Chiral effective field theory of nuclear interactions

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1 Chiral Effective Field Theory

Effective field theories are a general setting applicable whenever the energy range of interest is much lower than some heavy mass scale so that heavy particle degrees of freedom can be considered as "frozen"[1] There are a lot of examples of such a situation. One is the Euler-Heisenberg description of light-by-light scattering for energies much smaller than the mass of the lightest charged particle, the electron. In this limit, the interactions among the photons, which would involve exchanges of virtual electrons in a box diagram, can be considered as pointlike. We could say that highly virtual particles do not live enough to propagate for long[2] The pointlike, four-photon interaction is described by an additional term in the Lagrangian density of the effective theory of photons, which becomes

$$\mathcal{L}_{\rm EH} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + a(F_{\mu\nu}F^{\mu\nu})^2 + b(F^{\mu\nu}F_{\nu\rho}F^{\rho\sigma}F_{\sigma\mu}). \quad (1)$$

The new interactions, with respect to the term already present in QED, are parametrized by the coupling constants a and b, which are called "low-energy constants". Their values can be determined by a matching procedure with QED. Their form is dictated by symmetry considerations. In this case Lorentz symmetry and, most importantly the gauge symmetry. The latter forces the appearances of the photon field through $F_{\mu\nu}$ or covariant derivatives. The interactions above arethe simplest ones that can be written down respecting these symmetry. For example, it is easy to show that a term with 3 powers of the field-strength tensor $F_{\mu\nu}$ vanish. Of course there could be more complex terms involving more than 4 powers of $F\mu\nu$, which would contribute to more complex processesm and would involve higher number of derivatives. In general, the different terms can be ordered according to the power of derivatives involved, which translates into the power of momenta. The higher this power, the smaller their contributions at low energy.

Very similarly, the W boson exchange diagram relevant in β-decays shrinks to a 4-fermion contact interactions at energies much smaller than the mass of the exchanged boson, giving rise to the Fermi theory of weak interactions.

[1] There are a lot of excellent reviews about this subject, and I cannot aim at providing one of comparable quality, not even at being exhaustive in the reference list ... [1, 2, 3, 4, 5, 6].

[2] I like to recall a warning written in Griffith's book on particle physics: when a physicist invokes the uncertainty principle keep a hand on your wallet...

In the above examples, the low-energy version of the theory involves a subset of the fields entering the fundamental theory. The heaviest fields are simply "integrated out" and their effect is contained in the actual values of the low-energy constants. This situation is sometimes referred to as "decoupling" effective field theory. But that is not always the case. Sometimes the ingredients of the effective description are completely different than in the fundamental theory. One example is the Ginzburg-Landau theory of superconductivity [7], which is formulated in terms of a new ingredient, the order parameter, which bears little resemblance with the fundamental theory of the interactions of electrons and ions. Associated to this we have the phenomenon of spontaneous symmetry breakdown, i.e. the fact that the ground state of the system does not possess the same symmetry (in this case gauge symmetry) as the theory itself. It was precisely this situation that was translated to the realm of particle theory by Nambu [8] in the context of the chiral symmetry of strong interactions.

2 QCD chiral symmetry and its consequences

Chiral symmetry is approximately exhibited by QCD due to the small values of the light quark masses, as compared to the intrinsic scale of the theory, $\Lambda_{\rm QCD}$. Indeed, if we set the light quark masses to zero, then we can mix independently the left and right chiralities of the quarks,

$$\psi_L = \begin{pmatrix} u \\ d \\ \vdots \end{pmatrix}_L \to V_L \psi_L, \quad \psi_r = \begin{pmatrix} u \\ d \\ \vdots \end{pmatrix}_R \to V_L \psi_R, \tag{2}$$

where $\psi_{R/L} = (1 \pm \gamma_5)/2\psi$ and the transformation matrix $V_{R/L} \in$SU(N_f), N_f being the number of light flavours, in the case of interest $N_f = 2$. Without quark masses, this transformation leaves the Lagrangian invariant, since the kinetic term does not mix chiralities,

$$\bar{\psi} i \not{D} \psi = \bar{\psi}_R i \not{D} \psi_R + \bar{\psi}_R i \not{D} \psi_R. \tag{3}$$

In the absence of quark masses the left and right chiralities live their life independently.

This is a global symmetry, since if we allow V_L and V_R to depend on the spacetime point, then we could not shift the transformation matrix through the (covariant) derivative. It is also a continuous symmetry, characterized by parameters $\alpha^a_{R/L}$, such that the SU(N) transformation matrices can be written

$$V_{R/L} = \mathrm{e}^{-i\sum_a \frac{\lambda^a}{2}\alpha^a_{R/L}}, \tag{4}$$

in terms of the group generators λ^a in the defining representation (in the $N_f = 2$ case $\lambda^a = \tau^a$, the Pauli matrices, $a = 1, 2, 3$). As all continuous symmetries, chiral symmetry implies the existence of conserved Noether currents and associated charges. Their identification as

$$J^a_{\mu R/L} = \bar{\psi}_{R/L}\gamma_\mu \frac{\lambda^a}{2}\psi_{R/L} \tag{5}$$

is left as an exercise, together with the properties of the corresponding charges

$$Q^a_{R/L} = \int d^3\mathbf{x} J^a_{0R/L} \tag{6}$$

which, in force of the current conservation equation $\partial^\mu J_\mu = 0$, are time-independent. These Noether currents are the same as the electroweak currents, that is, to these currents are coupled weakly interacting particles. This is a particularly useful circumstance, which make the matrix elements of these Noether currents observable.

It is customary to define the vector and axial vector currents as the appropriate combinations

$$V^a_\mu = J^a_{\mu R} + J^a_{\mu L}, \quad A^a_\mu = J^a_{\mu R} - J^a_{\mu L}, \quad , \tag{7}$$

with corresponding vector and axial vector charges denoted as Q^a and Q^a_5 respectively. At this point it is worthwile to observe that actually the symmetry group of the Lagrangian is U(N_f)×U(N_f), so that the index a runs over the U(1) component too, say $a = 0$. But this is only possible for the vector symmetry, because the axial U(1) is affected by the QCD anomaly. The charges are the group generators and they satisfy the group algebra. Also the currents, by covariance, satisfy what is called the "current algebra",

$$[Q^a, V^b_\mu] = if^{abc}V^c_\mu, \quad [Q^a, A^b_\mu] = if^{abc}A^c_\mu, \tag{8}$$

$$[Q^a_5, V^b_\mu] = if^{abc}A^c_\mu, \quad [Q^a_5, A^b_\mu] = if^{abc}V^c_\mu. \tag{9}$$

On the basis of these relations a number of results were obtained in the '60, at a time when the theory of strong interaction was not yet known.

3 Chiral Ward identities

The symmetry, which at the classical level is simply expressed by the current conservation relation (or partial conservation), implies, at the quantum level, a whole hierarchy of relations among Green functions, i.e. vacuum correlation functions of time-ordered products of local operators involving the Noether currents themselves. These quantum-field theoretical objects are of interest because they are related, through the LSZ reduction formula, with matrix elements of the scattering operator.

There is a very convenient way to resume all these Ward identities. It consists of promoting the global symmetry to a local one. This means that the symmetry transformation parameters $\alpha^a_{R/L}$ are allowed to depend on the spacetime. Under such local transformations,

$$\psi_R \to V_R(x)\psi_R, \quad \psi_L \to V_L(x)\psi_L, \tag{10}$$

the Lagrangian is not left invariant, unless we equip it with external fields that transform as gauge fields, in order to absorb the non-invariant terms. We will thus write,

$$\bar{\psi}_R i \not{D} \psi_R + \bar{\psi}_R r^\mu \gamma_\mu \psi_R, \tag{11}$$

with the external field r^μ transforming under the symmetry transfornation as

$$r^\mu \to V_R r^\mu V_R^\dagger - i\partial^\mu V_R V_R^\dagger, \tag{12}$$

and, to gauge the left transformations, we have to introduce a corresponding external field ℓ^μ, transforming analogously. The external fields r^μ and ℓ^μ are matrices in flavor space, they belong to the group algebra, so that we can expand them in the basis of the generators,

$$r_\mu = \sum_a r^a_\mu \frac{\lambda^a}{2}, \quad \ell_\mu = \sum_a \ell^a_\mu \frac{\lambda^a}{2}. \tag{13}$$

We have thus promoted the global symmetry to a local one. The added terms amount to

$$\mathcal{L}_{\rm QCD} \to \mathcal{L}_{\rm QCD}[v_\mu, a_\mu] \equiv \mathcal{L}_{\rm QCD} + \bar{\psi}\gamma^\mu \left(v_\mu + \gamma_5 a_\mu\right)\psi, \tag{14}$$

where

$$v_\mu = \frac{1}{2}(\ell_\mu + r_\mu), \quad a_\mu = \frac{1}{2}(r_\mu - \ell_\mu), \tag{15}$$

are the vector and axial external sources which couple to the vector and axial currents V_μ^a and A_μ^a. We can also introduce scalar and pseudoscalar external sources, $s(x) = \sum_a s^a(x)\lambda^a/2$ and $p(x) = \sum_a p^a(x)\lambda^a/2$ coupling to the scalar and pseudoscalar bilinear densities,

$$S^a(x) = \bar{\psi}\frac{\lambda^a}{2}\psi, \quad P^a(x) = \bar{\psi}\frac{\lambda^a}{2}\gamma_5\psi, \tag{16}$$

and thus introducing a source-dependent Lagrangian

$$\mathcal{L}_{\rm QCD}[v_\mu, a_\mu, s, p] = \mathcal{L}^0_{\rm QCD} + \bar{\psi}\left\{\gamma^\mu\left[v_\mu + a_\mu\gamma_5\right] - s + ip\gamma_5\right\}\psi. \tag{17}$$

More precisely, $\mathcal{L}^0$ is the Lagrangian in the chiral limit. This Lagrangian is left invariant under local chiral transformations, provided the vector and axial sources transform following Eq. (12), and scalar and pseudoscalar ones as,

$$s + ip \to V_R(s+ip)V_L^\dagger, \quad s - ip \to V_L(s-ip)V_R^\dagger. \tag{18}$$

Besides rendering the theory chiral gauge invariant, the external sources are alsto useful because they can generate the Green functions of quark bilinears, through the generating functional $W[v_\mu, a_\mu, s, p]$,

$$\mathrm{e}^{iW[v_\mu,a_\mu,s,p]} = \int \mathcal{D}\mu_{\rm QCD}\mathrm{e}^{i\int dx \mathcal{L}_{\rm QCD}[v_\mu,a_\mu,s,p]}. \tag{19}$$

By differentiating W with respect to its arguments and evaluating the result at zero external sources, we get all connected Green functions of the quark bilinears. If, instead of evaluating the functional derivatives at zero value of the external sources we put $v_\mu = a_\mu = p = 0$ and $s = \mathcal{M}$, where $\mathcal{M}$ is the light quark mass matrix, we obtain the physical correlation functions, away from the chiral limit.

Now, what happens if we subject the sources to the chiral gauge transformations Eqs.(12)-(**??**) that we introduced?

$$\mathrm{e}^{iW[v'_\mu,a'_\mu,s',p']} = \int \mathcal{D}\mu_{\rm QCD}\mathrm{e}^{i\int dx \mathcal{L}_{\rm QCD}[v'_\mu,a'_\mu,s',p']} = \int \mathcal{D}\mu_{\rm QCD}\mathrm{e}^{i\int dx \tilde{\mathcal{L}}_{\rm QCD}[v_\mu,a_\mu,s,p]}, \tag{20}$$

where $\tilde{\mathcal{L}}$ is written in terms of the antitransformed quark fields,

$$\psi_{R/L} = V_{R/L}\tilde{\psi}_{R/L}, \tag{21}$$

using the invariance of the complete Lagrangian. But the quark fields are mere functional integration variables, dummy variables. Provided that the

QCD functional measure is invariant under the transformation, say $\mathcal{D}\mu_{\rm QCD} = \mathcal{D}\tilde{\mu}_{\rm QCD}$, we can conclude that

$$\mathrm{e}^{iW[v'_\mu,a'_\mu,s',p']} = \mathrm{e}^{iW[v_\mu,a_\mu,s,p]}. \tag{22}$$

which express the invariance of the generating functional under local chiral transformation. It should be remembered that, for axial transformation, the integration measure is not invariant, nevertheless the Jacobian can be expressed in closed form, and this has been done by Bardeen. So the correct equation is

$$W[v'_\mu, a'_\mu, s', p'] = W[v_\mu, a_\mu, s, p] + \Delta[v_\mu, a_\mu, s, p; V_L V_R^\dagger]. \tag{23}$$

It is called the chiral anomaly, but we need not discuss it here. By focusing on infinitesimal transformations, characterized by infinitesimal vector and axial parameters $\alpha^a = (\alpha_R^a + \alpha_L^a)/2$, $\beta^a = (\alpha_R^a - \alpha_L^a)/2$, and taking a derivative with respect to α^a and β^a one obtains two functional equations which contain all the constraints from the chiral Ward identities for quark bilinears. To obtain such equations is left as the second exercise.

Of course these are just formal manipulations, as we don't know how to calculate the functional integral. Perturbation theory in powers of the QCD coupling constants will certainly not work at large distances, due to the asymptotic freedom. We have to resort to some other strategy.

4 Spontaneous breakdown of chiral symmetry and non-linear realization

The chiral symmetry of QCD in the limit of two massless light quarks is spontaneously broken to the vectorial subgroup of isospin,

$$\mathrm{SU(2)_L} \times \mathrm{SU(2)_R} \to \mathrm{SU(2)_V}. \tag{24}$$

This means that, while the Lagrangian of the theory is invariant under the whole chiral group, the vacuum state only respects the isospin group, i.e. it is annihilated by the corresponding charge. This implies the existence in the spectrum of the theory of multiplets of particles degenerate in mass, which form irreducible representation of the isospin group, like the doublet of nucleons, the triplet of pions, the quadruplet of Δ etc. If the vacuum were also invariant under the whole group, then we would expect multiplets of degenerate hadrons with particles of different parities, since the chiral group does not respect parity. For example, the pion triplet should be accompanied by a scalar/isoscalar particle, which doesn't seem to be the case. The origin of the phenomenon of spontaneous breaking of chiral symmetry (SBChS) is entirely dynamical[3], it does not depend on any external agent, as it happens for the electroweak symmetry, broken by a scalar field (the Higgs field) to minimize its potential. This dynamics can be largely attributed to the gluons and their self-interactions, also responsible for the phenomenon of confinement and the generation of mass. These are so complex problems that, after more than 50 years, they remain unsolved[4]. However, there exists ample evidence for SBChS, both from phenomenology and from lattice QCD simulations. The spontaneous breaking has two consequences, which are expressed by the Goldstone's theorem [9]: the first is the existence of a massless particle for each one of the "broken generators", i.e. the generators which do not annihilate the vacuum state, which couples to the corresponding Noether current; the second is the fact that these "Goldstone bosons" interact weakly at low energy. This is also expressed in the form of "soft pions theorems" which were used in the past to derive results which are authomatically embedded in the ChEFT. The pions, which are identified with the Goldstone bosons, are not massless, because the chiral symmetry is not an exact one. But their mass is much smaller than the mass of any other hadrons. This corresponds to the fact that the light quarks are not exactly massless. Their mass[5] is approximately $m_{\rm up} \sim 2$ MeV and $m_{\rm down} \sim 5$ MeV, both values much smaller than the typical hadronic scale $\Lambda_{\rm H} \sim 1$ GeV. In any case, the coupling of pions to the axial current is dictated by Lorentz symmetry,

$$\langle 0|A^a_\mu(x)|\pi^b(\mathbf{p})\rangle = F_\pi p_\mu {\rm e}^{-ip\cdot x}, \tag{25}$$

where the dependence on x is a consequence of the translational invariance of the vacuum. It is parametrized in terms of a single constant F_π, the pion decay constant, which can be measured in the weak decay $\pi \to \mu + \bar{\nu}_\mu$, and is approximately $F_\pi \sim 92.4$ MeV. Using the current conservation, in the chiral

[3]Therefore a more appropriate qualification would be "dynamical" breaking.

[4]One of the "millennial questions" of the Clay Institute concerns the mechanism of the generation of mass in the theory of gluons.

[5]The above values refer to the current PDG compilation, at a renormalization scale of 2 GeV in the MS-bar scheme.

limit, we get as a consequence that

$$F_\pi M_\pi^2 \to 0. \tag{26}$$

Thus, two possibilities arise in principle, that $M_\pi \to 0$ in the chiral limit, as appropriate for a Goldstone boson, or that M_π stays different from zero but instead $F_\pi \to 0$. The latter possibility can be excluded, and one compelling reason comes from the neutron β decay. The involved hadronic matrix element, in this case is

$$\langle n(\mathbf{p_1})|A_\mu^1 - iA_\mu^2|p(\mathbf{p_2})\rangle, \tag{27}$$

where $\mathbf{p_1}$ and $\mathbf{p_2}$ represent the neutron and proton momenta, respectively, and the currents are evaluated at $x = 0$ (as usual, the x dependence follows from the translational symmetry). On general grounds, invoking Lorentz symmetry, the above matrix element is parametrized by two form factors as follows,

$$\bar{u}(\mathbf{p_1} \left\{ G_A(q^2)\gamma_\mu + H(q^2)q_\mu \right\} \gamma_5 u(\mathbf{p_2}), \tag{28}$$

where $u(\mathbf{p})$ denote the positive energy solutions of the Dirac equation, while $q = p_1 - p_2$ is the momentum transfer. In the chiral limit, current conservation implies

$$2m_N G_A(q^2) + q^2 H(q^2) = 0, \tag{29}$$

after using the Dirac equation. Therefore, by taking the limit $q \to 0$ one would have, in the chiral limit, $2m_N G_A(0) = 0$, a hard-to-believe constraint, since the axial coupling constant of the nucleon, $g_A = G_A(0) \sim 1.27$. The solution to this puzzle comes from the existence of a massless pole in the pseudoscalar form factor $H(q^2)$. Indeed, the latter can be decomposed into a pion-pole contribution plus a rest,

$$H(q^2) = \sqrt{2} g_{\pi NN} \frac{1}{M_\pi^2 - q^2} \sqrt{2} F_\pi + \tilde{H}(q^2). \tag{30}$$

What enters in the above expression are the coupling of the pion state to the nucleons, denoted as $g_{\pi NN}$ and to the axial current, that we already know from Eq. (25). Thus we get from Eq. (29), provided $M_\pi \to 0$ in the chiral limit,

$$2m_N G_A(0) = 2F_\pi g_{\pi NN}, \tag{31}$$

which is the famous Goldberger-Treiman relation. The pion-nucleon coupling constant which appears in this relation is the one parametrizing an interaction of the form

$$\mathcal{L}_{\pi NN} = g_{\pi NN} \bar{N} \pi^a \tau^a \gamma_5 N, \tag{32}$$

and amount approximately to $g_{\pi NN} \sim 13$. Thus, even from a purely phenomenological perspective, the pion has to be thought of as the Goldstone boson of SBChS, which in the chiral limit is massless and weakly interacting.

The two facts above allow to establish a calculational scheme for the generating functional (23). It is based on a representation in terms of the lightest vacuum excitations of the theory, the pions [10]. So the same functional $W[v_\mu, a_\mu, s, p]$ is written as a result of a theory of interacting pions,

$$\mathrm{e}^{iW[v_\mu,a_\mu,s,p]} = \int \mathcal{D}U \mathrm{e}^{i\int dx \mathcal{L}_{\mathrm{eff}}[U;v_\mu,a_\mu,s,p]}. \tag{33}$$

If we want it to respect the chiral Ward identities, we have to demand that the effective Lagrangian be invariant under the local chiral transformations of the sources[6]. But first we should ask: how to the pions transform under the chiral symmetry?

Let us denote by π the pion fields. Under a transformation g of the chiral group G =SU(2)$_L\times$SU(2)$_R$, it will be transformed as

$$\pi \xrightarrow{g} \pi' = f(\pi, g), \tag{34}$$

with a certain function f which must satisfy the group composition law, i.e.

$$\pi \xrightarrow{g_1} \pi' = f(\pi, g_1) \xrightarrow{g_2} \pi'' = f(\pi', g_2) = f(f(\pi, g_1), g_2) = f(\pi, g_2 g_1). \tag{35}$$

Consider now a transformation $h \in G$ that leaves the origin of the pion field manifold invariant, i.e.

$$f(0, h) = 0. \tag{36}$$

It can readily be checked that all such transformations form a subgroup $H \subset G$. Moreover, for any $g \in G$ and $h \in H$,

$$f(0, g) = f(0, gh), \tag{37}$$

in force of the group composition law (35). Thus, for any $\bar{g}$,

$$f(0, \bar{g}) = f(0, g), \quad \forall g \in \bar{g}H, \tag{38}$$

where $\bar{g}H$ denotes what is called a left coset of the subgroup H. Let us recall that given any subgroup H of a larger group G, the latter can be uniquely decomposed in left cosets,

$$G = H \cup g_1 H \cup g_2 H \cup g_3 H \cup ..., \tag{39}$$

[6] The effect of the chiral anomaly can be accounted for by considering specific terms to the effective Lagrangian.

with an infinity of factors in our case. We can view $f(0,...)$ as a function which takes from the set of all the left cosets of H (the factors in the above equation) to the manifold of the pion fields[7]. In addition, the correspondence is invertible, since, if $f(0,g) = f(0,g')$ then

$$f(0, g^{-1}g') = f(f(0,g'), g^{-1}) = f(f(0,g), g^{-1}) = f(0, g^{-1}g) = 0 \qquad (40)$$

which implies that $g^{-1}g' \in H$ so that there is a $h \in H$ such that

$$g^{-1}g' = h \Longrightarrow g' = gh, \qquad (41)$$

i.e. g and g' belong to the same left coset. So there is a one-to-one correspondence between the left cosets of H and the pion field manifold. Each pion field corresponds to some left coset of H, i.e. to one of the factors appearing in Eq. (39). The set of all left cosets of H is called the quotient space and it is denoted by G/H. The pion fields live in this space, they can be viewed as coordinates of this space. In the case of chiral symmetry breakdown G =SU(2)$_L$×SU(2)$_R$/SU(2)$_V$ the coset space is a group itself, the group SU(2). The pion field is thus essentially an SU(2) matrix. To set the correspondence between the coset space and SU(2) elements we have to choose a representative in each left coset, say

$$g = (g_L, g_R) \to (1, g_R g_L^{-1}). \qquad (42)$$

So, to every group transformation g, there corresponds a unique element of the coset space, represented by a SU(2) matrix $U = g_R g_L^{-1}$. We know that group transformation transform under group operation by the group composition law (by definition), so that

$$g = (g_L, g_R) \stackrel{(V_L,V_R)}{\longrightarrow} (V_L g_L, V_R g_R) \qquad (43)$$

which corresponds to the coset space element U',

$$U' = V_R g_R g_L^{-1} V_L^{-1} = V_R U V_L^{-1} \qquad (44)$$

so that the SU(2) matrix U, which can be taken to represent the pion field, transforms as

$$U \to U' = V_R U V_L^{\dagger}. \qquad (45)$$

[7]Remember that we can think of a group transformation g as a "field" of transformations, since we promoted the symmetry to a local one.

Different choices of the group representatives of the coset space elements (for instance $g \to (g_L g_R^{-1}, 1)$ would correspond to different transformation properties of the pion field matrix, which however lead to the same physical consequences, it would just amount to a change of coordinates of the coset space. The canonical choice for the parametrization of the SU(2) matrix U in terms of the pion field is

$$U = \mathrm{e}^{\frac{i}{F}\sum_a \pi^a \tau^a}, \tag{46}$$

where the constant F sets the scale of the pion field. Another frequently used parametrization is the σ-model one

$$U = \sigma + \frac{i}{F}\sum_a \pi^a \tau^a, \quad \sigma^2 + \frac{\boldsymbol{\pi}^2}{F^2} = 1, \tag{47}$$

You may prove, as exercise number 3, that the most general possible parametrization is

$$U = \sqrt{1 - \boldsymbol{\pi}^2 f(\boldsymbol{\pi}^2)} + i f(\boldsymbol{\pi}^2)\sum_a \pi^a \tau^a, \tag{48}$$

with a real scalar function f, using the fact that π^a is an isotriplet, the known transformation properties of U under isospin ($V_L = V_R = V$), and the requirement of definite transformation laws for U under the discrete symmetries. Show also that, up to four powers of the pion fields, the most general parametrization depends on only two arbitrary parameters, one of which sets the overall scale of the pion field.

Nothing must depend on the choice of the pion field, since U is a mere functional integration variable. Or, better said, the QCD Green functions generated by the functional W must be independent of the choice of the pion field, and the same is true for on-shell pion amplitudes, which constitute pole residues of those correlation functions. Off the mass-shell there can be differences though. At the level of the effective Lagrangian, a change of field variables φ induces additional terms in the action, which are proportional to the equations of motion, since, if the equation of motions are fulfilled, then the action is stationary with respect to change of fields,

$$\frac{\delta S}{\delta \varphi} = 0. \tag{49}$$

Such "equations of motion terms" will affect off-shell amplitudes, renormalization constants, but not physical quantities. They are therefore irrelevant.

5 Construction of the effective Lagrangian

We determined the required transformation properties of the SU(2) pion field matrix U. Remember that V_R and V_L are spacetime dependent, so the derivatives of the pion field matrix will lead to extra contributions

$$\partial^\mu U \to \partial^\mu V_R U V_L^\dagger + V_R \partial^\mu U V_L^\dagger + V_R U \partial^\mu V_L^\dagger \tag{50}$$

which can be eliminated by the use of the covariant derivative

$$D^\mu U = \partial^\mu - i r^\mu U + i U \ell^\mu, \tag{51}$$

which transform homogeneously under chiral symmetry

$$D^\mu U \to V_R D^\mu U V_L^\dagger. \tag{52}$$

The same can be done for derivatives of the χ source,

$$D^\mu \chi \to V_R D^\mu \chi V_L^\dagger. \tag{53}$$

We may form invariant structures by taking flavour traces (denoted as $\langle ... \rangle$) of alternate products of operators transforming covariantly like U or $U^\dagger$. In addition we may also use the left and right source curvatures

$$\mathcal{R}_{\mu\nu} \to V_R \mathcal{R}_{\mu\nu} V_R^\dagger, \quad \mathcal{L}_{\mu\nu} \to V_L \mathcal{L}_{\mu\nu} V_L^\dagger, \tag{54}$$

to form invariant structures like e.g.

$$\langle \mathcal{R}_{\mu\nu} U \mathcal{L}^{\mu\nu} U^\dagger \rangle. \tag{55}$$

Of course we can build an infinite variety of chiral invariant operators, but they can be ordered according to the number of (covariant) derivatives, and/or external sources involved. Since each derivatives brings down one power of momenta, this ordering corresponds to a low-momentum expansion. An important fact is that there are no possible invariants without derivatives or external sources at all: the only one would be $\langle U^\dagger U \rangle$ which is however constant since $U \in$SU(2). The first non trivial operator contains two derivatives

$$\langle D^\mu U^\dagger D_\mu U \rangle, \tag{56}$$

and belong to the leading order Lagrangian

$$\mathcal{L}^{(2)} = \frac{F_0^2}{4} \left[\langle D^\mu U^\dagger D_\mu U \rangle + 2B_0 \langle U^\dagger \chi + \chi^\dagger U \rangle \right], \tag{57}$$

together with a linear term in the scalar/pseudoscalar source $\chi = s + ip$, which provides a mass term for the pions, once evaluated at $s = \mathcal{M}$ the quark mass matrix. The superscript $^{(2)}$ in the Lagrangian specifies the so called "chiral power", according to the standard chiral counting

$$\partial^\mu \sim D^\mu \sim p^\mu \sim M_\pi \sim O(p), \quad \chi \sim O(p^2). \tag{58}$$

The constant F_0 which already appeared in the expression of U, is there to ensure a properly normalized pionic kinetic term. The other constant that appears at leading order (LO) is B_0, which is related to the vacuum quark-antiquark condensate in the SU(2) chiral limit. Notice that this Lagrangian already contains an infinite tower of pion self-interactions, which are always of derivative type, in the chiral limit. This is the manifestation of the good old soft pion theorems. Such interactions are also not renormalizable. But this causes no harm, as the renormalizability is recovered order by order in the chiral counting. This was Weinberg's original insight in the '70s, according to which a given diagram with L loops, I internal lines and n_i vertices of type i, each with chiral dimension d_i, will scale like p^ν with

$$\nu = 4L - 2I + \sum_i n_i d_i. \tag{59}$$

Using the topological identity relating L, I and $V = \sum_i n_i$, the total number of vertices,

$$L = I - V + 1, \tag{60}$$

we have that

$$\nu = 2L + 2 + \sum_i n_i(d_i - 2). \tag{61}$$

This implies that the loops are more and more suppressed in the chiral counting, since, as we have already noticed, $d_i \geq 2$. Notice that this depends crucially on the adoption of a mass-independent regularization scheme for the loop integrals, such as dimensional regularization, and on the fact that no hard scale is present in the integrands. This will change when there are nucleon propagators in the loops. But in this case we see that, up to a given chiral power ν only a finite number of diagrams needs to be calculated, and the divergences can be absorbed in the coefficients of the higher-order Lagrangians, which contain by construction all possible chiral invariant terms. What is the higher-order Lagrangian? Lorentz covariance and the adopted chiral counting implies that it may contain up to 4 (covariant) derivatives, or

2 derivatives and a χ source or source curvature, or two χ sources or source curvatures. The complete list of operators is [11]

$$
\begin{aligned}
\mathcal{L}^{(4)} \;=\; & \frac{\ell_1}{4}\langle D^\mu U^\dagger D_\mu U\rangle^2 + \frac{\ell_2}{4}\langle D^\mu U^\dagger D^\nu U\rangle\langle D_\mu U^\dagger D_\nu U\rangle \\
& +\ell_3 B_0^2\langle U^\dagger\chi + \chi^\dagger U\rangle^2 + \ell_4\frac{B_0}{2}\langle D^\mu U^\dagger D_\mu\chi + D^\mu\chi^\dagger D_\mu U\rangle \\
& +\ell_5\langle \mathcal{R}^{\mu\nu} U \mathcal{L}_{\mu\nu} U^\dagger\rangle + \frac{i}{2}\ell_6\langle \mathcal{R}^{\mu\nu} D_\mu U D_\nu U^\dagger + \mathcal{L}^{\mu\nu} D_\mu U^\dagger D_\nu U\rangle \\
& -\ell_7\frac{B_0^2}{4}\langle U^\dagger\chi - \chi^\dagger U\rangle^2 \qquad (62)
\end{aligned}
$$

Whenever possible the terms with a single trace have been expressed as products of traces, using the Cayley-Hamilton relations satisfied by any 2×2 matrix X,

$$X^2 - X\langle X\rangle + \det X = 0, \qquad (63)$$

that, applied to $X = A + B$ implies also

$$AB + BA - A\langle B\rangle - B\langle A\rangle - \langle AB\rangle + \langle A\rangle\langle B\rangle. \qquad (64)$$

Also, terms with squared covariant derivatives, like D^2U can be eliminated using the equation of motion, according to the discussion above. The constants ℓ_i are "low-energy constants" (LECs), whence their initial, while other pure source contact terms like $\langle\chi^\dagger\chi\rangle$ are multiplied by "high-energy constants" denoted by h_i.

6 The physics of the LECs

There is another aspect to discuss concerning the LECs, i.e. the fact that they contain information about the short-distance physics that is not explicitly included in the effective theory. This is called "resonance saturation". Indeed, heavier mesons can be included rather easily in the same picture, similarly to what will be explained for nucleons in the next lecture. Suppose, for simplicity, that there existed a scalar-isoscalar meson, described by a field ϕ, with mass m_S. Due to the particularly simple transformation properties, it is straightforward to write chiral invariant coupling of this field to the pions. One such coupling would be

$$h\phi\langle D^\mu U^\dagger D_\mu U\rangle, \qquad (65)$$

with a coupling h to be determined from the phenomenology. In the functional integral we will have to integrate over ϕ as well. A given term of the perturbative expansion will have e.g.

$$\frac{1}{2} i^2 \int d^4 z_1 h\phi(z_1) \langle D^\mu U^\dagger D_\mu U \langle (z_1) \int d^4 z_2 h\phi(z_2) \langle D^\nu U^\dagger D_\nu U \langle (z_2) \tag{66}$$

which will involve the ϕ propagator

$$\int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m_S^2 + i\epsilon} \mathrm{e}^{-ip\cdot(z_1 - z_2)}. \tag{67}$$

If the flowing momentum p is such that $p^2 << m_S^2$ the ϕ propagator becomes a Dirac δ function and its contribution amounts to a term

$$\frac{i^2}{2} h^2 \left(\frac{-i}{m_S^2} \right) \int d^4 z_1 \langle D^\mu U^\dagger D_\mu U \rangle^2 (z_1), \tag{68}$$

the same as would be given by the vertex proportional to ℓ_1. Therefore the heavy meson entails a contribution to the LEC ℓ_1

$$\ell_1^{(\phi)} = \frac{h^2}{2m_S^2}. \tag{69}$$

So, in general we can say that the LECs mimic the effect of virtual heavier particles, which have been "integrated out" from the theory. This works in practice much better for the vector meson (vector meson dominance), the principle is the same, the calculation a little more involved.

7 Extension to nucleons

In extending the effective chiral Lagrangian to include nucleons, the first thing we have to establish is how nucleons transform under the chiral symmetry. We know that they form an isodoublet under isospin, the vectorial SU(2),

$$N = \begin{pmatrix} p \\ n \end{pmatrix}, \tag{70}$$

but this leaves open many possibilities. One such possibility is that the respective chiralities transform linearly under the corresponding SU(2) transformations, e.g.,

$$N_L \to V_L N_L, \quad N_R \to V_R N_R, \tag{71}$$

where $N_{R/L} = (1 \pm \gamma_5)/2N$. When restricted to vector transformation $V_R = V_L = V$ then the isospin transformation properties are satisfied. However, the same is true for other choices, e.g.

$$N \to V_L N, \tag{72}$$

or

$$N \to V_R N. \tag{73}$$

It would seem that there is ample freedom in the choice of the representation. However, all the different choices are equivalent, since one can pass from one representation to the other by using the Goldstone bosons' field U. For instance, if we start from the transformation law (72), then the field $N' = UN$ transform according to (73),

$$N' = UN \to V_R U V_L^\dagger V_L N = V_R U N = V_R N'. \tag{74}$$

As we saw in the case of the pions, a field redefinition does not affect the physical consequences of the theory. Nevertheless, among all possible choices of transformation properties, there is a particularly convenient one. The reason is that all of the above choices lead to non-derivative couplings with pions, a fact that renders the crucial property at the basis of soft pion theorems (that would nevertheless arise), not immediately transparent from the power counting. This is clear, e.g. using the (parity respecting) representation (71), since in this case the nucleon mass term would take the chiral invariant form

$$m_N(\bar{N}_R U N_L + \bar{N}_L U^\dagger N_R) = m_N \left(\bar{N} \frac{U + U^\dagger}{2} N + \bar{N} \frac{U^\dagger - U}{2i} i\gamma_5 N \right), \tag{75}$$

leading to a tower of non-derivative pion-nucleon interactions, in addition to a pseudoscalar coupling. Notice that a pseudoscalar pion-nucleon coupling is a rather reasonable choice. A choice which is "equivalent" to the axial vector coupling, as stated by so-called "equivalence theorems. These theorems involve precisely some field redefinition (or equations of motion) to demonstrate the equivalence. For instance, it is left as the 4th exercise to prove that, starting from the πN interaction Lagrangian

$$\bar{N}(i\partial\!\!\!/ - m_N)N - ig\bar{N}\gamma_5 \boldsymbol{\tau} \cdot \boldsymbol{\pi} N, \tag{76}$$

the nucleon field redefinition,

$$N \to N' = \mathrm{e}^{-i\frac{g}{2m_N}\gamma_5 \boldsymbol{\tau}\cdot\boldsymbol{\pi}} N \tag{77}$$

leads to the replacement of the pseudoscalar coupling with the axial vector one,

$$\frac{g}{2m_N}\bar{N}\gamma^\mu\gamma_5\partial_\mu\boldsymbol{\tau}\cdot\boldsymbol{\pi}N, \tag{78}$$

in addition to more many-pion couplings. The same results can be obtained by partial integration of the axial vector coupling and by using the nucleon equation of motion. We see that, in order to eliminate the unwanted non-derivative pion couplings issuing from $\bar{N}_R U N_L + \bar{N}_L U^\dagger N_R$, we have to split the SU(2) matrix $U = uu$ and assign it partially to the left and right-handed nucleon fields, i.e.,

$$N'_L = uN_L, \quad N'_R = u^\dagger N_R, \tag{79}$$

so that the mass term does not involve the pion field U anymore. We have now to determine the transformation properties of the transformed fields, and therefore of the square root $u = \sqrt{U}$. We know that

$$u^2 \to u'^2 = V_R u^2 V_L^\dagger, \tag{80}$$

and require that there is a SU(2) matrix h such that

$$V_R u h^\dagger = h u V_L^\dagger, \tag{81}$$

so that

$$u' = V_R u h^\dagger = h u V_L^\dagger, \quad u'^\dagger = h u^\dagger V_R^\dagger = V_L u^\dagger h^\dagger. \tag{82}$$

h is called the compensator field and we can give an explicit expression for it from

$$u' = \sqrt{V_R U V_L^\dagger} = h u V_L^\dagger \Longrightarrow h = \sqrt{V_R U V_L^\dagger} V_L \sqrt{U^\dagger}. \tag{83}$$

In spite of the apparent uglyness, the transformation properties of the redefined nucleon fields are very simple,

$$N'_L = uN_L \to h u V_L^\dagger V_L N = h u N_L = h N'_L, \tag{84}$$

and the same for N'_R. Then finally the redefined nucleon field N transforms homogeneously,

$$N \to hN. \tag{85}$$

You can easily prove that the set of transformations

$$U \to V_R U V_L^\dagger, \quad N \to hN, \tag{86}$$

with the above definition of the compensator field h, defines a (non-linear) representation of the chiral group, in the sense that it respects the group composition law. This is the exercise #5. We have disposed of the (non-derivative) pion interaction. What about the derivative one? We saw that they must come from the covariant derivative of the pion field $D_\mu U$, which, however, transforms as U itself, $D_\mu U \to V_R D_\mu U V_L^\dagger$. We can put it inside a nucleon bilinear if we multiply it by appropriate factors of the u fields, e.g.

$$u^\dagger D_\mu U u^\dagger \to h u^\dagger V_R^\dagger V_R D_\mu U V_L V_L^\dagger u^\dagger h^\dagger = h(u^\dagger D_\mu U u^\dagger)h^\dagger, \tag{87}$$

and the same happens with $uD_\mu U^\dagger u$. It is convenient to define the object

$$u_\mu = iu^\dagger D_\mu U u^\dagger = -iuD_\mu U^\dagger u \to h u_\mu h^\dagger, \tag{88}$$

where the i ensures its hermiticity. An invariant operator is e.g.

$$\bar{N}\gamma^\mu\gamma_5 u_\mu N, \tag{89}$$

which gives a πNN derivative coupling of axial vector type. The presence of γ_5 is dictated by the parity invariance. Indeed, the pions inherits its properties under the discrete symmetries from the fact that it couples to the Noether axial current,

$$\langle 0|\bar{\psi}\gamma^\mu\gamma_5\frac{\tau^a}{2}\psi(x)|\pi^b(\mathbf{p})\rangle = ip^\mu \mathrm{e}^{-ip\cdot x}F_\pi, \tag{90}$$

therefore it is pseudoscalar and even under charge conjugation. So, e.g., under parity

$$U \xrightarrow{P} U^\dagger, \quad u^\mu \xrightarrow{P} -u_\mu. \tag{91}$$

So the pions can basically enter only through the field u_μ. Notice that, up to one spacetime derivative, we cant' have further invariant operator, as u_μ is traceless, so that e.g.

$$\bar{N}\gamma^\mu\gamma_5 N\langle u_\mu\rangle = 0. \tag{92}$$

We can also have derivatives of the nucleon fields, but we have to construct a chiral covariant derivative, since for local chiral transformation

$$N \to hN \Longrightarrow \partial^\mu N \to h\partial^\mu N + \partial^\mu h N. \tag{93}$$

This is done as usual, by introducing a connection with the duty to absorb the extra piece,

$$D^\mu N = (\partial^\mu + \Gamma^\mu)N, \tag{94}$$

where we require that
$$\Gamma^\mu \to h\Gamma^\mu h^\dagger - \partial^\mu h h^\dagger. \tag{95}$$
Now, we know that
$$u \to h u V_L^\dagger \Longrightarrow \partial^\mu u \to \partial^\mu h u V_L^\dagger + h\partial^\mu u V_L^\dagger + h u \partial^\mu V_L^\dagger, \tag{96}$$
so the field $\partial^\mu u$ can serve the purpose, but it has to be combined with $u^\dagger$,
$$\partial^\mu u u^\dagger \to h(\partial^\mu u u^\dagger)h^\dagger + \partial^\mu h h^\dagger + h u \partial^\mu V_L^\dagger V_L u^\dagger h^\dagger \tag{97}$$
the unwanted term depending on $\partial^\mu V_L^\dagger$ can be compensated by the inclusion of the external source ℓ^μ whose transformation properties involves precisely that term. Finally parity requires that also the right handed source be included. At the end we find, for the chiral connection
$$\Gamma^\mu = \frac{1}{2}\left(u^\dagger \partial^\mu u - \partial^\mu u u^\dagger\right) - \frac{i}{2} u \ell^\mu u^\dagger - \frac{i}{2} u^\dagger r^\mu r, \tag{98}$$
which ensures that
$$D^\mu N = (\partial^\mu + \Gamma^\mu)N \to h D^\mu N h^\dagger. \tag{99}$$
Also the scalar/pseudoscalar sources can be used to build homogeneously transforming building blocks, as
$$u^\dagger \chi u^\dagger, \quad u\chi^\dagger u, \tag{100}$$
and the curvatures,
$$u^\dagger \mathcal{R}_{\mu\nu} u, \quad u\mathcal{L}_{\mu\nu} u^\dagger, \tag{101}$$
that transform as u^μ. The chiral counting is modified, due to the fact that the nucleon mass m_N is not protected by chiral symmetry, it must be counted as order $O(1)$. Only the space part of nucleon four-momenta must be counted as a small parameter, therefore also covariant derivatives of nucleon fields must count as $O(1)$, while $\not{D} - m_N \sim O(p)$. The leading order πN Lagrangian is therefore of order $O(p)$,
$$\mathcal{L}_{\pi N}^{(1)} = \bar{N}\left(i\not{D} - m_N + \frac{1}{2} g_A \not{u} \gamma_5\right) N, \tag{102}$$
with a single LEC, g_A, that determines the nucleon coupling to the pion and also to the axial current. This is the celebrated Goldberger-Treiman relation,

which is authomatically built in in the effective theory. At the following order more LECs appear [12],

$$\begin{aligned}\mathcal{L}_{\pi N}^{(2)} &= \bar{N}\Big\{2B_0c_1\langle U^\dagger\chi+\chi^\dagger U\rangle - \frac{c_2}{4m_N^2}\langle u_\mu u_\nu\rangle(D^\mu D^\nu+\text{h.c.}) \\ &+\frac{c_3}{2}\langle u^\mu u_\mu\rangle + \frac{i}{4}c_4\sigma^{\mu\nu}[u_\mu,u_\nu] \\ &+2B_0c_5\left(u\chi^\dagger u + u^\dagger\chi u^\dagger - \langle U^\dagger\chi+\chi^\dagger U\rangle\right) \\ &+\frac{c_6}{8m_N}\sigma^{\mu\nu}(u^\dagger\mathcal{R}_{\mu\nu}u + u\mathcal{L}_{\mu\nu}u^\dagger) \\ &+\frac{c_7}{8m_N}\sigma^{\mu\nu}\langle\mathcal{R}_{\mu\nu}+\mathcal{L}_{\mu\nu}\rangle\Big\}N. \end{aligned} \tag{103}$$

The LEC c_1 is related to the πN σ-term, i.e. the light-quark condensate inside the nucleon, which also dictates the chiral expansion of the nucleon mass. c_2, c_3 and c_4 can be measured in πN scattering, and the first two play an important role since they get large contributions from Δ-resonance saturation, by a similar mechanism to what we have seen in the previous lecture for the ℓ_i. Other constants describe the structure of the nucleon, like the anomalous magnetic moment of the proton and neutron.

8 The problem of the heavy mass

After presenting the low-energy expansion of the counterterms, we would like to present the low-energy expansion of the loops, as done for the pions. We would like to say that a given loop has a “chiral dimension”, dictating its low-momentum behaviour,

$$D = 4L - 2I_\pi - I_N + \sum_i n_i d_i, \tag{104}$$

where L is the number of loops, I_π the number of pion internal lines (propagators), I_N the number of nucleon propagators and n_i the number of vertices of type i, each of dimension d_i in the chiral counting. This would follow from counting the nucleon propagators as $O(p^{-1})$, the inverse of $i\not{D}-m_N\sim O(p)$. Another way to put it is to say that the virtual nucleons always carry momentum $q=m_N v+k$ with v the four-velocity, $v^2=1$ and $k\sim O(p)$, so

$$\frac{1}{\not{q}-m_N+i\epsilon} = \frac{\not{q}+m_N}{q^2-m_N+i\epsilon} \sim \frac{\not{q}+m_N}{2m_N v\cdot k+i\epsilon} \sim O(p^{-1}). \tag{105}$$

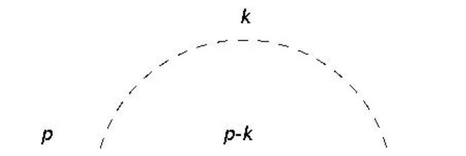

Figure 1: Pion loop contribution to the nucleon self-energy.

If that was the case then, using the already discussed topological identity which relates the number of loops L, the number of total internal lines $I = I_\pi + I_N$ and the number of vertices $V = \sum_i n_i$,

$$L = I_\pi + I_N - V + 1, \tag{106}$$

to remove the number of pion propagators, and the relation

$$2I_N + E_N = \sum_i n_i f_i, \tag{107}$$

which counts the total number of nucleon lines (external E_N or internal I_N) attached to the vertices, where to vertices of type i are attached f_i nucleon lines, we would arrive at

$$D = 2L + 2 - \frac{1}{2}E_N + \sum_i n_i \left(d_i - 2 + \frac{1}{2}f_i \right), \tag{108}$$

whence we would observe that D is bounded from below, for a given process, since chiral symmetry ensures that, for each vertex $d_i - 2 + f_i/2 \geq 0$. We would then have a well defined loop expansion as in the purely pionic case. Unfortunately, the presence of the nucleon mass m_N, which is a hard scale entering in the loop integrals, not suppressed in the chiral counting, complicates the life a bit. Indeed, you can prove as exercise # 6, that the nucleon self energy diagram depicted in Fig. 1, which should be of order $O(p^3)$ on the basis of the above couunting, scales instead, in ordinary dimensional regularization, as m_N^3, so it is $O(1)$. This is the consequence of the presence of a hard scale in the integrand, which was not the case for the pions, when using a mass-independent regularization scheme. Thus, it doesn't happen anymore that loop renormalize the higher order LECs. They also renormalize the lower order LECs! One way to cure these drawbacks is to use the so-called heavy-baryon formalism. This was originally introduced for the heavy quarks, but it works in the same way. The idea is that, in the limit of infinite mass, the four-velocity of each baryon is fixed, it will never be changed by processes that happen at low momenta. In other words, if we start from some (on-mass-shell) state with momentum

$$P_\mu = m_N v_\mu, \quad v^2 = 1, \tag{109}$$

then every soft transition will lead to a momentum

$$P'_\mu = m_N v_\mu + k_\mu = m_N v'_\mu, \quad v'^2 = 1, \tag{110}$$

so that in the limit $m_N \to \infty$ we have $v_\mu = v'_\mu$, So, each baryon has a definite 4-velocity in this limit, that we can keep track of, and that will never change. Then, in defining the theory, we can introduce velocity-dependent heavy fermion fields. Once a given velocity is picked up, it will not change. Formally the velocity-dependent fields are defined in terms of the original nucleon field N like

$$N = \mathrm{e}^{-im_N v\cdot x} \left(H_v + h_v\right), \tag{111}$$

with H_v and h_v representing eigenspinors of $\not{v} = \gamma^\mu v_\mu$,

$$\not{v} H_v = H_v, \quad \not{v} h_v = -h_v. \tag{112}$$

Notice that, since $\not{v}^2 = v^2 = 1$, then the eigenvalues of $\not{v}$ can either be $+1$ or -1. The eigenspinors can be obtained by the action of the projectors

$$P^\pm = \frac{1 \pm \not{v}}{2}. \tag{113}$$

In the nucleon rest frame, in which $v = (1, 0, 0, 0)$, the H_v and h_v fields are expressed, respectively, in terms of the large and small components of the nucleon Dirac spinors, whence the notation. The phase factor in Eq. (111) is meant to absorb most of the time-dependence of H_v due to the heavy mass. The relation (111) can be used to express the Lagrangian in terms of the heavy fermion fields. Using the defining properties (112) one obtains, e.g.,

$$\begin{aligned} &\bar{H}_v \mathrm{e}^{im_N v\cdot x} \left(i\not{D} - m_N + \frac{1}{2} g_A \not{u}\gamma_5\right) \mathrm{e}^{-im_N v\cdot x} H_v \qquad (114) \\ = \;& \frac{1}{2}\bar{H}_v \mathrm{e}^{im_N v\cdot x} \left[\not{v}\left(i\not{D} - m_N + \frac{1}{2} g_A \not{u}\gamma_5\right) + \left(i\not{D} - m_N + \frac{1}{2} g_A \not{u}\gamma_5\right)\not{v}\right] \mathrm{e}^{-im_N v\cdot x} H_v \end{aligned}$$

and using the Clifford algebra the above expression is equal to

$$\bar{H}_v \left(iv\cdot D + g_A S\cdot u\right) H_v \tag{115}$$

where the spin four-vector,

$$S^\mu = \frac{i}{2}\sigma^{\mu\nu}\gamma_5 v_\nu, \quad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]. \tag{116}$$

Notice that the heavy scale, the nucleon mass m_N, has disappeared from the Lagrangian involving H_v, which was the purpose of the formalism. The leading order Lagrangian becomes,

$$\begin{aligned}\mathcal{L}^{(1)}_{\pi N} &= \bar{H}_v\left(iv\cdot D + g_A S\cdot u\right) + \left[\bar{h}_v\left(i\not{d}^\perp - \frac{g_A}{2}v\cdot u\gamma_5\right)H_v + \text{h.c.}\right] \\ &\quad + \bar{h}_v\left(-2m_N - iv\cdot D - g_A S\cdot u\right)h_v,\end{aligned} \tag{117}$$

where $d^\perp_\mu = d_\mu - d\cdot v v_\mu$, such that $\left\{\not{d}^\perp, \not{v}\right\} = 0$. One can then "integrate out" the field h_v, by using iteratively the equations of motion, so that the dependence on m_N is reduced to additional vertices representing relativistic $1/m_N$ corrections. Notice however that, despite being formally Lorentz invariant, the dependence on v introduces a preferred reference frame. One way to restore the appropriate relativistic properties is to impose the so-called "reparametrization invariance", i.e. the freedom to relabel the four velocity by addition of small terms,

$$(v, k) \to (v + q/m_N, k - q), \quad \text{with } (v + q/m_N)^2 = 1, \tag{118}$$

which puts non-trivial constraints on the construction of the Lagrangian. We will not pursue this formalism, since in the case of more nucleons, as will be seen, it is natural, following Weinberg, to abandon the Lorentz-invariant perturbation theory, or Feynman diagrams, and use the old-fashioned time-ordered perturbation theory (TOPT). Indeed, TOPT can also be profitably used in the 1-nucleon sector. To understand this, let's come back to the example of the nucleon self energy diagram, Fig. 1, which involves the scalar integral

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{[k^2 - M^2 + i\epsilon]\,[(p-k)^2 - m_N^2 + i\epsilon]}, \tag{119}$$

which should count as $O(p)$ in the naïf counting. We can calculate thihs integral by first doing the integration over the temporal component k_0. This procedure actually generates the TOPT diagrams. In the complex k_0 plane the integrand has 4 poles located at

$$k_0^2 - \mathbf{k}^2 - M^2 + i\epsilon = 0 \implies k_0 = \pm\left(\sqrt{\mathbf{k}^2 + M^2} - i\epsilon\right), \tag{120}$$

$$(k_0 - p_0)^2 - (\mathbf{k} - \mathbf{p})^2 - m_N^2 + i\epsilon = 0 \implies k_0 = p_0 \pm \left(\sqrt{(\mathbf{k}-\mathbf{p})^2 + m_N^2} - i\epsilon\right), \tag{121}$$

two in the upper and two in the lower complex plane. We can close the contour at infinity, since the integrand goes like k_0^{-4}, and use Cauchy theorem. Choosing to complete the contour counterclockwise at positive imaginary infinity, we get contribution from the two poles at

$$k_0 \sim \sqrt{\mathbf{k}^2 + M^2} \equiv \omega_{\mathbf{k}}, \quad k_0 \sim p_0 - \sqrt{(\mathbf{k} - \mathbf{p})^2 + m_N^2} \sim O(p^2). \tag{122}$$

Neglecting $O(p^2)$ contributions the contour integration gives

$$2\pi i \frac{1}{4m_n \omega_{\mathbf{k}}^2}, \tag{123}$$

which, after integration over the spatial $\mathbf{k}$, restores the proper counting $\sim O(p)$.

9 Two and many nuclear forces

The addition of more nucleons in the effective theory does not change the principles for the construction of the effective Lagrangian. However, the counting implied by Eq. (108) is violated by diagrams involving purely nucleonic intermediate states [14]. In the language of Feynman diagrams this corresponds to infrared divergences, or "pinch singularities" [13]. On the other hand, if the perturbative framework implied by Eq. (108) worked one would never encounter nuclear bound states. Therefore Weinberg suggested to define a nuclear effective potential as consisting of only irreducible time-ordered diagrams, i.e. ignoring the purely nucleonic intermediate states. The neglected diagrams will be generated as iteration of the effective potential by the dynamical equation in the form of a Schroedinger or Lippmann -Schwinger equation. Thus, since the effective potential does not include the power-counting violating diagrams, it can be calculated perturbatively in the low-energy expansion following the counting (108). Various different methods to derive and regularize the effective potential have been developed, which are described in detail in many reviews [15, 16, 17, 18, 19]. From the point of view of vertices, the presence of many nucleons allows to write down many-nucleon contact interactions, as e.g.

$$\begin{array}{ll} o_1 = N^\dagger N N^\dagger N, & o_2 = N^\dagger \vec{\sigma} N \cdot N^\dagger \vec{\sigma} N, \\ o_3 = N^\dagger \tau^a N N^\dagger \tau^a N, & o_4 = N^\dagger \vec{\sigma} \tau^a N \cdot N^\dagger \vec{\sigma} \tau^a N. \end{array} \tag{124}$$

Notice that the absence of low-energy suppression due to derivatives doesn't impair the chiral counting (108) since $d_i - 2 - f_2/2 = 0$. The first one of the above structures is obviously invariant under the chiral transformation of the nucleon fields (85), as well as the second one, since the spin matrix $\vec{\sigma}$ commutes with the flavour matrix h [8]. A little more thought is required to show that also the last two structures are invariant under the transformation (85), since h does not commute with the isospin matrix τ^a. The proof is left as an exercise. The LECs multiplying these contact operators must be fixed from nucleon-nucleon data, contrary to the πN LECs, which can be taken from the 1-nucleon sector. Thus it is important to know their exact number at each order. Indeed, not all of the above contact operators are independent, as can be verified by simultaneously changing spin and isospin indeces of two nucleon fields (which are evaluated at the same space-time point) and using Fierz-like identities like

$$\delta_{s_1' s_1}\delta_{s_2' s_2} = \frac{1}{2}\delta_{s_1' s_2}\delta_{s_2' s_1} + \frac{1}{2}\boldsymbol{\sigma}_{s_1' s_2} \cdot \boldsymbol{\sigma}_{s_2' s_1}. \tag{125}$$

It is left as an exercise to show that $o_3 = -o_2 - 2o_!$ and $o_4 = -3o_1$. Thus at the leading order (LO) only two contact LECs appear, which parametrize the short-distance NN interaction. Together with the one-pion-exchange diagram, they form the LO NN interaction. The first correction can be expected when we add a derivative to the vertices. From parity symmetry, no such one-derivative NN contact operator exists. As for the πNN vertices, we can see from the explicit expression of the $\mathcal{L}_{\pi N}$ Lagrangian that no such terms can be written. The following non-zero contribution to the NN effective potential arises at $O(p^2)$ (NLO), and involves pion-loops describing two-pion exchanges, and a set of two-derivatives NN contact interactions. In the NN center of mass systems there are 7 independent terms of such type, each accompanied by a corresponding LEC. At N2LO ($O(p^3)$) the NN potential will not involve new "nucleonic" LECs. They will appear at N3LO ($O(p^4)$) stemming from four derivative NN contact interactions. In the NN center of mass systems there are 15 such terms, which are usually fixed from NN scattering data. The NN potential at N3LO thus involves about 24 free parameters to be fitted to data. The resulting χ^2 is very close to 1 per degree of freedom, thus at this order the chiral potentials are considered as "realistic", with considerably less fit parameters than more phenomenological

[8]The above operators can be viewed as the non relativistic limit of corresponding relativistic ones, thus the nucleon fields N in this case are two-component Pauli spinors.

approaches. Whether this can be viewed as a success of the ChEFT framework or simply due to the appearance of many fitting constants at N3LO is a matter of controversy. One important aspect that differentiates the ChEFT approach from the phenomenological ones is the possibility to check the pattern of the convergence of the low-energy expansion. Indeed, one can say that the true “prediction” of the EFT approaches is precisely about the convergence pattern. The validation of such a pattern also allows to estimate the theoretical uncertainty induced by the truncation of the expansion, an uncertainty which also affects the fitted values of the LECs and then propagates to the preditions. Various ways to apply this procedure have been proposed in the literature, also based on Bayesian inference [20, 21, 22, 23], and the expectation of the EFT framework, at least in the two-nucleon sector, seem confirmed.

The ChEFT framework also allows to justify the hierarchy of nuclear forces, i.e. the fact that the three-nucleon force is a small correction to the pairwise interactions and similarly for the A nucleon forces with respect to the $A-1$ ones. In order to see this, consider a generic N nucleon diagram, made out of c connected pieces and d disconnected 1-nucleon lines. The minimum chiral dimension of each of the connected pieces is

$$D_i = 2 - N_i,$$

where N_i $(i = 1, ..., c)$ is the number of nucleon lines participating in the i-th connected piece. Therefore, taking into account the three-momentum conserving δ function characterizing each connected subdiagrams and each disconnected 1-nucleon line, the total chiral dimension will be larger than

$$D = \sum_{i=1}^{c}(2 - N_i) - 3c - 3d = -c - 2d - N. \tag{126}$$

This formula implies the hierarchy of many-nucleon forces. For example, in the case of 3 nucleon, the largest contribution to the amplitude will come from $c = 0$, $d = 3$, amounting to $D = -9$. This contribution is not interesting since it does not represent interactions. Next we have $c = 1$, $d = 1$, whence $D = -6$, which represent the chiral power of two-body contributions in the $N = 3$ systems. The three-body contribution will correspond to $c = 1$, $d = 0$, then $D = -4$. Therefore we see that the three-body contribution is at least $O(p^2)$ suppressed with respect to the two-body one. Analogous reasoning applies to many-nucleon forces. In reality, the three-body contribution experiences a further $O(p)$ suppression, due to a cancellation occurring at the lowest order [24]. Therefore, in ChEFT, the three-body force starts to contribute at N2LO [25]. It is parametrized by one single three-nucleon contact interaction LEC, traditionally called c_E, and a LEC parametrizing a two-nucleon contact interaction with attached pion line, denoted as c_D. These LECs are usually fitted to the triton and ^{4}He binding energies, or to other observables in $A \geq 3$ systems. The common wisdom is that thirteen new three-nucleon LECs arise at $N4LO$ [26], although it has recently been pointed out that five of these LECs actually represent N3LO contributions [27].

Exercises

As exercise #0, please point out the mistakes you will spot in these notes!

1. Derive the Noether currents of the chiral symmetry of QCD. To do that, consider infinitesimal transformations and pretend the transformation parameters $\alpha^a_{L/R}$ depend on the space-time point. Then the Lagrangian is not left invariant, it acquires a variation $\delta\mathcal{L} \propto \partial^\mu \alpha^a_{L.R}(x)$, since it must reduce to zero for spacetime-independent parameters. The Noether currents are the proportionality factor, since the action changes as $\delta S = \int d^4x J^a_\mu \partial^\mu \alpha^a$, and we get that the current conservation as a consequence of the stationarity of the action if the fields satisfy the equations of motions (on the mass shell, as it is usually said).

2. Find how the vector, axial, scalar and pseudoscalar sources transform under infinitesimal local vector and axial transformations, with (infinitesimal) parameters $\alpha(x)$ and $\beta(x)$. Impose the invariance of the generating functional $W[v_\mu, a_\mu, s, p]$ of connected Green functions, with respect to such transformation to derive two functional equations (ignore the chiral anomaly for simplicity). All the chiral Ward identities, even away from the chiral limit, are derived from these equations by taking functional derivatives with respect to the external sources, and putting at the end $v_\mu = a_\mu = p = 0$ and $s = \mathcal{M}$.

3. Show that the most general parametrization of the SU(2) matrix U in terms of the isovector pion field $\boldsymbol{\pi}(x)$ is the following

$$U = f_0(\boldsymbol{\pi}^2) + i\sqrt{\frac{1 - f_0(\boldsymbol{\pi}^2)}{\pi^2}} \sum_a \pi^a \tau^a, \tag{127}$$

with a real scalar function f_0. Expand up to four powers of the pion fields, and give the most general expansion in terms of one parameter, besides F.

4. Show that, starting from the πN interaction Lagrangian

$$\bar{N}(i\partial\!\!\!/ - m_N)N - ig\bar{N}\gamma_5 \boldsymbol{\tau} \cdot \boldsymbol{\pi} N, \tag{128}$$

the nucleon field redefinition,

$$N \to N' = \mathrm{e}^{-i\frac{g}{2m_N}\gamma_5 \boldsymbol{\tau}\cdot\boldsymbol{\pi}} N \tag{129}$$

leads to the replacement of the pseudoscalar coupling with the axial vector one,

$$\frac{g}{2m_N}\bar{N}\gamma^\mu\gamma_5\partial_\mu\boldsymbol{\tau}\cdot\boldsymbol{\pi}N, \tag{130}$$

in addition to more many-pion couplings.

5. Show that the transformation

$$N \to hN, \quad U \to V_R U V_L^\dagger, \tag{131}$$

with h defined in Eq. (83), realizes a (non-linear) representation of the chiral group, in the sense that it respects the group composition law. Show also that, when restricted to vector transformations, you recover the proper isospin transformation law.

6. Show by explicit calculation of the Feynman diagram, in dimensional regularization, that the pion loop contribution to the nucleon self-energy of Fig. 1 contains contributions of order $O(m_N^3)$ which therefore are not suppressed in the chiral counting.

7. Show that the operator $\bar{N}\tau^a N\bar{N}\tau^a N$ is invariant under the transformation (85) of the nucleon fields. As an intermediate step, you may prove that,

$$h^\dagger\tau^a h = \hat{\mathbf{n}}\cdot\boldsymbol{\tau}\hat{n}^a + \cos(2\alpha)\left(\tau^a - \hat{\mathbf{n}}\cdot\boldsymbol{\tau}\right) - \sin(2\alpha)\left(\hat{\mathbf{n}}\times\boldsymbol{\tau}\right)^a, \tag{132}$$

where the SU(2) matrix h is decomposed as

$$h = \cos\alpha\mathbf{1} + i\sin\alpha\hat{\mathbf{n}}\cdot\boldsymbol{\tau}. \tag{133}$$

8. Starting from the contact operators defined in Eq. (124), use the Fierz-like identities for both spin and isospin indeces to show that

$$o_3 = -o_2 - 2o_1, \quad o_4 = -3o_1. \tag{134}$$

Remember to pay the minus sign when you interchange the two nucleon fields.

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