}^\dagger-\mathbf{b}^\dagger\right) \qquad \beta^\dagger = \tfrac{1}{\sqrt{2}i}\left(-\mathbf{a}+\mathbf{b}^\dagger\right)$$

By interacting with the system at $x = 0$ we populate ψ_0.

As $L \to \infty$ the state is an entangled state of fractional fermion number (Dirac fermions).

Measurement of charge will collapse the state either in ψ_0 or ψ_L since $\frac{1}{2}$ is a sharp eigenvalue!

Majorana fermions

The situation is different with Majorana fermions!

- ψ_+ and ψ_- correspond to the same eigenstate.
- This state is either occupied or unoccupied.
- Convention
$$(-1)^F = -1 \qquad \text{unoccupied state}$$
$$(-1)^F = 1 \qquad \text{occupied state}$$
- ψ_0 and ψ_L are then superpositions of occupied and unoccupied states and than violate fermion parity!
- Thus a consistent (no strange things happening with 2π-rotations) theory of Majorana fermions interacting with soliton-antisoliton background does not admit ψ_0 and ψ_L as acceptable states.
- If we have something at $x = 0$ "automatically" we find something at $x = L$ since wavefunction have peaks at $x = 0$ and $x = L$.

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$$\begin{aligned} (-1)^F &= -1 && \text{unoccupied state} \\ (-1)^F &= 1 && \text{occupied state} \end{aligned}$$

- ψ_0 and ψ_L are then superpositions of occupied and unoccupied states and than violate fermion parity!
- Thus a consistent (no strange things happening with 2π-rotations) theory of Majorana fermions interacting with soliton-antisoliton background does not admit ψ_0 and ψ_L as acceptable states.
- If we have something at $x = 0$ "automatically" we find something at $x = L$ since wavefunction have peaks at $x = 0$ and $x = L$.

"Exotic entanglement"

Majorana fermions

The situation is different with Majorana fermions!

- ψ_+ and ψ_- correspond to the same eigenstate.
- This state is either occupied or unoccupied.
- Convention

$$\begin{array}{ll} (-1)^F = -1 & \text{unoccupied state} \\ (-1)^F = 1 & \text{occupied state} \end{array}$$

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"Exotic entanglement"

Conventional second quantization of complex fermion

Let's assume that - in some approximation - it makes sense to have single non interacting particle whose wavefunction satisfy Schrödinger equation:

$$i\hbar\frac{\partial\Psi}{\partial t} = H_0\Psi(x,t)$$

with H_0 usually a matrix and $\Psi = \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}$ a column vector.

For instance, in $3+1$ dimensions:

$$H_0 = i\vec{\alpha}\vec{\nabla} + \beta m$$

with $\vec{\alpha}$ and β set of 4 hermitian matrices (anticommuting).

Conventional second quantization of complex fermion

Furthermore

$$\exists \Gamma : \gamma \vec{\alpha} \Gamma = \vec{\alpha}^* \ , \ \Gamma \beta \Gamma = -\beta^*$$

so that

$$\Gamma H_0 \Gamma = -H_0^* \text{ and } \Gamma \Psi_E^* = \Psi_{-E}$$

$\Rightarrow \Gamma$ induces a $1-1$ mapping from $\Psi_E \rightarrow \Psi_{-E}$.

If $\vec{\alpha} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix}$, $\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ than $\Gamma = \begin{pmatrix} 0 & -i\sigma^2 \\ i\sigma^2 & \end{pmatrix}$. Note $\Gamma = \Gamma^*$ and $\Gamma^2 = 1$.

$$\underline{\text{Majorana fermion: } \Psi(x,t) = \Gamma \Psi^*(x,t)}$$

Conventional second quantization of complex fermion

$\Psi(x,t)$ satisfies $\{\Psi(x,t),\Psi^\dagger(y,t)\} = \delta(x-y)$
Writing the second quantized Hamiltonian

$$H = \int dx \ \ :\Psi^\dagger(x,t) H_0 \Psi(x,t):$$

one can derive the wave equation (1) from

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = [\Psi(x,t), H_0]$$

H_0 is “single particle” Hamiltonian which is hermitian with a real spectrum

$$H_0 \Psi_E = E \Psi_E$$

$$\int dx \ \Psi_E^\dagger(x) \Psi_{E'}(x) = \delta_{EE'} \ , \ \sum_E \Psi_E(x) \Psi_E^\dagger(x) = \delta(x-y)$$

Conventional second quantization of complex fermion

$$\int dx\ \Psi_E^\dagger(x)\Psi_{E'}(x) = \delta_{EE'}\ ,\ \sum_E \Psi_E(x)\Psi_E^\dagger(x) = \delta(x-y) \tag{2}$$

Second quantized:

$$\Psi(x,t) = \sum_{E>0} \Psi_E(x)e^{-iEt/\hbar}\mathbf{a}_E + \sum_{E<0} \Psi_E(x)e^{-iEt/\hbar}\mathbf{b}^\dagger_{-E}$$

$\mathbf{a}_E$: annihilation operator for particle with energy E
$\mathbf{b}^\dagger_{-E}$: creation operator for hole with energy $-E$

$$\left\{\mathbf{a}_E, \mathbf{a}^\dagger_{E'}\right\} = \delta_{EE'}\ ,\ \left\{\mathbf{b}_{-E}, \mathbf{b}^\dagger_{-E'}\right\} = \delta_{EE'} \tag{3}$$

$$(2) + (3) \Rightarrow \left\{\Psi(x,t), \Psi^\dagger(y,t)\right\} = \delta(x-y).$$

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Conventional second quantization of complex fermion

Ground state $|0\rangle$ is the state where all the positive energy levels are empty and all negative energies are filled (all hole states are empty)

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$$\mathbf{a}_E|0\rangle = \mathbf{b}_{-E}|0\rangle = 0$$

Excited states: excitations created by $\mathbf{a}^\dagger_E$ are particle while those created by $\mathbf{b}^\dagger_{-E}$ are the "antiparticle" or holes.

$$\mathbf{a}^\dagger_{E_1} \ldots \mathbf{a}^\dagger_{E_m} \mathbf{b}^\dagger_{E_1} \ldots \mathbf{b}^\dagger_{E_n} |0\rangle \quad \text{generic n-particle state.}$$

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Majorana fermion

Particle and holes have identical spectra

$$\Downarrow$$

Particle and hole with the same energy are single (i.e. the same) excitation

$$\Downarrow$$

Fields operator given

$$\Phi(x,t) = \sum_{E>0} \left(\psi_E(x) e^{-iEt/\hbar} \mathbf{a}_E + \Gamma \psi_E^*(x) e^{iEt/\hbar} \mathbf{a}_E^\dagger \right)$$

Ground state: $\mathbf{a}_E|0\rangle = 0$
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Fields operator is “pseudo-real”:

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Practically:

- If $\Psi(x,t)$ is complex fermion and the Hamiltonian allows for particle-hole symmetry, the complex fermion can be decomposed in 2 Majorana fermions

$$\Phi_1(x,t) = \frac{1}{\sqrt{2}}\left(\Psi(x,t) + \Gamma\Psi^*(x,t)\right)$$

$$\Phi_2(x,t) = \frac{1}{\sqrt{2}i}\left(\Psi(x,t) - \Gamma\Psi^*(x,t)\right)$$

- Where are Majorana fermions in Nature?
  1. Not in Q.E.D.! Since interaction of fermions with photon is not diagonal in the separation between real and imaginary parts Φ_1 and Φ_2

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     If you split Ψ it remixes!
  2. In superconductors? The electromagnetic field is screened

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     Bogoliubov quasi-electron behave like neutral particles. But beware of spin!

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Majorana fermions in superconductors

In s-wave superconductor quasi electron operator is

$$\begin{pmatrix} \Psi_\uparrow(x) \\ \Psi^*_\downarrow(x) \end{pmatrix}$$

spin up-down

$$\Gamma\Psi^* \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \Psi_\uparrow(x) \\ \Psi^*_\downarrow(x) \end{pmatrix}^* = \begin{pmatrix} \Psi_\downarrow(x) \\ \Psi^*_\uparrow(x) \end{pmatrix}$$

is not Majorana condition since it entails conjugation and spin-flip!

We need to consider a superconductor where the condensate has Cooper pair with the same spin so that quasi-electron is

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Thus candidates are p-wave superconductors such as Strontium Ruthenate.

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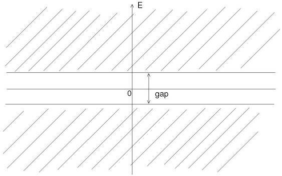
Zero energy states



Second quantization such that:

$$H_0\Psi_0 = 0 \quad \Psi_0 \equiv \text{zero mode}$$

for complex fermion:

$$\Psi(x,t) = \psi_0(x)\alpha + \sum_{E>0} \psi_E e^{-iWt/\hbar}\mathbf{a}_E + \sum_{E<0} \psi_E e^{-iWt/\hbar}\mathbf{b}_{-E}$$

$$\left\{\alpha, \alpha^\dagger\right\} = 1 \ \ , \ \ \{\alpha, \mathbf{a}_E\} = \{\alpha, \mathbf{b}_{-E}\} = \cdots = 0$$

Existence of a zero mode leads to degeneracy of fermion spectrum

$$\mathbf{a}_E|0\rangle = \mathbf{b}_{-E}|0\rangle = 0$$

but now it must carry a representation of algebra $\left\{\alpha, \alpha^\dagger\right\} = 1$

Zero energy states

The minimal representation is 2-dimensional ($|\uparrow\rangle, |\downarrow\rangle$)

$$\mathbf{a}_E|\uparrow\rangle = \mathbf{a}_E|\downarrow\rangle = 0 = \mathbf{b}_E|\uparrow\rangle = \mathbf{b}_E|\downarrow\rangle$$

$$\begin{aligned} \alpha^\dagger|\downarrow\rangle &= |\uparrow\rangle & \alpha^\dagger|\uparrow\rangle &= 0 \\ \alpha|\downarrow\rangle &= 0 & \alpha|\uparrow\rangle &= |\downarrow\rangle \end{aligned}$$

(Jackiw-Rebbi)

Two tower of excited states

$$\mathbf{a}^\dagger_{E_1}\ldots\mathbf{a}^\dagger_{E_m}\mathbf{b}^\dagger_{E_1}\ldots\mathbf{b}^\dagger_{E_m}|\uparrow\rangle$$

$$\mathbf{a}^\dagger_{E_1}\ldots\mathbf{a}^\dagger_{E_m}\mathbf{b}^\dagger_{E_1}\ldots\mathbf{b}^\dagger_{E_m}|\downarrow\rangle$$

having the same energy $\sum_i E_i$

For complex fermion one has states with fractional fermion number ($\theta = eN \Rightarrow$ fractional charge).

In fact: $Q = \int dx \frac{1}{2}\left[\Psi^*(x,t), \Psi(x,t)\right] = \sum_{E>0}\left(\mathbf{a}^\dagger_E\mathbf{a}_E - \mathbf{b}^\dagger_{-E}\mathbf{b}_{-E}\right) + \alpha^\dagger\alpha - \frac{1}{2}$

has fractional eigenvalues

$$Q|\uparrow\rangle = \frac{1}{2}|\uparrow\rangle \qquad Q|\downarrow\rangle = -\frac{1}{2}|\downarrow\rangle$$

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Majorana fermion with a zero mode

No fractional charge since field is real!

$$\Phi(x,t) = \psi_0(x)\alpha + \sum_{E>0} \psi_E(x) e^{-iEt/\hbar} \mathbf{a}_E + \sum_{E<0} \psi_E(x) e^{-iEt/\hbar} \mathbf{a}^\dagger_{-E}$$

Zero mode operator real $\Leftrightarrow \alpha = \alpha^\dagger$

$$\left\{ \mathbf{a}_E, \mathbf{a}^\dagger_{E'} \right\} = \delta_{EE'}$$

$$\alpha^2 = \frac{1}{2} \ ; \ \{\alpha, \mathbf{a}_{E'}\} = \left\{ \alpha, \mathbf{a}^\dagger_E \right\} = 0$$

A minimal representation can be constructed by: $\mathbf{a}|0\rangle = 0 \quad \forall E > 0$
with $\alpha = \frac{1}{\sqrt{2}}(-1)^{\sum_{E>0} \mathbf{a}^\dagger_E \mathbf{a}_E}$ (Klein operator) $\Rightarrow \alpha = \alpha^\dagger$ and, since $\sum_{E>0} \mathbf{a}^\dagger_E \mathbf{a}_E |0\rangle = 0$

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A basis for Hilbert space: $|0\rangle, \mathbf{a}^\dagger_{E_1} \dots \mathbf{a}^\dagger_{E_k}|0\rangle \dots$ and all are eigenvalue of $\sum_{E>0} \mathbf{a}^\dagger_E \mathbf{a}_E$ with integer eigenvalue. In this basis $\alpha^2 = \frac{1}{2}$ and thus $\{\alpha, \alpha^\dagger\} = 1$ is satisfied!

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Another inequivalent representation can be obtained starting with

$$\tilde{\alpha} = -\frac{1}{\sqrt{2}}(-1)^{\sum_{E>0} \mathbf{a}_E^\dagger \mathbf{a}_E}$$

$$\Downarrow$$

two minimal representations leading to states which are orthogonal to each other and without an automorphism relating them.

BUT

FERMION PARITY, i.e. symmetry of the fermion theory under $\Phi(x,t) \to -\Phi(x,t)$, is broken by these minimal representations. (rotation by 2π!!)

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At the quantum level fermion parity leads to conservation for the number of fermions modulo 2, i.e. any physical process leads to creation or destruction of an EVEN number of fermions

$$\exists(-1)^F : \quad \begin{array}{l} (-1)^F \Phi(x,t) + \Phi(x,t)(-1)^F = 0 \\ (-1)^F H = H(-1)^F \end{array}$$

In both representations

$$\begin{aligned} \langle 0|\Phi|0\rangle &= \frac{1}{\sqrt{2}}\Psi_0 \\ \langle 0|\Phi|0\rangle &= -\frac{1}{\sqrt{2}}\Psi_0 \end{aligned} \qquad \Rightarrow \text{fermion parity broken!}$$

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Reducible representation

To restore fermion parity symmetry introduce β:

$$\left\{\beta,\beta^{\dagger}\right\}=1 \ , \ \beta^2=\frac{1}{2}$$

$$\{\beta,\alpha\}=\{\beta,\mathbf{a}_E\}=\left\{\beta,\mathbf{a}_E^{\dagger}\right\}=0$$

Then the algebra of α and β would have a two dimensional representation which can be represented in terms of fermionic oscillators as

$$\mathbf{a}=\frac{1}{\sqrt{2}}(\alpha+i\beta) \ , \ \mathbf{a}^{\dagger}=\frac{1}{\sqrt{2}}(\alpha-i\beta)$$

$$\Downarrow$$

$$\alpha=\frac{1}{\sqrt{2}}(\mathbf{a}+\mathbf{a}^{\dagger}) \ , \ \beta=\frac{1}{\sqrt{2}i}(\mathbf{a}-\mathbf{a}^{\dagger})\ \mathbf{a}^2=0 \ , \ \mathrm{a}^{\dagger 2} \ , \ \left\{\mathbf{a},\mathbf{a}^{\dagger}\right\}=1$$

$$\begin{aligned} \mathbf{a}|-\rangle &= 0 & \mathbf{a}^{\dagger}|-\rangle &= |+\rangle \\ \mathbf{a}|+\rangle &= |-\rangle & \mathbf{a}^{\dagger}|+\rangle &= 0 \end{aligned}$$

Both $|-\rangle$ and $|+\rangle$ are eigenvalues of $(-1)^F$ and parity restored.

Fermion parity symmetry $\Longleftrightarrow$ hidden variable β!!

Reducible representation

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$$\mathbf{a} = \frac{1}{\sqrt{2}}(\alpha + i\beta) \ , \ \mathbf{a}^{\dagger} = \frac{1}{\sqrt{2}}(\alpha - i\beta)$$

$$\Downarrow$$

$$\alpha = \frac{1}{\sqrt{2}}(\mathbf{a} + \mathbf{a}^{\dagger}) \ , \ \beta = \frac{1}{\sqrt{2}i}(\mathbf{a} - \mathbf{a}^{\dagger})\ \mathbf{a}^2 = 0 \ , \ \mathrm{a}^{\dagger 2} \ , \ \left\{\mathbf{a}, \mathbf{a}^{\dagger}\right\} = 1$$

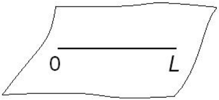
$$\begin{aligned} \mathbf{a}|-\rangle &= 0 & \mathbf{a}^{\dagger}|-\rangle &= |+\rangle \\ \mathbf{a}|+\rangle &= |-\rangle & \mathbf{a}^{\dagger}|+\rangle &= 0 \end{aligned}$$

Both $|-\rangle$ and $|+\rangle$ are eigenvalues of $(-1)^F$ and parity restored.

Fermion parity symmetry $\Longleftrightarrow$ hidden variable β!!

The model

Quantum wire in p-wave superconductor



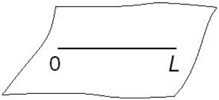
$$H = \sum_{n=1}^{L} \left(\frac{t}{2} \mathbf{a}_{n+1}^{\dagger} \mathbf{a}_{n} + \frac{t^{*}}{2} \mathbf{a}_{n}^{\dagger} \mathbf{a}_{n+1} + \frac{\Delta}{2} \mathbf{a}_{n+1}^{\dagger} \mathbf{a}_{n}^{\dagger} + \frac{\Delta^{*}}{2} \mathbf{a}_{n} \mathbf{a}_{n+1} + \mu \mathbf{a}_{n}^{\dagger} \mathbf{a}_{n} \right)$$

sites of wire $1 \ldots n$

- $\mathbf{a}_n$, $\mathbf{a}_n^{\dagger}$: operators creating electrons on sites n
- t: hopping electrons
- Δ, Δ^{*} arise from the presence of superconducting environment and describe the amplitude for a pair of electron to leave or enter the wire from the environment (assumption about size and coherence of Cooper pairs ...)
- μ chemical potential, i.e. energy of an electron sitting at site n

The model

Quantum wire in p-wave superconductor



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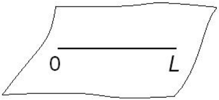

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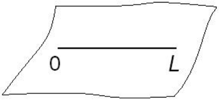
sites of wire 1 ... n

- $\mathbf{a}_n$, $\mathbf{a}^{\dagger}_n$: operators creating electrons on sites n
- t: hopping electrons
- Δ, Δ^* arise from the presence of superconducting environment and describe the amplitude for a pair of electron to leave or enter the wire from the environment (assumption about size and coherence of Cooper pairs ...)
- μ chemical potential, i.e. energy of an electron sitting at site n

The model

Quantum wire in p-wave superconductor



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sites of wire $1 \ldots n$

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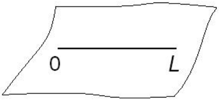
The model

Quantum wire in p-wave superconductor



$$H = \sum_{n=1}^{L} \left(\frac{t}{2} \mathbf{a}_{n+1}^\dagger \mathbf{a}_n + \frac{t^*}{2} \mathbf{a}_n^\dagger \mathbf{a}_{n+1} + \frac{\Delta}{2} \mathbf{a}_{n+1}^\dagger \mathbf{a}_n^\dagger + \frac{\Delta^*}{2} \mathbf{a}_n \mathbf{a}_{n+1} + \mu \mathbf{a}_n^\dagger \mathbf{a}_n \right)$$

sites of wire $1 \dots n$

- $\mathbf{a}_n$, $\mathbf{a}_n^\dagger$: operators creating electrons on sites n
- t: hopping electrons
- Δ, Δ^* arise from the presence of superconducting environment and describe the amplitude for a pair of electron to leave or enter the wire from the environment (assumption about size and coherence of Cooper pairs ...)
- μ chemical potential, i.e. energy of an electron sitting at site n

The model

We assume:

- $|t| > \Delta$ hopping on the wire favored respect to hopping from (to) the bulk
- $|\mu| < |t|$ electron bond has substantial filling

$$i\hbar\dot{\mathbf{a}}_n = [\mathbf{a}_n, H]$$

$$i\hbar\frac{d}{dt}\mathbf{a}_n = \frac{t}{2}\left(\mathbf{a}_{n+1} + \mathbf{a}_{n-1}\right) - \frac{\Delta}{2}\left(\mathbf{a}^\dagger_{n+1} - \mathbf{a}^\dagger_{n-1}\right) + \mu\mathbf{a}_n \qquad n = 2\ldots L-1$$

$n = 1$, $n = L$ boundary (open boundary condition)

$$\mathbf{a}_0(t) = 0 \quad , \quad \mathbf{a}_{L+1}(t) = 0$$

extend chain by one site and impose Dirichlet boundary condition

$$\psi^+_n = \phi_n \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \phi_{L+1-n} \begin{pmatrix} i \\ 0 \end{pmatrix}$$

$$\psi^-_n = \phi_n \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \phi_{L+1-n} \begin{pmatrix} i \\ 0 \end{pmatrix}$$

The model

$$\phi_n = i\sqrt{\frac{\Delta}{2t}\frac{t^2-\mu^2}{t^2-\Delta^2-\mu^2}}\left(\frac{\left(-\mu+i\sqrt{t^2-\Delta^2-\mu^2}\right)^n-(-\mu-i\sqrt{\mu})^n}{(t+\Delta)^n}\right)$$

$$\phi_n = \phi_n^* \qquad \sum_n |\phi|^2 = \frac{1}{2}$$

$$\psi_n^- = \psi_n^{+*} \qquad \sum_n \psi_n^{\pm}\Psi_n^{\pm} = 1$$

They have support near $n = 1$ and $n = L$ and are exponentially small inside the wire. They are complex Majorana fermion

$$\Psi_n(t) = \psi_n^+ e^{-i\omega}\mathbf{a} + \psi_n^- e^{i\omega}\mathbf{a}^\dagger + \ldots$$

The model

They are complex Majorana fermion

$$\Psi_n(t) = \psi_n^+ e^{-i\omega} \mathbf{a} + \psi_n^- e^{i\omega} \mathbf{a}^\dagger + \dots$$

$$\psi_n(t) = \underbrace{\phi_n \begin{pmatrix} 0 \\ 1 \end{pmatrix} \left(\mathbf{a} + \mathbf{a}^\dagger\right)}_{support \;\; near \;\; 0} + \underbrace{\phi_{L+1-n} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \frac{1}{i} \left(\mathbf{a} - \mathbf{a}^\dagger\right)}_{support \;\; near \;\; L} + \dots$$

$$\alpha = \frac{1}{\sqrt{2}} \left(\mathbf{a} + \mathbf{a}^\dagger\right) \qquad \beta = \frac{1}{\sqrt{2}i} \left(\mathbf{a} - \mathbf{a}^\dagger\right) \qquad \left\{\mathbf{a}, \mathbf{a}^\dagger\right\} = 1$$

$$\begin{array}{llll} \mathbf{a}|-\rangle & = 0 & \mathbf{a}^\dagger|-\rangle & = |+\rangle \\ \mathbf{a}|+\rangle & = |-\rangle & \mathbf{a}^\dagger|+\rangle & = 0 \end{array}$$

$|-\rangle$ and $|+\rangle$ eigenvalues of $(-1)^F$.
Imagine at $t = 0$ the system being in $|-\rangle$ and inject electron so that at $t = 0$ is sitting at site $\#1$.

The model

$\mathbf{a}_1^\dagger|0\rangle$? What is the transition amplitude for the transition of this state - after time T is elapsed- to a stat in which the electron is at site L? Final state: $\mathbf{a}_L^\dagger|-\rangle$

$$A = \langle -|\mathbf{a}_L e^{iHt}\mathbf{a}_1^\dagger|-\rangle = \underbrace{|\Phi_1^0|^2}_{\substack{unusual\ \ zero\ \ mode\ \ contribution, \\ ``exotic\ \ entanglement''}} + \underbrace{(T-L\ \ dependent\ \ terms)}_{\substack{usual\ \ via\ \ excited \\ quasi-electrons\ \ in\ \ thecore}}$$

$$A_{EE} = \left(\frac{2\Delta}{t}\right)\left(\frac{t^2-\Delta^2-\mu^2}{(t+\Delta)^2}\right) \Rightarrow P = \sum_n |\phi_n|^2|\phi_0|^2 = \frac{1}{2}A_{EE}$$

Concluding remarks

What we do now?
Once the quantum wire p-wave superconductor is prepared, the extended Majorana state of the electron is there, ready to use. The system has 2-degeneracy $|+\rangle$, $|-\rangle$.
Each preserves fermion parity

- $\alpha|+\rangle + \beta|-\rangle$ not allowed! $\Rightarrow$ $|+\rangle$ and $|-\rangle$ classical bit
  $\Rightarrow |+\rangle \equiv |ON\rangle \;\;,\;\; |-\rangle \equiv |OFF\rangle$
  $\vec{e} + |ON\rangle = |OFF\rangle$
  $|OFF\rangle + \vec{e} = |ON\rangle$
- If one would allow superposition

$$|\vartheta, \varphi\rangle = \cos\frac{\vartheta}{2}|-\rangle + e^{i\varphi}\sin\frac{\vartheta}{2}|+\rangle$$

$$\Downarrow$$

relative sign of the combination should not be observed

$\varphi \sim \varphi + \pi$ half a block sphere
peculiar q-bit
proximity effect $|ON, ON\rangle$ $|OFF, OFF\rangle$
$|ON, OFF\rangle$ $|OFF, ON\rangle$