



Exotic hadrons

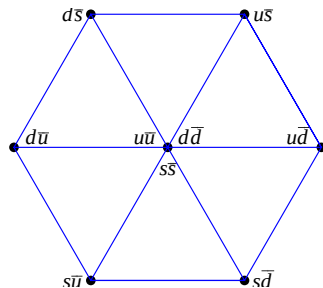
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GGI School on Frontiers in Nuclear and Hadronic Physics 2025

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Light meson SU(3) [u, d, s] multiplets (octet + singlet):



• Vector mesons

meson	quark content	mass (MeV)
ρ^+ / ρ^-	$u\bar{d} / d\bar{u}$	775
ρ^0	$(u\bar{u} - d\bar{d}) / \sqrt{2}$	775
K^{*+} / K^{*-}	$u\bar{s} / s\bar{u}$	892
$K^{*0} / \bar{K}^{*0}$	$d\bar{s} / s\bar{d}$	896
ω	$(u\bar{u} + d\bar{d}) / \sqrt{2}$	783
ϕ	$s\bar{s}$	1019

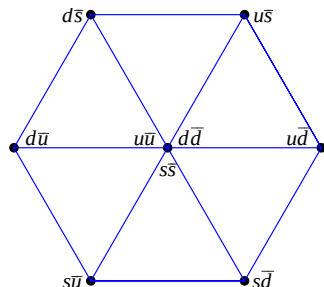
👉 approximate SU(3) symmetry

👉 very good isospin SU(2) symmetry

$$m_{\rho^0} - m_{\rho^\pm} = (-0.7 \pm 0.8) \text{ MeV}, \quad m_{K^{*0}} - m_{K^{*\pm}} = (6.7 \pm 1.2) \text{ MeV}$$

Light vector and pseudoscalar mesons

Light meson SU(3) $[u, d, s]$ multiplets (octet + singlet):



- Pseudoscalar mesons

meson	quark content	mass (MeV)
π^+ / π^-	$u\bar{d} / d\bar{u}$	140
π^0	$(u\bar{u} - d\bar{d}) / \sqrt{2}$	135
K^+ / K^-	$u\bar{s} / s\bar{u}$	494
$K^0 / \bar{K}^0$	$d\bar{s} / s\bar{d}$	498
η	$\sim (u\bar{u} + d\bar{d} - 2s\bar{s}) / \sqrt{6}$	548
η'	$\sim (u\bar{u} + d\bar{d} + s\bar{s}) / \sqrt{3}$	958

👉 very good isospin SU(2) symmetry

$$m_{\pi^\pm} - m_{\pi^0} = (4.5936 \pm 0.0005) \text{ MeV}, \quad m_{K^0} - m_{K^\pm} = (3.937 \pm 0.028) \text{ MeV}$$

👉 Why are the pions so light? Pseudo-Nambu-Goldstone bosons of the spontaneous breaking of chiral symmetry: $\text{SU}(2)_L \times \text{SU}(2)_R \rightarrow \text{SU}(2)_V$

Lectures by Juan Nieves

- J^{PC} of regular $q\bar{q}$ mesons

L : orbital angular momentum

$S = (0, 1)$: total spin of q and $\bar{q}$

$$P = (-1)^{L+1} \quad [Y_{Lm}(\theta - \pi, \phi + \pi) = (-1)^L Y_{Lm}(\theta, \phi)]$$

$$C = (-1)^{L+S} = (-1)^{L+1+S+1} \text{ for flavor-neutral mesons}$$

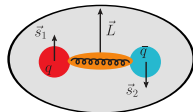
$$S = 0: \frac{1}{\sqrt{2}} |\uparrow_q \downarrow_{\bar{q}} - \downarrow_q \uparrow_{\bar{q}}\rangle; \quad S = 1: \left\{ |\uparrow_q \uparrow_{\bar{q}}\rangle, \frac{1}{\sqrt{2}} |\uparrow_q \downarrow_{\bar{q}} + \downarrow_q \uparrow_{\bar{q}}\rangle, |\downarrow_q \downarrow_{\bar{q}}\rangle \right\}$$

☞ For $S = 0$, the meson spin $J = L$, one has $P = (-1)^{J+1}$ and $C = (-1)^J$. Hence,
 $J^{PC} = \text{even}^{-+} \text{ and odd}^{+-}$

☞ For $S = 1$, one has $P = C = (-1)^{L+1}$. Hence,
 $J^{PC} = 1^{--}, \{0, 1, 2\}^{++}, \{1, 2, 3\}^{--}, \dots$

- Exotic J^{PC} for mesons:

$$J^{PC} = 0^{--}, \text{even}^{+-} \text{ and odd}^{-+}$$



Digression: Mesons with exotic quantum numbers

- π_1 listed as established particles by the Particle Data Group (PDG)

$$\pi_1(1400) \quad I^G(J^{PC}) = 1^-(1^{+-})$$

See also the mini-review under non- $q\bar{q}$ candidates in PDG 2006, Journal of Physics G33 1 (2006).

$\pi_1(1400)$ MASS	1354 ± 25 MeV (S = 1.8)
$\pi_1(1400)$ WIDTH	330 ± 35 MeV

Decay Modes

Mode	Fraction (Γ_i / Γ)	Scale Factor/ Conf. Level	P (MeV/c)
$\Gamma_1 \quad \eta\pi^0$	seen		557
$\Gamma_2 \quad \eta\pi^-$	seen		556
$\Gamma_3 \quad \eta' \pi$			318
$\Gamma_4 \quad \rho(770)\pi$	not seen		442

$$\pi_1(1600) \quad I^G(J^{PC}) = 1^-(1^{+-})$$

$\pi_1(1600)$ MASS	1660^{+15}_{-11} MeV (S = 1.2)
$\pi_1(1600)$ WIDTH	257 ± 60 MeV (S = 1.9)

Decay Modes

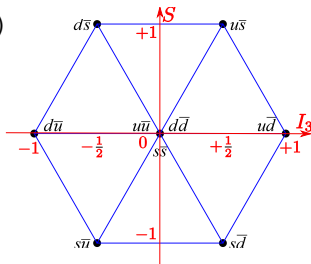
Mode	Fraction (Γ_i / Γ)	Scale Factor/ Conf. Level	P (MeV/c)
$\Gamma_1 \quad \pi\pi\pi$	seen		802
$\Gamma_2 \quad \rho^0 \pi^-$	seen		640
$\Gamma_3 \quad f_2(1270)\pi^-$	not seen		316
$\Gamma_4 \quad b_1(1235)\pi$	seen		355
$\Gamma_5 \quad \eta'(958)\pi^-$	seen		542
$\Gamma_6 \quad f_1(1285)\pi$	seen		312

- $J^{PC} = 1^{+-} \quad \eta_1(1855) \rightarrow \eta\eta'$ recently observed by BESIII BESIII, PRL129(2022)192002

It is unclear what they are: hybrids? hadronic molecules? or sth. else?

Some trivial facts about additive quantum numbers of **regular mesons**

- Light-flavor mesons (here S =strangeness)
 - Nonstrange mesons: $S = 0, I = 0, 1$
 - Strange mesons: $S = \pm 1, I = \frac{1}{2}$
- Open-flavor heavy mesons
 - $Q\bar{q}(q = u, d)$: $S = 0, I = 1/2$
 - $Q\bar{s}$: $S = 1, I = 0$
- Heavy quarkonia ($Q\bar{Q}$): $S = 0, I = 0$, **neutral**

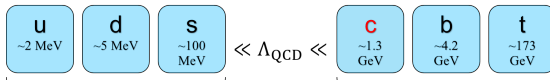


Charge, isospin, strangeness etc. which cannot be achieved in the $q\bar{q}$ and qqq scheme would be a smoking gun for an exotic nature

$$\mathcal{L}_{\text{QCD}} = \sum_{f=u,d,s,c,b,t} \bar{q}_f (i \not{D} - m_f) q_f - \frac{1}{4} G_{\mu\nu}^a G^{\mu\nu,a} + \frac{g_s^2 \theta}{64\pi^2} \epsilon^{\mu\nu\rho\sigma} G_{\mu\nu}^a G_{\rho\sigma}^a$$

- Exact: Lorentz-invariance, $\text{SU}(3)_c$ gauge, C (for $\theta = 0$ w/ real m_f , P , T as well)

- Approximate:



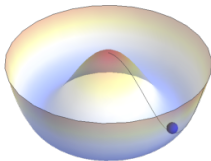
Light quarks

Heavy quarks

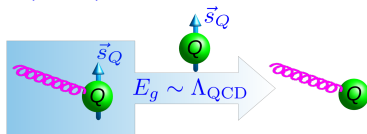
- ☞ Spontaneously broken **chiral**

symmetry:

$$\text{SU}(N_f)_L \times \text{SU}(N_f)_R \xrightarrow{\text{SSB}} \text{SU}(N_f)_V$$



- ☞ Heavy quark spin symmetry (HQSS)
- ☞ Heavy quark flavor symmetry (HQFS)
- ☞ Heavy quark-diquark symmetry (HADS)



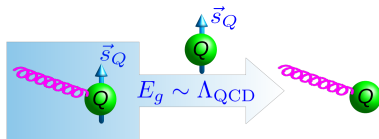
Heavy quark symmetries

- For heavy quarks (charm, bottom) in a hadron, typical momentum transfer Λ_{QCD}

☞ heavy quark spin symmetry (HQSS):

$$\text{chromomag. interaction} \propto \frac{\vec{\sigma} \cdot \vec{B}}{m_Q}$$

spin of the heavy quark decouples



Let total angular momentum $\mathbf{J} = \mathbf{s}_Q + \mathbf{s}_\ell$,

s_Q : heavy quark spin,

s_ℓ : spin of the light degrees of freedom (including orbital angular momentum)

✓ HQSS:

s_ℓ and s_Q are conserved separately in the heavy quark limit!

✓ spin multiplets:

for singly heavy mesons, e.g. $\{D, D^*\}$, $\{B, B^*\}$ with $s_\ell^P = \frac{1}{2}^-$;

for heavy quarkonia, e.g. S -wave: $\{\eta_c, J/\psi\}$, $\{\eta_b, \Upsilon\}$;

P -wave: $\{h_c, \chi_{c0,c1,c2}\}$, $\{h_b, \chi_{b0,b1,b2}\}$

- For heavy quarks (charm, bottom) in a hadron, typical momentum transfer Λ_{QCD}

👉 heavy quark flavor symmetry (HQFS) for any hadron containing **one** heavy quark:

velocity remains unchanged in the limit $m_Q \rightarrow \infty$: $\Delta v = \frac{\Delta p}{m_Q} = \frac{\Lambda_{\text{QCD}}}{m_Q}$

$\Rightarrow$ heavy quark is like a **static** color triplet source, m_Q is irrelevant

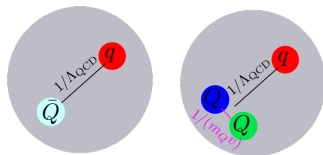
👉 heavy anti-quark–diquark symmetry

$m_{Q\bar{Q}} \gg \Lambda_{\text{QCD}}$,

the diquark serves as a **point-like color- $\bar{3}$** source, like a heavy anti-quark.

It relates doubly-heavy baryons to anti-heavy mesons

Savage, Wise (1990)



- In the heavy quark limit $m_Q \rightarrow \infty$, consider the quark propagator

$$i \frac{\not{p} + m_Q}{p^2 - m_Q^2 + i\epsilon} = i \frac{m_Q(\not{v} + 1) + \not{k}}{2m_Q v \cdot k + k^2 + i\epsilon} \xrightarrow{m_Q \rightarrow \infty} \frac{1 + \not{v}}{2} \frac{i}{v \cdot k + i\epsilon}$$

here $p = m_Q v + k$, with a residual momentum $k \sim \Lambda_{\text{QCD}}$.

- Decompose heavy quark field into v -dep. fields

$$Q(x) = e^{-im_Q v \cdot x} [Q_v(x) + q_v(x)]:$$

$$Q_v(x) = e^{im_Q v \cdot x} \frac{1 + \not{v}}{2} Q(x), \quad q_v(x) = e^{im_Q v \cdot x} \frac{1 - \not{v}}{2} Q(x)$$

- At leading order (LO) of the $1/m_Q$ expansion:

$$\mathcal{L}_Q = \bar{Q}(i\not{D} - m_Q)Q = \bar{Q}_v(iv \cdot D)Q_v + \mathcal{O}(m_Q^{-1})$$

👉 No Dirac gamma matrices in the LO Lagrangian, so **invariant under spin rotation** $\Rightarrow$ **HQSS**

👉 No heavy quark mass term $\Rightarrow$ **HQFS**

Applications to states of current interest

Examples of HQSS phenomenology:

- In the Review of Particle Physics (RPP) by the Particle Data Group (PDG), there are two D_1 ($J^P = 1^+$) mesons with very different widths

☞ $\Gamma[D_1(2420)] = (27.4 \pm 2.5) \text{ MeV} \ll \Gamma[D_1(2430)] = (384_{-110}^{+130}) \text{ MeV}$

☞ $s_\ell = s_q + \mathbf{L} \Rightarrow$ for P -wave charmed mesons: $s_\ell^P = \frac{1}{2}^+$ or $\frac{3}{2}^+$

☞ for decays $D_1 \rightarrow D^* \pi$:

$$\frac{1}{2}^+ \rightarrow \frac{1}{2}^- + 0^- \text{ in } S\text{-wave} \Rightarrow \text{large width}$$

$$\frac{3}{2}^+ \rightarrow \frac{1}{2}^- + 0^- \text{ in } D\text{-wave} \Rightarrow \text{small width}$$

☞ thus, dominant components: $D_1(2420)$: $s_\ell = \frac{3}{2}$, $D_1(2430)$: $s_\ell = \frac{1}{2}$

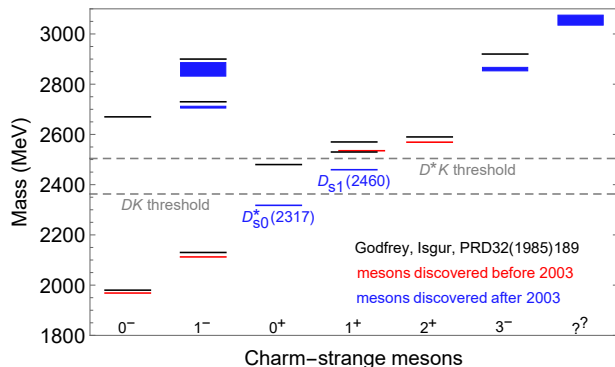
- Suppression of the S -wave production of $\frac{3}{2}^+ + \frac{1}{2}^-$ heavy meson pairs in e^+e^- annihilation

Table VI in E.Eichten et al., PRD17(1978)3090; X. Li, M. Voloshin, PRD88(2013)034012

Exercise: Try to understand this statement as a consequence of HQSS.

Hint: in e^+e^- collisions, the leading production mechanism of heavy meson pairs is from the vector current $\bar{Q}\gamma^\mu Q$ which couples to the virtual photon, *i.e.*, $e^+e^- \rightarrow \gamma^* \rightarrow \bar{Q}Q$ with the $Q\bar{Q}$ pair in an S -wave.

Charm-strange mesons (1)



- $D_{s0}^*(2317)$: BaBar (2003)
 $J^P = 0^+$, $\Gamma < 3.8$ MeV
- $D_{s1}(2460)$: CLEO (2003)
 $J^P = 1^+$, $\Gamma < 3.5$ MeV
- no isospin partner observed, tiny widths
 $\Rightarrow I = 0$

- **Mystery 1:** Mass problem: Why are $D_{s0}^*(2317)$ and $D_{s1}(2460)$ so light?
- **Mystery 2:** Naturalness problem:

$$\text{Why } \underbrace{M_{D_{s1}(2460)} - M_{D_{s0}^*(2317)}}_{(141.8 \pm 0.8) \text{ MeV}} \simeq \underbrace{M_{D^{*\pm}} - M_{D^{\pm}}}_{(140.67 \pm 0.08) \text{ MeV}} ?$$

Charm-strange mesons (2)

- HQFS: for a **singly**-heavy hadron, $M_{H_Q} = m_Q + A + \mathcal{O}(m_Q^{-1})$
- rough estimates of bottom analogues **whatever the D_{sJ} states**: Defining the spin-averaged mass for charmed mesons:

$$\bar{M}_c = \frac{1}{4} [M_{D_{s0}^*}(2317) + 3M_{D_{s1}}(2460)] \simeq 2.424 \text{ GeV, then}$$

$$M_{B_{s0}^*} = \bar{M}_c + \Delta_{b-c} + (M_{D_{s0}^*} - \bar{M}_c) \frac{m_c}{m_b} \simeq 5.71 \text{ GeV}$$

$$M_{B_{s1}} = \bar{M}_c + \Delta_{b-c} + (M_{D_{s1}} - \bar{M}_c) \frac{m_c}{m_b} \simeq 5.76 \text{ GeV}$$

here $\Delta_{b-c} \equiv m_b - m_c \simeq \bar{M}_{B_s} - \bar{M}_{D_s} \simeq 3.33 \text{ GeV}$, where

$\bar{M}_{B_s} = 5.403 \text{ GeV}$, $\bar{M}_{D_s} = 2.076 \text{ GeV}$: spin-averaged g.s. $Q\bar{s}$ meson masses

 comparing with the lattice QCD results:

Lang et al., PLB750(2015)17

$$M_{B_{s0}^*}^{\text{lat.}} = (5.711 \pm 0.013 \pm 0.019) \text{ GeV}$$

$$M_{B_{s1}}^{\text{lat.}} = (5.750 \pm 0.017 \pm 0.019) \text{ GeV}$$

 both to be discovered ¹

- more precise predictions can be made in a given model, e.g. **hadronic molecules**

¹The established meson $B_{s1}(5830)$ is probably the bottom partner of $D_{s1}(2536)$.

- In the hadronic molecular model, the main component: $D_{s0}^*(2317) : DK(I=0)$,
 $D_{s1}(2460) : D^*K(I=0)$
as a consequence of HQSS:

similar binding energies $M_D + M_K - M_{D_{s0}^*} \simeq 45 \text{ MeV}$

$$M_{D_{s1}(2460)} - M_{D_{s0}^*(2317)} \simeq M_{D^*} - M_D \text{ is natural}$$

- predicting the bottom-partner masses in one minute:

$$M_{B_{s0}^*} \simeq M_B + M_K - 45 \text{ MeV} \simeq 5.73 \text{ GeV}$$

$$M_{B_{s1}} \simeq M_{B^*} + M_K - 45 \text{ MeV} \simeq 5.78 \text{ GeV}$$

to be compared with lattice results for the lowest positive-parity bottom-strange mesons:

Lang, Mohler, Prelovsek, Woloshyn, PLB750(2015)17

$$M_{B_{s0}^*}^{\text{lat.}} = (5.711 \pm 0.013 \pm 0.019) \text{ GeV}$$

$$M_{B_{s1}}^{\text{lat.}} = (5.750 \pm 0.017 \pm 0.019) \text{ GeV}$$

Applications: from heavy baryons to doubly-heavy tetraquarks (1)

Development inspired by the LHCb discovery of the $\Xi_{cc}(3620)^{++}$

- Heavy antiquark-diquark symmetry (HADS):

repalcing $\bar{Q}$ in $\bar{Q}q$ by QQ [$\bar{3}_{\text{color}}$] $\Rightarrow QQq$;

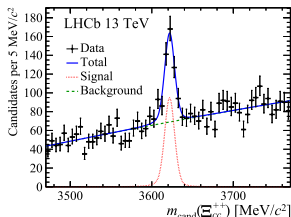
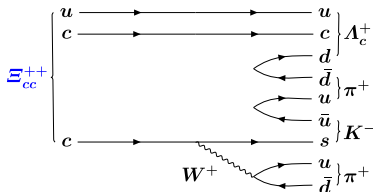
repalcing $\bar{Q}$ in $\bar{Q}\bar{q}\bar{q}$ by QQ [$\bar{3}_{\text{color}}$] $\Rightarrow QQ\bar{q}\bar{q}$;

$$\begin{aligned} \bar{Q}q &\Rightarrow QQq, & \bar{Q}\bar{q}\bar{q} &\Rightarrow QQ\bar{q}\bar{q} \\ \text{mass} \approx m_Q + A &\Rightarrow m_{QQ} + A, & m_Q + B &\Rightarrow m_{QQ} + B \end{aligned}$$

Prediction: $M_{QQ\bar{q}\bar{q}} - M_{\bar{Q}\bar{q}\bar{q}} \simeq M_{QQq} - M_{\bar{Q}q}$

- Doubly-charmed baryon discovered by LHCb

PRL119(2017)112001 [arXiv:1707.01621]



$M_{\Xi_{cc}^{++}} = (3621.40 \pm 0.78) \text{ MeV}$ can be used as input

Applications from heavy baryons to doubly-heavy tetraquarks (2)

TABLE II. Expectations for the ground-state tetraquark masses, in MeV.^a The column labeled “HQS relation” is the result of our heavy-quark symmetry relations and is explicitly given by the sum of the right-hand side of Eq. (1) and the kinetic-energy mass shifts of Eq. (7). Here q denotes an up or down quark. For stable tetraquark states the Q value is highlighted in a box.

State	J^P	j_ℓ	$m(Q_i Q_j q_m)$ (c.g.)	HQS relation	$m(Q_i Q_j \bar{q}_k \bar{q}_l)$	Decay channel	Q (MeV)
$\{cc\}[\bar{u}\bar{d}]$	1^+	0	3663 ^b	$m(\{cc\}u) + 315$	3978	$D^+ D^{*0}$ 3876	102
$\{cc\}[\bar{q}_k \bar{s}]$	1^+	0	3764 ^c	$m(\{cc\}s) + 392$	4156	$D^+ D_s^{*-}$ 3977	179
$\{cc\}[\bar{q}_k \bar{q}_l]$	$0^+, 1^+, 2^+$	1	3663	$m(\{cc\}u) + 526$	4146, 4167, 4210	$D^+ D^0, D^+ D^{*0}$ 3734, 3876	412, 292, 476
$[bc][\bar{u}\bar{d}]$	0^+	0	6914	$m([bc]u) + 315$	7229	$B^- D^+ / B^0 D^0$ 7146	83
$[bc][\bar{q}_k \bar{s}]$	0^+	0	7010 ^d	$m([bc]s) + 392$	7406	$B_s D$ 7236	170
$[bc][\bar{q}_k \bar{q}_l]$	1^+	1	6914	$m([bc]u) + 526$	7439	$B^* D / BD^*$ 7190/7290	249
$\{bc\}[\bar{u}\bar{d}]$	1^+	0	6957	$m(\{bc\}u) + 315$	7272	$B^* D / BD^*$ 7190/7290	82
$\{bc\}[\bar{q}_k \bar{s}]$	1^+	0	7053 ^d	$m(\{bc\}s) + 392$	7445	DB_s^* 7282	163
$\{bc\}[\bar{q}_k \bar{q}_l]$	$0^+, 1^+, 2^+$	1	6957	$m(\{bc\}u) + 526$	7461, 7472, 7493	$BD / B^* D$ 7146/7190	317, 282, 349
$\{bb\}[\bar{u}\bar{d}]$	1^+	0	10 176	$m(\{bb\}u) + 306$	10 482	$B^- \bar{B}^{*0}$ 10 603	-121
$\{bb\}[\bar{q}_k \bar{s}]$	1^+	0	10 252 ^c	$m(\{bb\}s) + 391$	10 643	$\bar{B} \bar{B}_s^* / \bar{B}_s \bar{B}^*$ 10 695/10 691	-48
$\{bb\}[\bar{q}_k \bar{q}_l]$	$0^+, 1^+, 2^+$	1	10 176	$m(\{bb\}u) + 512$	10 674, 10 681, 10 695	$B^- B^0, B^- B^{*0}$ 10 559, 10 603	115, 78, 136

^aMasses of the unobserved doubly heavy baryons are taken from Ref. [14]; for lattice evaluations of b -baryon masses, see Ref. [15].

^bBased on the mass of the LHCb Ξ_{cc}^{++} candidate, 3621.40 MeV, Ref. [10].

^cUsing the s/d mass differences of the corresponding heavy-light mesons.

^dEvaluated as $\frac{1}{2}[m(c\bar{s}) - m(c\bar{d}) + m(b\bar{s}) - m(b\bar{d})] + m(bcd)$.

Eichten, Quigg, PRL119(2017)202002

- HADS $\Rightarrow$ stable doubly-bottom tetraquarks (only decay weakly) are likely to exist

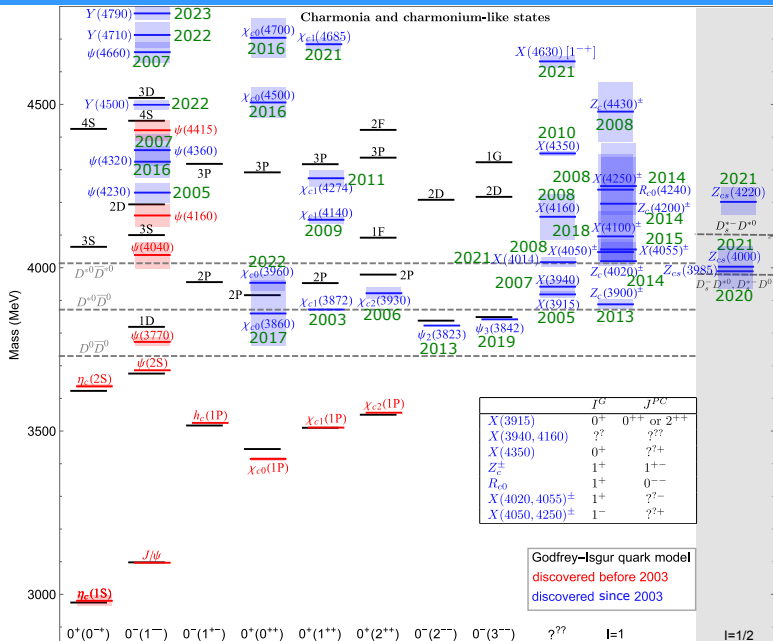
see also Carlson, Heller, Tjon, PRD37(1988)744; Manohar, Wise, NPB399(1993)17; Karliner, Rosner,

PRL119(2017)202001; Czarnecki, Leng, Voloshin, PLB778(2018)233; ...

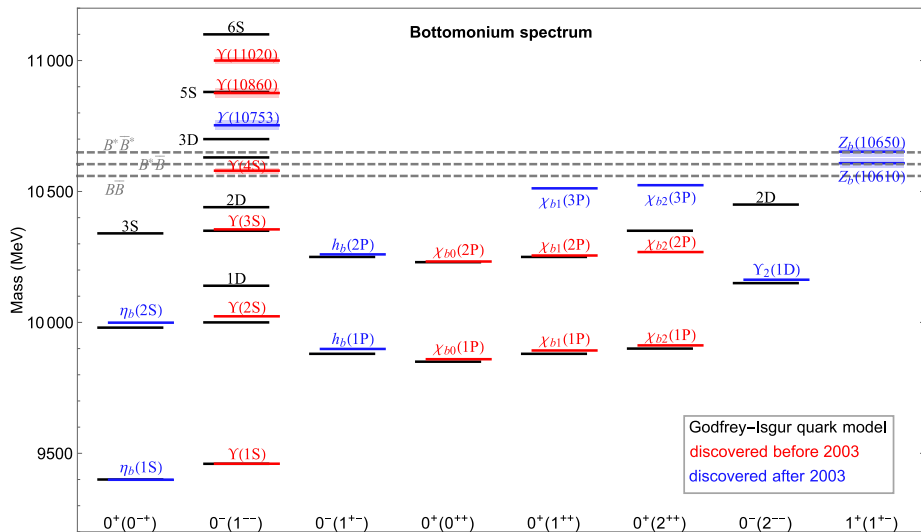
- support from lattice QCD Francis, Hudspith, Lewis, Maltman, PRL118(2017)142001; ...

- Possible detecting method: T. Gershon, A. Poluektov, *Displaced B_c mesons as an inclusive signature of weakly decaying double beauty hadrons*, JHEP 01 (2019) 019

Charmonium spectrum: current status



Bottomonium spectrum: current status



Naming convention

For states with properties in conflict with naive quark model (normally):

- X : $I = 0$, J^{PC} other than 1^{--} or unknown
- Y : $I = 0$, $J^{PC} = 1^{--}$
- Z : $I = 1$

PDG naming scheme:

$J^{PC} =$	$\begin{cases} 0^{-+} \\ 2^{-+} \\ \vdots \end{cases}$	$\begin{cases} 1^{+-} \\ 3^{+-} \\ \vdots \end{cases}$	$\begin{cases} 1^{--} \\ 2^{--} \\ \vdots \end{cases}$	$\begin{cases} 0^{++} \\ 1^{++} \\ \vdots \end{cases}$
Minimal quark content				
$u\bar{d}, w\bar{u} - d\bar{d}, d\bar{u}$ ($I = 1$)	π	b	ρ	a
$d\bar{d} + u\bar{u}$ ($I = 0$)	η, η'	h, h'	ω, ϕ	f, f'
and/or $s\bar{s}$				
$c\bar{c}$	η_c	h_c	$\psi^\dagger$	χ_c
$b\bar{b}$	η_b	h_b	Υ	χ_b
$I = 1$ with $c\bar{c}$	(Π_c)	Z_c	R_c	(W_c)
$I = 1$ with $b\bar{b}$	(Π_b)	Z_b	(R_b)	(W_b)

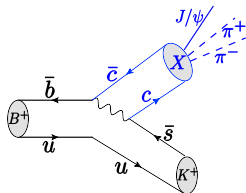
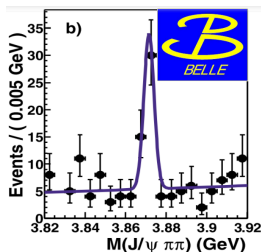
[†]The J/ψ remains the J/ψ .

“Young man, if I could remember the names of these particles, I would have been a botanist.”

— Enrico Fermi

$X(3872)$ (1)

Belle, PRL91(2003)262001 [hep-ex/0309032]



- The beginning of the XYZ story, discovered in $B^\pm \rightarrow K^\pm J/\psi \pi \pi$

$$M_X = (3871.69 \pm 0.17) \text{ MeV}$$

- $\Gamma < 1.2 \text{ MeV}$ Belle, PRD84(2011)052004
- Confirmed in many experiments: Belle, BaBar, BESIII, CDF, CMS, D0, LHCb, ...
- 10 years later, $J^{PC} = 1^{++}$

LHCb, PRL110(2013)222001

$\Rightarrow S$ -wave coupling to $D\bar{D}^*$

Mysterious properties:

- Mass coincides with the $D^0 \bar{D}^{*0}$ threshold:

$$M_{D^0} + M_{D^{*0}} - M_X = (0.01 \pm 0.14) \text{ MeV}$$

LHCb, PRD102(2020)092005

Mysterious properties (cont.):

- Large coupling to $D^0 \bar{D}^{*0}$:

$$\mathcal{B}(X \rightarrow D^0 \bar{D}^{*0}) > 30\% \quad \text{Belle, PRD81(2010)031103}$$

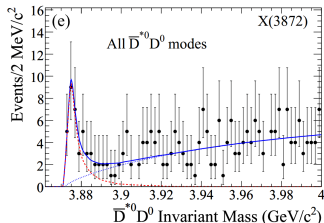
$$\mathcal{B}(X \rightarrow D^0 \bar{D}^0 \pi^0) > 40\% \quad \text{Belle, PRL97(2006)162002}$$

- No isospin partner observed $\Rightarrow I = 0$

but, large isospin breaking:

$$\frac{\mathcal{B}(X \rightarrow \omega J/\psi)}{\mathcal{B}(X \rightarrow \pi^+ \pi^- J/\psi)} = 0.8 \pm 0.3$$

$$C(X) = +, C(J/\psi) = - \Rightarrow C(\pi^+ \pi^-) = - \Rightarrow I(\pi^+ \pi^-) = 1$$



BaBar, PRD77(2008)011102

Exercise:

Why is the isospin of the negative C -parity $\pi^+ \pi^-$ system equal to 1?

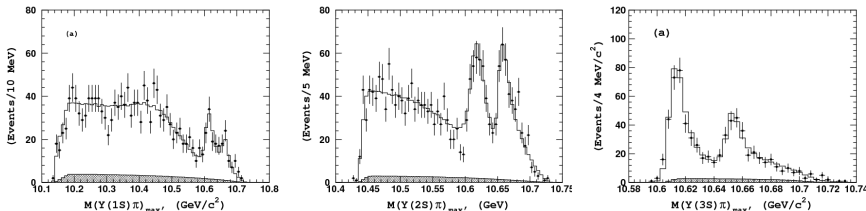
More exotic structures: $Z_c^\pm$ and $Z_b^\pm$ with **hidden** $Q\bar{Q}$

- $Z_c^\pm, Z_b^\pm$: **charged** structures in **heavy quarkonium** mass region, $Q\bar{Q}\bar{d}u, Q\bar{Q}\bar{u}d$
 $Z_c(3900), Z_c(4020), Z_c(4200), Z_c(4430), \dots$

- $Z_b(10610)$ and $Z_b(10650)$ (the latter also called Z_b' sometimes):

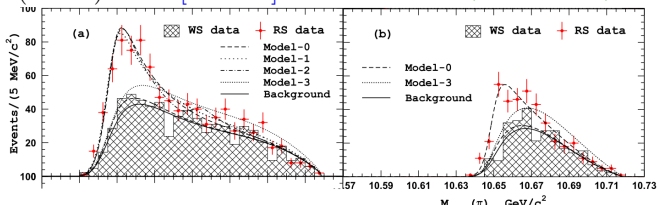
Belle, arXiv:1105.4583; PRL108(2012)122001

observed in $\Upsilon(10860) \rightarrow \pi^\mp [\pi^\pm \Upsilon(1S, 2S, 3S)/h_b(1P, 2P)]$



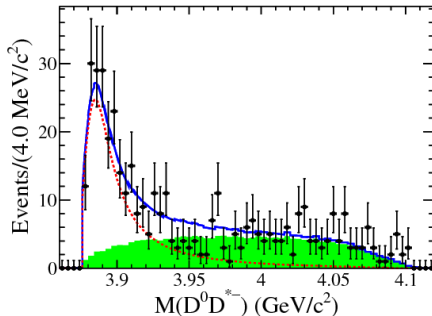
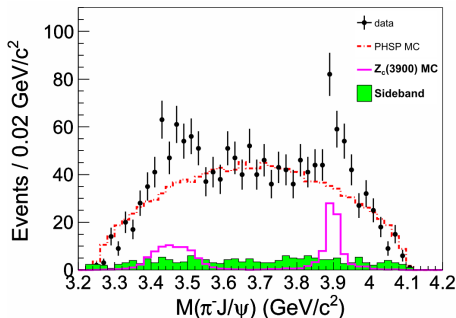
also in $\Upsilon(10860) \rightarrow \pi^\mp [B^{(*)}\bar{B}^*]^\pm$

Belle, arXiv:1209.6450; PRL116(2016)212001



$Z_c^\pm$ and $Z_b^\pm$ with **hidden** $Q\bar{Q}$ (2)

- $Z_c(3900/3885)^\pm$: structure around 3.9 GeV seen in $J/\psi\pi$ by BESIII and Belle in
 $Y(4260) \rightarrow J/\psi\pi^+\pi^-$, BESIII, PRL110(2013)252001; Belle, PRL110(2013)252002
 and in $D\bar{D}^*$ by BESIII in $Y(4260) \rightarrow \pi^\pm(D\bar{D}^*)^\mp$ BESIII, PRD92(2015)092006



- $Z_c(4020)^\pm$ observed in $h_c\pi^\pm$ and $(\bar{D}^*D^*)^\pm$ distributions

BESIII, PRL111(2013)242001; PRL112(2014)132001

- Hadronic molecular model: $X(3872): D\bar{D}^*$; $Z_b(10610, 10650): B\bar{B}^*$ and $B^*\bar{B}^*$
- Consider S -wave interaction between a pair of $s_\ell^P = \frac{1}{2}^-$ (anti-)heavy mesons:

$$0^{++} : D\bar{D}, \quad D^*\bar{D}^*$$

$$1^{+-} : \frac{1}{\sqrt{2}} (D\bar{D}^* + D^*\bar{D}), \quad D^*\bar{D}^*$$

$$1^{++} : \frac{1}{\sqrt{2}} (D\bar{D}^* - D^*\bar{D}); \quad 2^{++} : D^*\bar{D}^*$$

here, phase convention: $D \xrightarrow{C} +\bar{D}$, $D^* \xrightarrow{C} -\bar{D}^*$

- Heavy quark spin irrelevant $\Rightarrow$ interaction matrix elements:

$$\begin{aligned} & \left\langle s_{1c}, s_{2c}, s_{c\bar{c}}; s_{1\ell}, s_{2\ell}, s_L; J \left| \hat{H} \right| s'_{1c}, s'_{2c}, s'_{c\bar{c}}; s'_{1\ell}, s'_{2\ell}, s'_L; J' \right\rangle \\ &= \left\langle s_{1\ell}, s_{2\ell}, s_L \left| \hat{H} \right| s'_{1\ell}, s'_{2\ell}, s'_L \right\rangle \delta_{s_{c\bar{c}}, s'_{c\bar{c}}} \delta_{s_L, s'_L} \delta_{JJ'} \end{aligned}$$

For each isospin, 2 independent terms

$$\left\langle \frac{1}{2}, \frac{1}{2}, 0 \left| \hat{H} \right| \frac{1}{2}, \frac{1}{2}, 0 \right\rangle, \quad \left\langle \frac{1}{2}, \frac{1}{2}, 1 \left| \hat{H} \right| \frac{1}{2}, \frac{1}{2}, 1 \right\rangle$$

$\Rightarrow$ 6 pairs grouped in 2 multiplets with $s_L = 0$ and 1, respectively

- For the HQSS consequences, convenient to use the basis of states: $s_L^{PC} \otimes s_{c\bar{c}}^{PC}$
 - S -wave: $s_L^{PC}, s_{c\bar{c}}^{PC} = 0^{-+}$ or 1^{--}
 - multiplet with $s_L = 0$:

$$0_L^{-+} \otimes 0_{c\bar{c}}^{-+} = 0^{++}, \quad 0_L^{-+} \otimes 1_{c\bar{c}}^{--} = 1^{+-}$$

- multiplet with $s_L = 1$:

$$1_L^{--} \otimes 0_{c\bar{c}}^{-+} = 1^{+-}, \quad 1_L^{--} \otimes 1_{c\bar{c}}^{--} = 0^{++} \oplus \boxed{1^{++}} \oplus 2^{++}$$

- Multiplets in strict heavy quark limit:

- $X(3872)$ has three partners with 0^{++} , 2^{++} and 1^{+-}

Hidalgo-Duque et al., PLB727(2013)432; Baru et al., PLB763(2016)20

- Z_b, Z'_b as $B^{(*)}\bar{B}^*$ molecules would imply 6 $I = 1$ hadronic molecules:

$$Z_b[1^{+-}], Z'_b[1^{+-}] \text{ and } W_{b0}[0^{++}], W'_{b0}[0^{++}], W_{b1}[1^{++}] \text{ and } W_{b2}[2^{++}]$$

Bondar et al., PRD84(2011)054010; Voloshin, PRD84(2011)031502;

Mehen, Powell, PRD84(2011)114013

- Recall the exercise on page 50:

Is $\Upsilon\pi^+\pi^-$ a good choice of final states for the search of X_b , the $J^{PC} = 1^{++}$ bottom analogue of the $X(3872)$?

Answer: No. $X_b \rightarrow \Upsilon\pi\pi$ breaks isospin symmetry

FKG, Hidalgo-Duque, Nieves, Valderrama, PRD88(2013)054007; Karliner, Rosner, PRD91(2015)014014

$$M_{B^0} - M_{B^\pm} = (0.31 \pm 0.06) \text{ MeV} \quad [M_{D^\pm} - M_{D^0} = (4.822 \pm 0.015) \text{ MeV}]$$

- Negative results:

CMS, *Search for a new bottomonium state decaying to $\Upsilon(1S)\pi^+\pi^-$ in pp collisions at $\sqrt{s} = 8 \text{ TeV}$* , PLB727(2013)57;

ATLAS, *Search for the X_b and other hidden-beauty states in the $\pi^+\pi^-\Upsilon(1S)$ channel at ATLAS*, PLB740(2015)199

- The results can be reinterpreted as for the search of W_{bJ} ($I = 1, J^{++}$)

$$1_L^{--} \otimes 1_{c\bar{c}}^{--} = 0^{++} \oplus \boxed{1^{++}} \oplus 2^{++}$$

- Heavy quark spin selection rule for $X(3872)$:
for $X(3872)$ being a $1^{++} D\bar{D}^*$ molecule, $s_L = 1$, $s_{c\bar{c}} = 1$
- spin structure of $Q\bar{Q}$:

Voloshin, PLB604(2004)69

	s_L	$s_{c\bar{c}}$	J^{PC}	$c\bar{c}$
S -wave	0	0	0^{-+}	η_c
	0	1	1^{--}	J/ψ
P -wave	1	0	1^{+-}	h_c
	1	1	$(0, 1, 2)^{++}$	$\chi_{c0}, \chi_{c1}, \chi_{c2}$

- allowed: $X(3872) \rightarrow J/\psi\pi\pi$, $X(3872) \rightarrow \chi_{cJ}\pi$, $X(3872) \rightarrow \chi_{cJ}\pi\pi$
suppressed: $X(3872) \rightarrow \eta_c\pi\pi$, $X(3872) \rightarrow h_c\pi\pi$
 - Interesting feature of $Z_b^{(')}$: observed with similar rates in both $\Upsilon\pi\pi[s_{b\bar{b}} = 1]$ and $h_b\pi\pi[s_{b\bar{b}} = 0]$
Bondar, Garmash, Milstein, Mizuk, Voloshin, PRD84(2011)054010
- $$Z_b \sim B\bar{B}^* \sim 0_{b\bar{b}}^- \otimes 1_{q\bar{q}}^- - 1_{b\bar{b}}^- \otimes 0_{q\bar{q}}^-, \quad Z_b' \sim B^*\bar{B}^* \sim 0_{b\bar{b}}^- \otimes 1_{q\bar{q}}^- + 1_{b\bar{b}}^- \otimes 0_{q\bar{q}}^-$$

unitary transformation from two-meson basis to $|\mathbf{s}_{1c}, \mathbf{s}_{2c}, s_{c\bar{c}}; \mathbf{s}_{1\ell}, \mathbf{s}_{2\ell}, \mathbf{s}_L; J\rangle$:

$$|\mathbf{s}_{1c}, \mathbf{s}_{1\ell}, j_1; \mathbf{s}_{2c}, \mathbf{s}_{2\ell}, j_2; J\rangle = \sum_{s_{c\bar{c}}, s_L} \sqrt{(2j_1+1)(2j_2+1)(2s_{c\bar{c}}+1)(2s_L+1)} \\ \times \begin{Bmatrix} s_{1c} & s_{2c} & s_{c\bar{c}} \\ s_{1\ell} & s_{2\ell} & s_L \\ j_1 & j_2 & J \end{Bmatrix} |\mathbf{s}_{1c}, \mathbf{s}_{2c}, s_{c\bar{c}}; \mathbf{s}_{1\ell}, \mathbf{s}_{2\ell}, \mathbf{s}_L; J\rangle$$

$j_{1,2}$: meson spins;

J : the total angular momentum of the whole system

$s_{1c}(2c) = \frac{1}{2}$: spin of the heavy quark in meson 1 (2)

$s_{1\ell}(2\ell) = \frac{1}{2}$: angular momentum of the light quarks in meson 1 (2)

- $s_{c\bar{c}} = 0, 1$: total spin of $c\bar{c}$, conserved but decoupled
- $s_L = 0, 1$: total angular momentum of the light-quark system, conserved
- only two independent $\langle s_{\ell 1}, s_{\ell 2}, s_L | \hat{\mathcal{H}} | s'_{\ell 1}, s'_{\ell 2}, s_L \rangle_I$ terms for each isospin I :

$$F_{I0} = \left\langle \frac{1}{2}, \frac{1}{2}, 0 \left| \hat{\mathcal{H}} \right| \frac{1}{2}, \frac{1}{2}, 0 \right\rangle_I, \quad F_{I1} = \left\langle \frac{1}{2}, \frac{1}{2}, 1 \left| \hat{\mathcal{H}} \right| \frac{1}{2}, \frac{1}{2}, 1 \right\rangle_I$$

Exercise:

Show that the combinations of the LO contact terms in the S -wave interaction matrix elements for $D^{(*)}\bar{D}^{(*)}$ are as follows:

$$\begin{aligned} \begin{pmatrix} D\bar{D} \\ D^*\bar{D}^* \end{pmatrix} : \quad V^{(0^{++})} &= \begin{pmatrix} C_{IA} & \sqrt{3}C_{IB} \\ \sqrt{3}C_{IB} & C_{IA} - 2C_{IB} \end{pmatrix}, \\ \begin{pmatrix} D\bar{D}^* \\ D^*\bar{D}^* \end{pmatrix} : \quad V^{(1^{+-})} &= \begin{pmatrix} C_{IA} - C_{IB} & 2C_{IB} \\ 2C_{IB} & C_{IA} - C_{IB} \end{pmatrix}, \\ D\bar{D}^* : \quad V^{(1^{++})} &= C_{IA} + C_{IB}, \\ D^*\bar{D}^* : \quad V^{(2^{++})} &= C_{IA} + C_{IB}, \end{aligned}$$

here, $C_{IA} = \frac{1}{4}(3F_{I1} + F_{I0})$, $C_{IB} = \frac{1}{4}(F_{I1} - F_{I0})$

- This would suggest spin multiplets. Good candidates:

☞ $X(3872)$ and $X_2(4013)$ (not observed yet!); $Z_c(3900)$ and $Z_c(4020)$
 Nieves, Valderrama, PRD86(2012)056004; ...

$$M_{X_2(4013)} - M_{X(3872)} \approx M_{Z_c(4020)} - M_{Z_c(3900)} \approx M_{D^*} - M_D$$

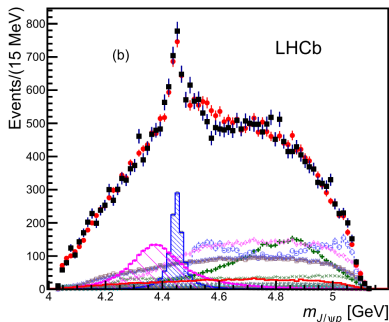
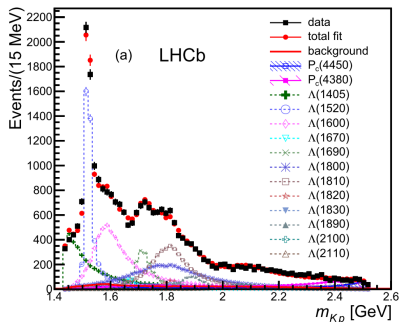
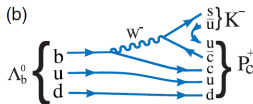
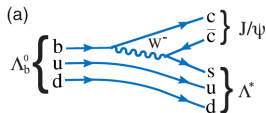
☞ $Z_b(10610)$ and $Z_b(10650)$: Bondar et al., PRD84(2011)054010; ...

$$M_{Z_b(10650)} - M_{Z_b(10610)} \approx M_{B^*} - M_B$$

☞ Z_c and Z_b states need a suppression of coupled-channel effect (reason?)

Discovered in $\Lambda_b^0 \rightarrow J/\psi p K^-$

LHCb, PRL115(2015)072001 [arXiv:1507.03414]



Two Breit–Wigner resonances needed:

$$M_1 = (4380 \pm 8 \pm 29) \text{ MeV},$$

$$M_2 = (4449.8 \pm 1.7 \pm 2.5) \text{ MeV},$$

$$\Gamma_1 = (205 \pm 18 \pm 86) \text{ MeV},$$

$$\Gamma_2 = (39 \pm 5 \pm 19) \text{ MeV}.$$

- In $J/\psi p$ invariant mass distribution, with hidden charm
 $\Rightarrow$ pentaquarks if they are hadron resonances

- Quantum numbers not fully determined, for ($P_c(4380)$, $P_c(4450)$):
 $(3/2^-, 5/2^+)$, $(3/2^+, 5/2^-)$, $(5/2^+, 3/2^-)$, ...

LHCb, PRL115(2015)072001

From a reanalysis using an extended Λ^* model:

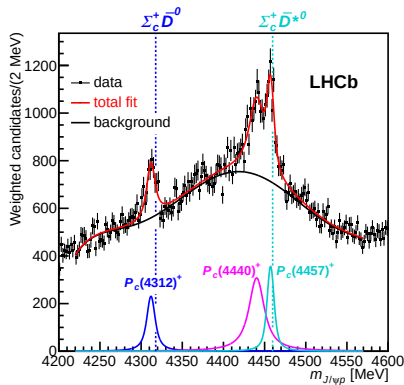
N. Jurik, CERN-THESIS-2016-086

$J^p(4380, 4450)$	$(\sqrt{\Delta(-2 \ln \mathcal{L})})^2$	$P_c(4380)$		$P_c(4450)$	
		M_0	Γ_0	M_0	Γ_0
$(3/2^-, 5/2^+)$ solution					
$3/2^-, 5/2^+$	--	4359	151	4450.1	49
Δ from $(3/2^-, 5/2^+)$ solution					
$5/2^+, 3/2^-$	-3.6^2	10	-7	-1.6	-6
$5/2^-, 3/2^+$	-2.7^2	-4	-9	-3.6	-2
$3/2^-, 5/2^+$	-	-	-	-	-

- Early prediction:

Prediction of narrow N^ and Λ^* resonances with hidden charm above 4 GeV,*

J.-J. Wu, R. Molina, E. Oset, B.-S. Zou, PRL105(2010)232001



State	M [MeV]	Γ [MeV]	(95% CL)	$\mathcal{R}$ [%]
$P_c(4312)^+$	$4311.9 \pm 0.7^{+6.8}_{-0.6}$	$9.8 \pm 2.7^{+3.7}_{-4.5}$	(< 27)	$0.30 \pm 0.07^{+0.34}_{-0.09}$
$P_c(4440)^+$	$4440.3 \pm 1.3^{+4.1}_{-4.7}$	$20.6 \pm 4.9^{+8.7}_{-10.1}$	(< 49)	$1.11 \pm 0.33^{+0.22}_{-0.10}$
$P_c(4457)^+$	$4457.3 \pm 0.6^{+4.1}_{-1.7}$	$6.4 \pm 2.0^{+5.7}_{-1.9}$	(< 20)	$0.53 \pm 0.16^{+0.15}_{-0.13}$

$$\mathcal{R} \equiv \mathcal{B}(\Lambda_b^0 \rightarrow P_c^+ K^-) \mathcal{B}(P_c^+ \rightarrow J/\psi p) / \mathcal{B}(\Lambda_b^0 \rightarrow J/\psi p K^-)$$

The LHCb P_c states might be $\Sigma_c^{(*)} \bar{D}^{(*)}$ molecules predicted in Wu, Molina, Oset, Zou (2010)

$$P_c(4312) \sim \Sigma_c \bar{D}, P_c(4440, 4457) \sim \Sigma_c \bar{D}^*$$

Consider S -wave pairs of $\Sigma_c^{(*)} \bar{D}^{(*)}$ [$J_{\Sigma_c} = \frac{1}{2}, J_{\Sigma_c^*} = \frac{3}{2}$]:

$$J^P = \frac{1}{2}^- : \Sigma_c \bar{D}, \Sigma_c \bar{D}^*, \Sigma_c^* \bar{D}^*$$

$$J^P = \frac{3}{2}^- : \Sigma_c^* \bar{D}, \Sigma_c \bar{D}^*, \Sigma_c^* \bar{D}^*$$

$$J^P = \frac{5}{2}^- : \Sigma_c^* \bar{D}^*$$

Spin of the light degrees of freedom s_ℓ : $s_\ell(D^{(*)}) = \frac{1}{2}$, $s_\ell(\Sigma_c^{(*)}) = 1$. Thus, $s_L = \frac{1}{2}, \frac{3}{2}$

For each isospin, 2 independent terms

$$\left\langle 1, \frac{1}{2}, \frac{1}{2} \left| \hat{\mathcal{H}} \right| 1, \frac{1}{2}, \frac{1}{2} \right\rangle, \quad \left\langle 1, \frac{1}{2}, \frac{3}{2} \left| \hat{\mathcal{H}} \right| 1, \frac{1}{2}, \frac{3}{2} \right\rangle$$

Thus, the 7 pairs are in two spin multiplets: 3 with $s_L = \frac{1}{2}$ and 4 with $s_L = \frac{3}{2}$

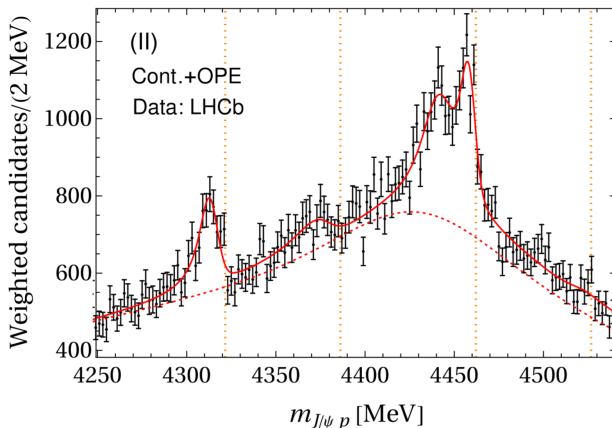
Seven P_c generally expected in this hadronic molecular model Xiao, Nieves, Oset (2013); Liu et al. (2018, 2019); Sakai et al. (2019); ...

Predictions using the masses of $P_c(4440, 4457)$ as inputs Liu et al., PRL122(2019)242001

Scenario	Molecule	J^P	B (MeV)	M (MeV)
A	$\bar{D}\Sigma_c$	$\frac{1}{2}^-$	7.8 – 9.0	4311.8 – 4313.0
A	$\bar{D}\Sigma_c^*$	$\frac{3}{2}^-$	8.3 – 9.2	4376.1 – 4377.0
A	$\bar{D}^*\Sigma_c$	$\frac{1}{2}^-$	Input	4440.3
A	$\bar{D}^*\Sigma_c$	$\frac{3}{2}^-$	Input	4457.3
A	$\bar{D}^*\Sigma_c^*$	$\frac{1}{2}^-$	25.7 – 26.5	4500.2 – 4501.0
A	$\bar{D}^*\Sigma_c^*$	$\frac{3}{2}^-$	15.9 – 16.1	4510.6 – 4510.8
A	$\bar{D}^*\Sigma_c^*$	$\frac{5}{2}^-$	3.2 – 3.5	4523.3 – 4523.6
B	$\bar{D}\Sigma_c$	$\frac{1}{2}^-$	13.1 – 14.5	4306.3 – 4307.7
B	$\bar{D}\Sigma_c^*$	$\frac{3}{2}^-$	13.6 – 14.8	4370.5 – 4371.7
B	$\bar{D}^*\Sigma_c$	$\frac{1}{2}^-$	Input	4457.3
B	$\bar{D}^*\Sigma_c$	$\frac{3}{2}^-$	Input	4440.3
B	$\bar{D}^*\Sigma_c^*$	$\frac{1}{2}^-$	3.1 – 3.5	4523.2 – 4523.6
B	$\bar{D}^*\Sigma_c^*$	$\frac{3}{2}^-$	10.1 – 10.2	4516.5 – 4516.6
B	$\bar{D}^*\Sigma_c^*$	$\frac{5}{2}^-$	25.7 – 26.5	4500.2 – 4501.0

Fit to the LHCb measured $J/\psi p$ invariant mass distribution using hadronic molecular model with HQSS

M.-L. Du, V. Baru, F.-K. Guo, C. Hanhart, U.-G. Meißner, J. A. Oller, Q. Wang, PRL124(2020)072001

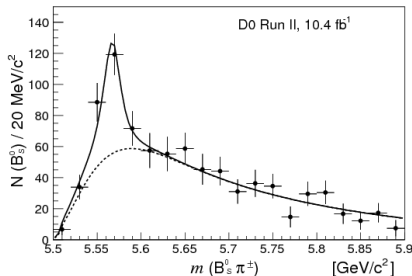


Scenario B in the previous page is favored in the fit after considering the one-pion exchange.

Explicitly exotic: $X(5568)$ (1)

- $X(5568)$ by D0 Collaboration ($p\bar{p}$ collisions)

PRL117(2016)022003



$$M = (5567.8 \pm 2.9_{-1.9}^{+0.9}) \text{ MeV}$$

$$\Gamma = (21.9 \pm 6.4_{-2.5}^{+5.0}) \text{ MeV}$$

- Observed in $B_s^{(*)0} \pi^+$, sizeable width
 $\Rightarrow I = 1$:
minimal quark contents is $\bar{b}s\bar{d}u$!
- a favorite multiquark candidate:
explicitly flavor exotic, minimal number
of quarks ≥ 4

Estimate of isospin breaking decay width:

$$\Gamma_I \sim \left(\left(\frac{m_d - m_u}{\Lambda_{\text{QCD}}} \right)^2 \right) \times \mathcal{O}(100 \text{ MeV})$$
$$= \mathcal{O}(10 \text{ keV})$$

$X(5568)$ (2)

FKG, Meißner, Zou, *How the $X(5568)$ challenges our understanding of QCD*, Commun.Theor.Phys. 65 (2016) 593

- **mass too low** for $X(5568)$ to be a $\bar{b}s\bar{u}d$: $M \simeq M_{B_s} + 200 \text{ MeV}$
 - ☞ $M_\pi \simeq 140 \text{ MeV}$ because **pions are pseudo-Goldstone bosons**
 - ☞ For any matter field: $M_R \gg M_\pi$; we expect $M_{\bar{q}q} \sim M_R \gtrsim M_\sigma$

$$M_{\bar{b}s\bar{u}d} \gtrsim M_{B_s} + 500 \text{ MeV} \sim 5.9 \text{ GeV}$$

- **HQFS** predicts an isovector X_c :

$$M_{X_c} = M_{X(5568)} - \Delta_{b-c} + \mathcal{O}\left(\Lambda_{\text{QCD}}^2 \left(\frac{1}{m_c} - \frac{1}{m_b}\right)\right) \simeq (2.24 \pm 0.15) \text{ GeV}$$

but in $D_s\pi$, only the **isoscalar** $D_{s0}^*(2317)$ was observed!

BaBar (2003)

- ☞ negative results reported by LHCb,
by CMS,
by CDF,
by ATLAS

LHCb, PRL117(2016)152003

CMS, PRL120(2018)202005

CDF, PRL120(2018)202006

ATLAS, PRL120(2018)202007

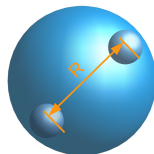
Hadronic molecules

FKG, C. Hanhart, U.-G. Meißner, Q. Wang, Q. Zhao, B.-S. Zou, *Hadronic molecules*, Rev. Mod. Phys. 90 (2018) 015004

- Hadronic molecule:
dominant component is a composite state of 2 or more hadrons
- **Concept at large distances**, so that can be approximated by system of multi-hadrons **at low energies**

Consider a 2-body bound state with a mass $M = m_1 + m_2 - E_B$

size: $\sim \frac{1}{\sqrt{2\mu E_B}} \gg r_{\text{hadron}}$



- **scale separation** $\Rightarrow$ (nonrelativistic) EFT applicable!
- Only **narrow** hadrons can be considered as components of hadronic molecules,
 $\Gamma_h \ll 1/r$, r : range of forces

Filin *et al.*, PRL105(2010)019101; FKG, Meißner, PRD84(2011)014013

Compositeness (1)

S. Weinberg, PR137(1965)B672; V. Baru *et al.*, PLB586(2004)53; T. Hyodo, IJMPA28(2013)1330045, ...

Model-independent result for *S*-wave loosely bound composite states:

Consider a system with Hamiltonian

$$\mathcal{H} = \mathcal{H}_0 + V$$

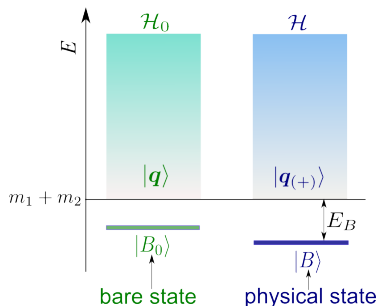
$\mathcal{H}_0$: free Hamiltonian, V : interaction potential

- Compositeness:**

the probability of finding the physical state $|B\rangle$ in the 2-body continuum $|q\rangle$

$$1 - Z = \int \frac{d^3\mathbf{q}}{(2\pi)^3} |\langle q|B\rangle|^2$$

- $Z = |\langle B_0|B\rangle|^2$, $0 \leq (1 - Z) \leq 1$
 - $Z = 0$: pure bound (composite) state
 - $Z = 1$: pure elementary state



Compositeness : $1 - Z = \int \frac{d^3 \mathbf{q}}{(2\pi)^3} |\langle \mathbf{q} | B \rangle|^2$

- Schrödinger equation

$$(\mathcal{H}_0 + V)|B\rangle = -E_B|B\rangle$$

multiplying by $\langle \mathbf{q} |$ and using $\mathcal{H}_0|\mathbf{q}\rangle = \frac{\mathbf{q}^2}{2\mu}|\mathbf{q}\rangle$:
 $\Rightarrow$ momentum-space wave function:

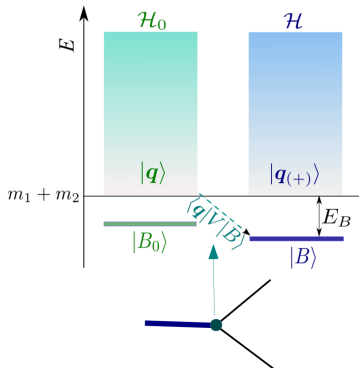
$$\langle \mathbf{q} | B \rangle = -\frac{\langle \mathbf{q} | V | B \rangle}{E_B + \mathbf{q}^2/(2\mu)}$$

- S*-wave, small binding energy so that $R = 1/\sqrt{2\mu E_B} \gg r$, r : range of forces

$$\langle \mathbf{q} | V | B \rangle = g_{\text{NR}} [1 + \mathcal{O}(r/R)]$$

- Compositeness:

$$1 - Z = \int \frac{d^3 \mathbf{q}}{(2\pi)^3} \frac{g_{\text{NR}}^2}{[E_B + \mathbf{q}^2/(2\mu)]^2} \left[1 + \mathcal{O}\left(\frac{r}{R}\right) \right] = \frac{\mu^2 g_{\text{NR}}^2}{2\pi \sqrt{2\mu E_B}} \left[1 + \mathcal{O}\left(\frac{r}{R}\right) \right]$$



- Coupling constant measures the compositeness** for an S -wave shallow bound state

$$g_{\text{NR}}^2 \approx (1 - Z) \frac{2\pi}{\mu^2} \sqrt{2\mu E_B} \leq \frac{2\pi}{\mu^2} \sqrt{2\mu E_B}$$

bounded from the above

It can be shown that g_{NR}^2 is the residue of the T -matrix element at the pole $E = -E_B$ ($E \equiv \sqrt{s} - m_1 - m_2$):

$$g_{\text{NR}}^2 = \lim_{E \rightarrow -E_B} (E + E_B) \langle \mathbf{k} | T_{\text{NR}} | \mathbf{k} \rangle$$

here nonrelativistic normalization is used, comparing with the T -matrix using relativistic normalization in the first part of this lecture (also a sign difference in the definition), $T_{\text{NR}} = -\frac{T}{4\mu\sqrt{s}} \simeq -\frac{T}{4m_1 m_2}$.

Hint: use the Lippmann–Schwinger equation $T_{\text{NR}} = V + V \frac{1}{E - \mathcal{H}_0 + i\epsilon} T_{\text{NR}}$ and the completeness relation $|B\rangle\langle B| + \int \frac{d^3q}{(2\pi)^3} |\mathbf{q}_{(+)}\rangle\langle \mathbf{q}_{(+)}| = 1$ to derive the Low equation (noticing $T_{\text{NR}}|\mathbf{q}\rangle = V|\mathbf{q}_{(+)}\rangle$):

$$\langle \mathbf{k}' | T_{\text{NR}} | \mathbf{k} \rangle = \langle \mathbf{k}' | V | \mathbf{k} \rangle + \frac{\langle \mathbf{k}' | V | B \rangle \langle B | V | \mathbf{k} \rangle}{E + E_B + i\epsilon} + \int \frac{d^3q}{(2\pi)^3} \frac{\langle \mathbf{k}' | T_{\text{NR}} | \mathbf{q} \rangle \langle \mathbf{q} | T_{\text{NR}}^\dagger | \mathbf{k} \rangle}{E - \mathbf{q}^2/(2\mu) + i\epsilon}$$

- Z can be related to scattering length a and effective range r_e

Weinberg (1965)

$$a_0 = \frac{2R(1-Z)}{2-Z} \left[1 + \mathcal{O}\left(\frac{r}{R}\right) \right], \quad r_{e0} = -\frac{RZ}{1-Z} \left[1 + \mathcal{O}\left(\frac{r}{R}\right) \right]$$

Effective range expansion (S -wave): $f_0^{-1}(k) = -1/a_0 + r_{e0}k^2/2 - ik + \mathcal{O}(k^4)$

Derivation:

$$T_{\text{NR}}(E) \equiv \langle k | T_{\text{NR}} | k \rangle = -\frac{2\pi}{\mu} f_0(k) \Rightarrow \text{Im } T_{\text{NR}}^{-1}(E) = \frac{\mu}{2\pi} \sqrt{2\mu E} \theta(E)$$

Twice-subtracted dispersion relation for $t^{-1}(E)$

$$\begin{aligned} T_{\text{NR}}^{-1}(E) &= \frac{E + E_B}{g_{\text{NR}}^2} + \frac{(E + E_B)^2}{\pi} \int_0^{+\infty} dw \frac{\text{Im } T_{\text{NR}}^{-1}(w)}{(w - E - i\epsilon)(w + E_B)^2} \\ &= \frac{E + E_B}{g_{\text{NR}}^2} + \frac{\mu R}{4\pi} \left(\frac{1}{R} - \sqrt{-2\mu E - i\epsilon} \right)^2 \end{aligned}$$

- Purely composite: $Z = 0 \Rightarrow a_0 = \frac{1}{\sqrt{2\mu E_B}}, r_{e0} = 0$
- Purely elementary: $Z = 1 \Rightarrow a_0 = 0, r_{e0} = -\infty$

- Classic example: deuteron as pn bound state. Exp.: $E_B = 2.2$ MeV,
 $a_0[{}^3S_1] = 5.4$ fm, $r_{e0}[{}^3S_1] = 1.8$ fm

$$a_{Z=1} \simeq 0 \text{ fm}, \quad a_{Z=0} \simeq 4.3 \text{ fm}$$

- However, problematic for systems with **positive effective range** I. Matuschek, V. Baru, FKG, C. Hanhart, EPJA 57 (2021) 101; Y. Li, FKG, J.-Y. Pang, J.-J. Wu, PRD 105 (2022) L071502

$r_{e0} > 0 \Rightarrow Z \notin (0, 1)$, thus Z or $1 - Z = \sqrt{\frac{a_0}{a_0 + 2r_{e0}}}$ in Weinberg's relations loses a probability interpretation

A generalization is suggested in Y. Li, FKG, J.-Y. Pang, J.-J. Wu, PRD 105 (2022) L071502

$$Z = \exp \left(\frac{1}{\pi} \int_0^\infty dE \frac{\delta_0(E)}{E - E_B} \right) \in [0, 1]$$

which reduces to Weinberg's relations for $r_{e0} < 0$.

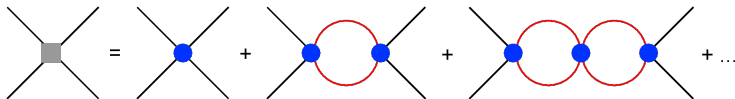
We consider a system of two particles of masses m_1, m_2

- in the near-threshold region, a **momentum expansion** for the interactions with the LO being a constant

$$\mathcal{L} = \sum_{i=1,2} \phi_i^\dagger \left(i\partial_0 - m_i + \frac{\nabla^2}{2m_i} \right) \phi_i - C_0 \phi_1^\dagger \phi_2^\dagger \phi_1 \phi_2 + \dots$$

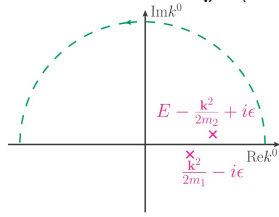
nonrelativistic propagator: $\frac{i}{p^0 - m_i - \mathbf{p}^2/(2m_i) + i\epsilon}$

- to have a near-threshold bound state (hadronic molecule)



$$\begin{aligned} T_{\text{NR}}(E) &= C_0 + C_0 G_{\text{NR}}(E) C_0 + C_0 G_{\text{NR}}(E) C_0 G_{\text{NR}}(E) C_0 + \dots \\ &= \frac{1}{C_0^{-1} - G_{\text{NR}}(E)} \end{aligned}$$

- The loop integral is linearly divergent (E defined relative to $m_1 + m_2$), regularized with, e.g., a sharp cut



$$\begin{aligned}
 G_{\text{NR}}(E) &= i \int \frac{d^3 \mathbf{k} dk^0}{(2\pi)^4} \left[\left(k^0 - \frac{\mathbf{k}^2}{2m_1} + i\epsilon \right) \left(E - k^0 - \frac{\mathbf{k}^2}{2m_2} + i\epsilon \right) \right]^{-1} \\
 &= -i2\mu(2\pi i) \int^\Lambda \frac{d^3 \mathbf{k}}{(2\pi)^4} \frac{1}{2\mu E - \mathbf{k}^2 + i\epsilon} \\
 &= -\frac{\mu}{\pi^2} \left(\Lambda - \sqrt{-2\mu E - i\epsilon} \arctan \frac{\Lambda}{\sqrt{-2\mu E - i\epsilon}} \right) \\
 &= -\frac{\mu}{\pi^2} \Lambda + \frac{\mu}{2\pi} \sqrt{-2\mu E - i\epsilon} + \mathcal{O}(\Lambda^{-1})
 \end{aligned}$$

for real E , $\sqrt{-2\mu E - i\epsilon} = \sqrt{-2\mu E} \theta(-E) - i\sqrt{2\mu E} \theta(E)$

- Renormalization: T_{NR} is Λ -independent,

$$\begin{aligned}
 T_{\text{NR}}(E) &= \frac{1}{C_0^{-1} - G_{\text{NR}}} \\
 &= \left(\underbrace{\frac{1}{C_0} + \frac{\mu}{\pi^2} \Lambda}_{\equiv 1/C_0^r} - \frac{\mu}{2\pi} \sqrt{-2\mu E - i\epsilon} \right)^{-1} \\
 &= \frac{2\pi/\mu}{2\pi/(\mu C_0^r) - \sqrt{-2\mu E - i\epsilon}}
 \end{aligned}$$

- Other regularization can be used as well, equivalent to the sharp cutoff up to $1/\Lambda$ suppressed terms, e.g.

☞ with a Gaussian regulator $\exp(-\mathbf{k}^2/\Lambda_G^2)$, $\Lambda_G = \sqrt{2/\pi} \Lambda$

☞ with the power divergence subtraction (PDS) scheme in dimensional regularization by letting, $\Lambda_{\text{PDS}} = 2\Lambda/\pi$

Kaplan, Savage, Wise (1998)

$$T_{\text{NR}}(E) = \frac{2\pi/\mu}{2\pi/(\mu C_0^r) - \sqrt{-2\mu E - i\epsilon}} = \frac{2\pi/\mu}{2\pi/(\mu C_0^r) + i k}$$

- from matching to effective range expansion,

$$f_0^{-1}(k) = -\frac{2\pi}{\mu} T_{\text{NR}}^{-1} = -\frac{1}{a_0} + \frac{1}{2} r_{e0} k^2 - i k + \mathcal{O}(k^4)$$

$2\pi/(\mu C_0^r) = 1/a_0$; higher terms are necessary to match both a and r_e

- pole below threshold at $E = -E_B$ with $E_B > 0$

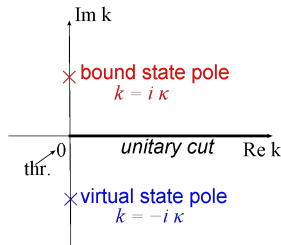
$$\kappa \equiv |\sqrt{2\mu E_B}|$$

👉 **bound state pole**, in the **1st Riemann sheet**

$$\Rightarrow 2\pi/(\mu C_0^r) = \kappa$$

👉 **virtual state pole**, in the **2nd Riemann sheet**

$$\Rightarrow 2\pi/(\mu C_0^r) = -\kappa$$



👉 unable to get a resonance pole at LO with a single channel

Bound state and virtual state

- If the same binding energy, **bound** and **virtual** states cannot be distinguished above threshold ($E > 0$):

$$|T_{\text{NR}}(E)|^2 \propto \left| \frac{1}{\pm\kappa + i\sqrt{2\mu E}} \right|^2 = \frac{1}{\kappa^2 + 2\mu E}$$

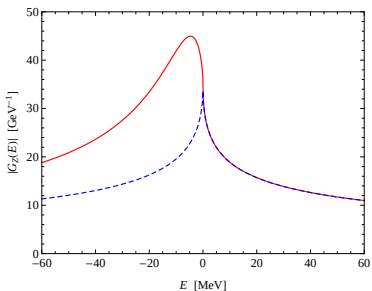
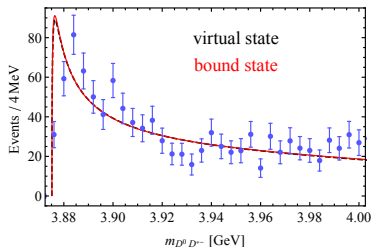
- Bound state** and **virtual state** are different below threshold ($E < 0$):

👉 **bound state**: peaked below threshold

$$|T_{\text{NR}}(E)|^2 \propto \frac{1}{(\kappa - \sqrt{-2\mu E})^2}$$

👉 **virtual state**: a sharp **cusp** at threshold

$$|T_{\text{NR}}(E)|^2 \propto \frac{1}{(\kappa + \sqrt{-2\mu E})^2}$$



Lower Fig.: **bound state** and **virtual state** with $E_B = 5$ MeV and a small width to the **inelastic** channel

For complexity of near-threshold line shapes, see [X.-K. Dong, FKG, B.-S. Zou, PRL126\(2021\)152001](#)

Coupling constant for S -wave bound state

$$T_{\text{NR}}(E) = \frac{2\pi/\mu}{2\pi/(\mu C_0^r) - \sqrt{-2\mu E - i\epsilon}}$$

At LO, effective coupling strength for bound state

$$\begin{aligned} g_{\text{NR}}^2 &= \lim_{E \rightarrow -E_B} (E + E_B) T_{\text{NR}}(E) = -\frac{2\pi}{\mu} \left(\frac{d}{dE} \sqrt{-2\mu E - i\epsilon} \right)_{E=-E_B}^{-1} \\ &= \frac{2\pi}{\mu^2} \sqrt{2\mu E_B} \end{aligned}$$

Recall the compositeness formula:

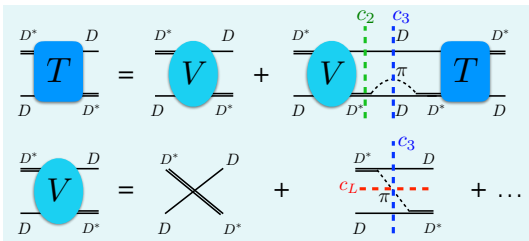
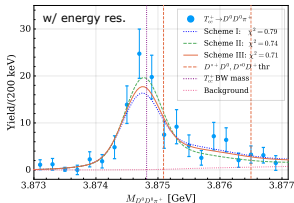
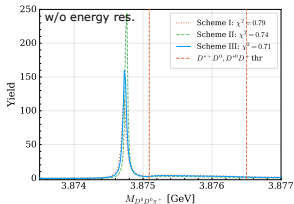
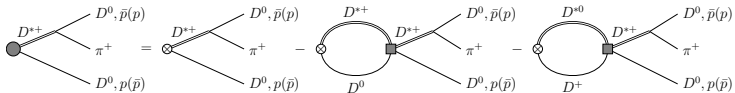
$$g_{\text{NR}}^2 = (1 - Z) \frac{2\pi}{\mu^2} \sqrt{2\mu E_B}$$

This means that the pole obtained at LO NREFT with only a constant contact term corresponds to a purely composite state ($Z = 0$)

Range corrections: other components at shorter distances

- energy/momentum-dependent interactions: higher order
- coupling to additional states/channels

- Consider $D^0 \bar{D}^{*+} - D^+ D^{*0}$ coupled-channel system
- Contact term (two low-energy constants $C_{I=0}$ and $C_{I=1}$, assuming $C_{I=1}$ vanish) + one-pion exchange; 3-body effects ($2m_D + m_\pi \simeq m_{D^*} + m_{D^0}$)



- Pole w.r.t. $D^{*+} D^0$ th: $-356_{-38}^{+39} - i(28 \pm 1)$ keV
- Compositeness of $T_{cc}(3875)^+$:
 $X_{D^{*+} D^0} = 0.73 \pm 0.11$, $X_{D^{*0} D^+} = 0.27 \pm 0.02$

Lots of new hadron resonances and resonance-like structures were found since 2003

- H.-X. Chen et al., *The hidden-charm pentaquark and tetraquark states*, Phys. Rept. 639 (2016) 1 [arXiv:1601.02092]
- A. Hosaka et al., *Exotic hadrons with heavy flavors — X , Y , Z and related states*, Prog. Theor. Exp. Phys. 2016, 062C01 [arXiv:1603.09229]
- R. F. Lebed, R. E. Mitchell, E. Swanson, *Heavy-quark QCD exotica*, Prog. Part. Nucl. Phys. 93 (2017) 143 [arXiv:1610.04528]
- A. Esposito, A. Pilloni, A. D. Polosa, *Multiquark resonances*, Phys. Rept. 668 (2017) 1 [arXiv:1611.07920]
- F.-K. Guo, C. Hanhart, U.-G. Meißner, Q. Wang, Q. Zhao, B.-S. Zou, *Hadronic molecules*, Rev. Mod. Phys. 90 (2018) 015004 [arXiv:1705.00141]
- S. L. Olsen, T. Skwarnicki, *Nonstandard heavy mesons and baryons: Experimental evidence*, Rev. Mod. Phys. 90 (2018) 015003 [arXiv:1708.04012]
- M. Karliner, J. L. Rosner, T. Skwarnicki, *Multiquark states*, Ann. Rev. Nucl. Part. Sci. 68 (2018) 17 [arXiv:1711.10626]
- C.-Z. Yuan, *The XYZ states revisited*, Int. J. Mod. Phys. A 33 (2018) 1830018 [arXiv:1808.01570]
- N. Brambilla et al., *The XYZ states: experimental and theoretical status and perspectives*, Phys. Rept. 873 (2020) 154 [arXiv:1907.07583]
- F.-K. Guo, X.-H. Liu, S. Sakai, *Threshold cusps and triangle singularities in hadronic reactions*, Prog. Part. Nucl. Phys. 112 (2020) 103757 [arXiv:1912.07030]
- ...

Thank you for your attention!

Bispinor fields for heavy mesons (1)

- $\frac{1}{2}(1 + \not{v})$ projects onto the particle component of the heavy quark spinor.
- Convenient to introduce heavy mesons as bispinors:

$$H_a = \frac{1 + \not{v}}{2} [P_a^{*\mu} \gamma_\mu - P_a \gamma_5], \quad \bar{H}_a = \gamma_0 H_a^\dagger \gamma_0$$

$P = \{Q\bar{u}, Q\bar{d}, Q\bar{s}\}$: pseudoscalar heavy mesons, P^* : vector heavy mesons

For hadrons with arbitrary spin, see A. Falk, NPB378(1992)79

Charge conjugation:

- ☞ H_a destroys mesons containing a Q , but does **not** create mesons with a $\bar{Q}$
- ☞ Free to choose the phase convention for charge conjugation. If we use, e.g.,

$$P_a \xrightarrow{C} +P_a^{(\bar{Q})}, \quad P_{a,\mu}^* \xrightarrow{C} -P_{a,\mu}^{*(\bar{Q})},$$

then the fields annihilating mesons containing a $\bar{Q}$ is ($\mathcal{C} = i\gamma^2\gamma^0$)

$$\begin{aligned} H_a^{(\bar{Q})} &= \mathcal{C} \left[\frac{1 + \not{v}}{2} \left(\underbrace{-P_{a,\mu}^{*(\bar{Q})}}_{P_{a,\mu}^* \xrightarrow{C}} \gamma^\mu - \underbrace{P_a^{(\bar{Q})}}_{P_a \xrightarrow{C}} \gamma_5 \right) \right]^T \mathcal{C}^{-1} \\ &= \left(+P_{a,\mu}^{*(\bar{Q})} \gamma^\mu - P_a^{(\bar{Q})} \gamma_5 \right) \frac{1 - \not{v}}{2} \end{aligned}$$

Bispinor fields for heavy mesons (2)

- Free heavy-meson Lagrangian:

$$\mathcal{L}_{\text{free}} = -i \text{Tr} [\bar{H}_a v_\mu \partial^\mu H_a] = 2i P_a^\dagger v_\mu \partial^\mu P_a - 2i P_a^{*\dagger} v_\mu \partial^\mu P_a^{*\nu}$$

Tr: trace in the spinor space, a, b : indices in the light flavor space

- Notice that the mass dimension of H_a is $3/2$.

Nonrelativistic normalization: $H_a \simeq \sqrt{M_H} H_a^{\text{rel.}}$

D -meson propagator:

$$\begin{aligned} \frac{i}{2v \cdot k + i\epsilon} &\simeq \sqrt{M_H} \times \underbrace{\frac{i}{p^2 - M_H^2 + i\epsilon}}_{(p = M_H v + k)} \\ &= \frac{i}{2M_H v \cdot k + i\epsilon} [1 + \mathcal{O}(k^2/M_H^2)] \end{aligned}$$

In some papers, the normalization factor is $\sqrt{2M_H}$, instead of $\sqrt{M_H}$. Then the corresponding propagator should be

$$\frac{i}{v \cdot k + i\epsilon} \quad \text{or} \quad \frac{i}{k^0 - \vec{k}^2/(2M_H) + i\epsilon}$$

Simplified two-component notation

The superfield for pseudoscalar and vector heavy mesons: (in this page, we use $^{(4)}$ to mean the usual 4-component notation)

$$H_a^{(4)} = \frac{1 + \not{v}}{2} [P_a^{*\mu} \gamma_\mu - P_a \gamma_5]$$

In the rest frame of heavy meson, $v^\mu = (1, \mathbf{0})$. We take the Dirac basis

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}.$$

Simplifications: $\frac{1 + \not{v}}{2} = \frac{1 + \gamma^0}{2} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}$

$$H_a^{(4)} = \begin{pmatrix} 0 & -(P_a + \mathbf{P}_a^* \cdot \boldsymbol{\sigma}) \\ 0 & 0 \end{pmatrix}, \quad \bar{H}_a^{(4)} = \begin{pmatrix} 0 & 0 \\ (\mathbf{P}_a^\dagger + \mathbf{P}_a^{*\dagger} \cdot \boldsymbol{\sigma}) & 0 \end{pmatrix}$$

Thus, it is convenient to simply use the **two-component notation**

$$H_a = P_a + \mathbf{P}_a^* \cdot \boldsymbol{\sigma}, \quad H_a^{(4)} \rightarrow -H_a, \quad \bar{H}_a^{(4)} \rightarrow H_a^\dagger$$

- Spin-dependent term $\Rightarrow$ mass difference between vector and pseudoscalar mesons ($\sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu]$)

$$\begin{aligned}\mathcal{L}_\Delta &= \frac{\lambda_2}{m_Q} \text{Tr} \left[\bar{H}_a^{(4)} \sigma_{\mu\nu} H_a^{(4)} \sigma^{\mu\nu} \right] = -\frac{2\lambda_2}{m_Q} \text{Tr} \left[H_a^\dagger \sigma^i H_a \sigma^i \right] \\ &= \frac{4\lambda_2}{m_Q} \left(\mathbf{P}_a^{*\dagger} \cdot \mathbf{P}_a^* - 3P_a^\dagger P_a \right) \quad H_a = P_a + \mathbf{P}_a^* \cdot \boldsymbol{\sigma}\end{aligned}$$

$$\Rightarrow \quad M_{P_a^*} - M_{P_a} = -\frac{8\lambda_2}{m_Q}$$

- Thus, we expect

$$\frac{M_{B^*} - M_B}{M_{D^*} - M_D} \simeq \frac{m_c}{m_b} \simeq 0.3$$

measured values:

$$M_{D^*} - M_D \simeq 140 \text{ MeV}, \quad M_{B^*} - M_B \simeq 46 \text{ MeV}$$

LO effective Lagrangian for interaction of heavy meson pair

- At LO of nonrelativistic expansion, constant contact terms for S -wave interaction between a pair of heavy mesons

$$\begin{aligned}\mathcal{L}_{4H} = & -\frac{1}{4} \text{Tr} [H^{a\dagger} H_b] \text{Tr} [\bar{H}^c \bar{H}_d^\dagger] \left(F_A \delta_a^b \delta_c^d + F_A^\lambda \vec{\lambda}_a^b \cdot \vec{\lambda}_c^d \right) \\ & + \frac{1}{4} \text{Tr} [H^{a\dagger} H_b \sigma^m] \text{Tr} [\bar{H}^c \bar{H}_d^\dagger \sigma^m] \left(F_B \delta_a^b \delta_c^d + F_B^\lambda \vec{\lambda}_a^b \cdot \vec{\lambda}_c^d \right)\end{aligned}$$

- Using the completeness relations for $SU(N)$ generators T_m satisfying $\text{Tr} [T_m T_n] = C \delta_{mn}$:

$$\delta_i^l \delta_k^j = \frac{1}{N} \delta_i^j \delta_k^l + \frac{1}{C} (T_m)_i^j (T_m)_k^l$$

Proof: Any $N \times N$ complex matrix can be expanded as $X = X_0 + X_m T_m$ ($m = 1, \dots, N^2 - 1$). Using $\text{Tr} [T_m T_n] = C \delta_{mn}$, one gets

$$X_0 = \frac{1}{N} \text{Tr}[X] = \frac{1}{N} X_i^i, \quad X_m = \frac{1}{C} \text{Tr} [X T_m] = \frac{1}{C} X_l^k (T_m)_k^l.$$

Then

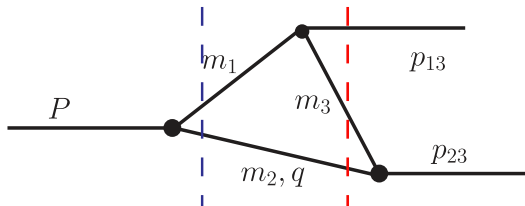
$$X_l^k \delta_i^l \delta_k^j = \frac{1}{N} X_l^k \delta_k^l \delta_i^j + \frac{1}{C} X_l^k (T_m)_k^l (T_m)_i^j.$$

one finds that the two-trace terms can be rewritten in the form of single-trace terms:

$$\begin{aligned}
 \text{Tr} \left[H^{a\dagger} H_a \right] \text{Tr} \left[\bar{H}^b \bar{H}_b^\dagger \right] &= \left(H^{a\dagger} H_a \right)_l^i \delta_i^l \left(\bar{H}^b \bar{H}_b^\dagger \right)_j^k \delta_k^j \\
 &= \left(H^{a\dagger} H_a \right)_l^i \left(\bar{H}^b \bar{H}_b^\dagger \right)_j^k \left[\frac{1}{2} \delta_k^l \delta_i^j + \frac{1}{2} (\sigma_m)_k^l (\sigma_m)_i^j \right] \\
 &= \frac{1}{2} \text{Tr} \left[H^{a\dagger} H_a \bar{H}^b \bar{H}_b^\dagger \right] + \frac{1}{2} \text{Tr} \left[H^{a\dagger} H_a \sigma_m \bar{H}^b \bar{H}_b^\dagger \sigma_m \right], \\
 \text{Tr} \left[H^{a\dagger} H_a \sigma_m \right] \text{Tr} \left[\bar{H}^b \bar{H}_b^\dagger \sigma_m \right] &= \left(H^{a\dagger} H_a \right)_l^k (\sigma_m)_k^l \left(\bar{H}^b \bar{H}_b^\dagger \right)_j^i (\sigma_m)_i^j \\
 &= \left(H^{a\dagger} H_a \right)_l^k \left(\bar{H}^b \bar{H}_b^\dagger \right)_j^i \left(2\delta_i^l \delta_k^j - \delta_k^l \delta_i^j \right) \\
 &= 2 \text{Tr} \left[H^{a\dagger} H_a \bar{H}^b \bar{H}_b^\dagger \right] - \text{Tr} \left[H^{a\dagger} H_a \right] \text{Tr} \left[\bar{H}^b \bar{H}_b^\dagger \right] \\
 &= \frac{3}{2} \text{Tr} \left[H^{a\dagger} H_a \bar{H}^b \bar{H}_b^\dagger \right] - \frac{1}{2} \text{Tr} \left[H^{a\dagger} H_a \sigma_m \bar{H}^b \bar{H}_b^\dagger \sigma_m \right].
 \end{aligned}$$

Triangle singularity

FKG, X.-H. Liu, S. Sakai, *Threshold cusps and triangle singularities in hadronic reactions*, Prog. Part. Nucl. Phys. 112 (2020) 103757



Consider the scalar three-point loop integral

$$I = i \int \frac{d^4 q}{(2\pi)^4} \frac{1}{[(P - q)^2 - m_1^2 + i\epsilon] (q^2 - m_2^2 + i\epsilon) [(p_{23} - q)^2 - m_3^2 + i\epsilon]}$$

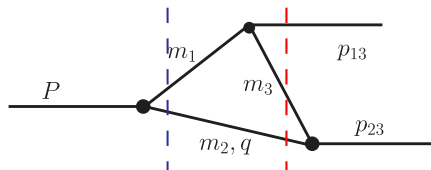
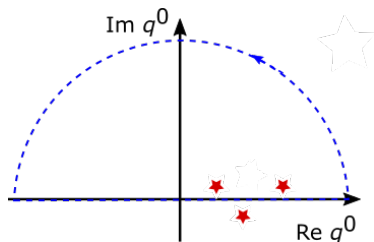
Rewriting a propagator into two poles:

$$\frac{1}{q^2 - m_2^2 + i\epsilon} = \frac{1}{(q^0 - \omega_2 + i\epsilon)(q^0 + \omega_2 - i\epsilon)} \quad \text{with} \quad \omega_2 = \sqrt{m_2^2 + \mathbf{q}^2}$$

Nonrelativistically, on the positive-energy poles

$$I \simeq \frac{i}{8m_1 m_2 m_3} \int \frac{dq^0 d^3 \mathbf{q}}{(2\pi)^4} \frac{1}{(P^0 - q^0 - \omega_1 + i\epsilon)(q^0 - \omega_2 + i\epsilon)(p_{23}^0 - q^0 - \omega_3 + i\epsilon)}$$

TS: some details (2)



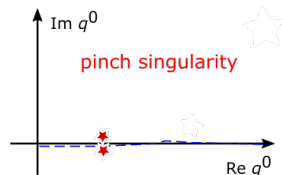
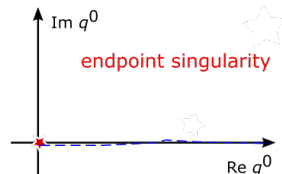
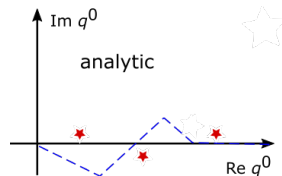
$$\begin{aligned}
 I &\propto \int \frac{d^3 \mathbf{q}}{(2\pi)^3} \frac{1}{[P^0 - \omega_1(q) - \omega_2(q) + i\epsilon][E_{23} - \omega_2(q) - \omega_3(\mathbf{p}_{23} - \mathbf{q}) + i\epsilon]} \\
 &\propto \int_0^\infty dq \frac{q^2}{P^0 - \omega_1(q) - \omega_2(q) + i\epsilon} f(q)
 \end{aligned}$$

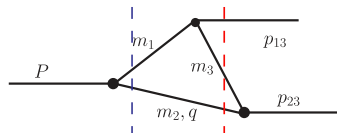
The second cut:

$$f(q) = \int_{-1}^1 dz \frac{1}{E_{23} - \omega_2(q) - \sqrt{m_3^2 + q^2 + p_{23}^2 - 2p_{23}qz} + i\epsilon}$$

Relation between singularities of integrand and integral

- singularity of integrand does **not necessarily** give a singularity of integral:
integral contour can be deformed to avoid the singularity
- Two cases that a singularity cannot be avoided:
 - ☞ **endpoint singularity**
 - ☞ **pinch singularity**





$$I \propto \int_0^\infty dq \frac{q^2}{P^0 - \omega_1(q) - \omega_2(q) + i\epsilon} f(q)$$

$$f(q) = \int_{-1}^1 dz \frac{1}{A(q, z)} \equiv \int_{-1}^1 dz \frac{1}{E_{23} - \omega_2(q) - \sqrt{m_3^2 + q^2 + p_{23}^2 - 2p_{23}qz} + i\epsilon}$$

Singularities of the **integrand** in the rest frame of initial particle:

- **First cut:** $M - \omega_1(l) - \omega_2(l) + i\epsilon = 0 \Rightarrow q_{\text{on}+} \equiv \frac{1}{2M} \sqrt{\lambda(M^2, m_1^2, m_2^2)} + i\epsilon$
- **Second cut:** $A(q, \pm 1) = 0 \Rightarrow$ **endpoint singularities of $f(q)$**

$$z = +1 : \quad q_{a+} = \gamma(\beta E_2^* + p_2^*) + i\epsilon, \quad q_{a-} = \gamma(\beta E_2^* - p_2^*) - i\epsilon,$$

$$z = -1 : \quad q_{b+} = \gamma(-\beta E_2^* + p_2^*) + i\epsilon, \quad q_{b-} = -\gamma(\beta E_2^* + p_2^*) - i\epsilon$$

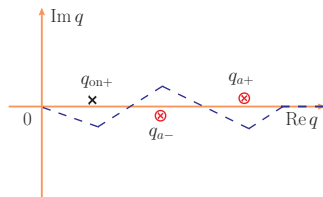
$$\beta = |\mathbf{p}_{23}|/E_{23}, \quad \gamma = 1/\sqrt{1 - \beta^2} = E_{23}/m_{23}$$

$E_2^*(p_2^*)$: energy (momentum) of particle-2 in the cmf of the (2,3) system

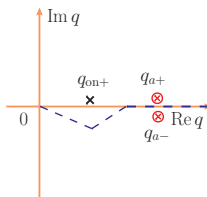
All singularities of the integrand:

$$q_{\text{on}+}, \quad q_{a+} = \gamma (\beta E_2^* + p_2^*) + i \epsilon, \quad q_{a-} = \gamma (\beta E_2^* - p_2^*) - i \epsilon,$$

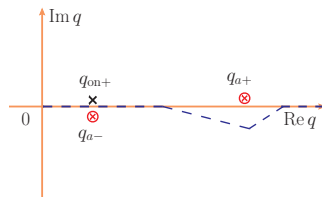
$$q_{b+} = -q_{a-}, \quad q_{b-} = -q_{a+} < 0 \text{ (for } \epsilon = 0 \text{)}$$



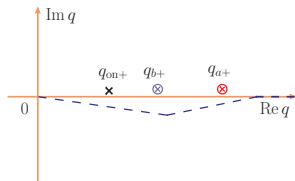
(a)



(b)



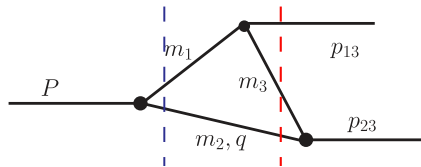
(c)



2-body threshold
singularity at
 $m_{23} = m_2 + m_3$

triangle singularity at

$$q_{\text{on}+} = q_{a-}$$



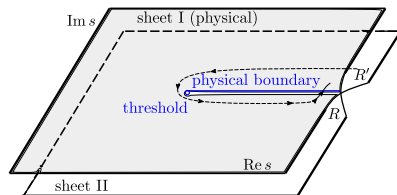
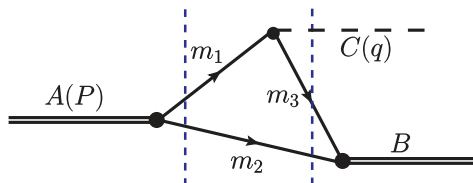
Rewrite $q_{a-} = p_2 - i\epsilon$, $p_2 \equiv \gamma(\beta E_2^* - p_2^*)$

Kinematics for $p_2 > 0$, which is relevant to triangle singularity:

- $p_3 = \gamma(\beta E_3^* + p_2^*) > 0 \Rightarrow$
particles 2 and 3 move in the same direction in the rest frame of initial particle
- velocities in the rest frame of the initial particle: $v_3 > \beta > v_2$

$$v_2 = \beta \frac{E_2^* - p_2^*/\beta}{E_2^* - \beta p_2^*} < \beta, \quad v_3 = \beta \frac{E_3^* + p_2^*/\beta}{E_3^* + \beta p_2^*} > \beta$$

particle 3 moves faster than particle 2 in the rest frame of initial particle



- Coleman–Norton theorem:

S. Coleman and R. E. Norton, *Nuovo Cim.* 38 (1965) 438

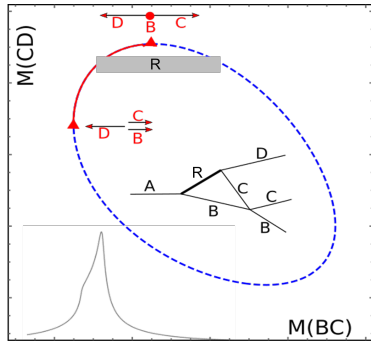
The singularity is on the **physical boundary** if and only if the diagram can be interpreted as a classical process in space-time.

☞ **physical boundary**: upper edge (lower edge) of the unitary cut in the first (second) Riemann sheet

- Translation:

☞ all three intermediate states can go **on shell**

☞ $p_2 \parallel p_3, m_3$ can catch up with the m_2 to rescatter



- For fixed $m_{B,C}, m_A$ and m_D , TS on the physical boundary only when

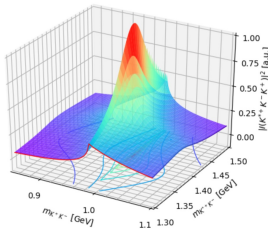
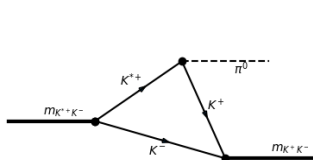
$$m_R^2 \in \left[\frac{m_A^2 m_C + m_D^2 m_B}{m_B + m_C} - m_B m_C, (m_A - m_B)^2 \right]$$

- TS on the physical boundary only when

$$m_A^2 \in \left[(m_R + m_B)^2, (m_R + m_B)^2 + \frac{m_B}{m_C} \left[(m_R - m_C)^2 - m_D^2 \right] \right],$$

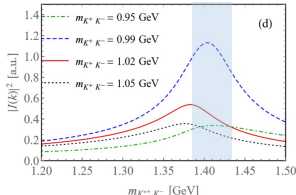
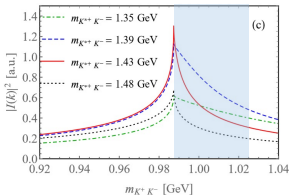
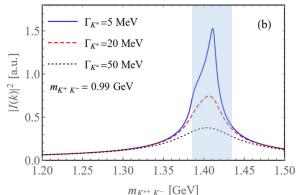
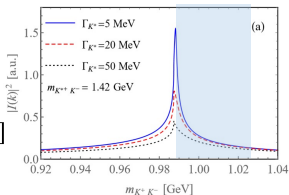
$$m_{BC}^2 \in \left[(m_B + m_C)^2, (m_B + m_C)^2 + \frac{m_B}{m_R} \left[(m_R - m_C)^2 - m_D^2 \right] \right]$$

TS: invariant mass spectra



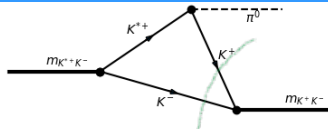
- TS structure is sensitive to energy
- While a resonance would persist independent of energy

TS only when
 $m_{K\bar{K}} \in [987, 1026]$
 MeV



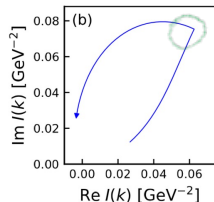
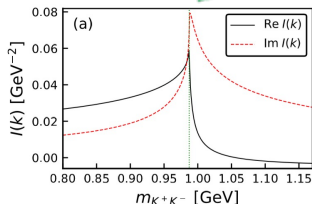
TS: Argand plot

- Phase motion of triangle diagram in the presence of a TS

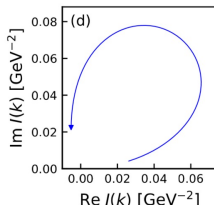
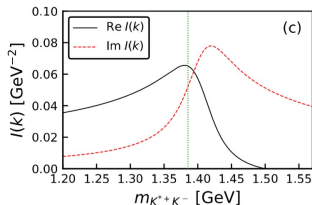


$K\bar{K}$ threshold

$$m_{K^*\bar{K}} = 1.42 \text{ GeV}$$



$$m_{K\bar{K}} = 0.99 \text{ GeV}$$



The argand plot is **counterclockwise**, resembling that of a resonance

Exercises for Lectures on Exotic Hadrons

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1. Try to understand this statement as a consequence of heavy quark spin symmetry.
Hint: in e^+e^- collisions, the leading production mechanism of heavy meson pairs is from the vector current $\bar{Q}\gamma^\mu Q$ which couples to the virtual photon, *i.e.*, $e^+e^- \rightarrow \gamma^* \rightarrow \bar{Q}Q$ with the $Q\bar{Q}$ pair in an S -wave.
2. Why is the isospin of the negative C -parity $\pi^+\pi^-$ system equal to 1?
3. The $X(3872)$ was found in the $J/\psi\pi^+\pi^-$ final state. It is expected to have a bottom partner X_b in various models. Suppose the X_b has a mass tens of MeV below the $B\bar{B}^*$ threshold. Are the $\Upsilon(1S, 2S, 3S)\pi^+\pi^-$ final states suitable for searching for the X_b ? Why?
4. Show that the combinations of the LO contact terms in the S -wave interaction matrix elements for $D^{(*)}\bar{D}^{(*)}$ scattering are as follows:

$$\begin{aligned} \begin{pmatrix} D\bar{D} \\ D^*\bar{D}^* \end{pmatrix} : \quad V^{(0^{++})} &= \begin{pmatrix} C_{IA} & \sqrt{3}C_{IB} \\ \sqrt{3}C_{IB} & C_{IA} - 2C_{IB} \end{pmatrix}, \\ \begin{pmatrix} D\bar{D}^* \\ D^*\bar{D} \end{pmatrix} : \quad V^{(1^{+-})} &= \begin{pmatrix} C_{IA} - C_{IB} & 2C_{IB} \\ 2C_{IB} & C_{IA} - C_{IB} \end{pmatrix}, \\ D\bar{D}^* : \quad V^{(1^{++})} &= C_{IA} + C_{IB}, \\ D^*\bar{D} : \quad V^{(2^{++})} &= C_{IA} + C_{IB}, \end{aligned}$$

where $C_{IA} = \frac{1}{4}(3F_{I1} + F_{I0})$, $C_{IB} = \frac{1}{4}(F_{I1} - F_{I0})$. Here, F_{Is_L} are defined as $F_{I0} = \langle \frac{1}{2}, \frac{1}{2}, \mathbf{0} | \hat{\mathcal{H}} | \frac{1}{2}, \frac{1}{2}, \mathbf{0} \rangle_I$, $F_{I1} = \langle \frac{1}{2}, \frac{1}{2}, \mathbf{1} | \hat{\mathcal{H}} | \frac{1}{2}, \frac{1}{2}, \mathbf{1} \rangle_I$ in terms of $\langle s_{\ell 1}, s_{\ell 2}, \mathbf{s}_L | \hat{\mathcal{H}} | s'_{\ell 1}, s'_{\ell 2}, \mathbf{s}_L \rangle_I$ with $s_{\ell i}$ the angular momentum of the light (anti)quark in meson- i and s_L the total spin of the light quark-antiquark pair.

5. Show that the Lagrangian for the $D^{(*)}\bar{D}^{(*)}$ interaction in S -wave

$$\begin{aligned}\mathcal{L}_{4H} = & -\frac{1}{4} \text{Tr} [H^{a\dagger} H_b] \text{Tr} [\bar{H}^c \bar{H}_d^\dagger] \left(F_A \delta_a^b \delta_c^d + F_A^\lambda \vec{\lambda}_a^b \cdot \vec{\lambda}_c^d \right) \\ & + \frac{1}{4} \text{Tr} [H^{a\dagger} H_b \sigma^m] \text{Tr} [\bar{H}^c \bar{H}_d^\dagger \sigma^m] \left(F_B \delta_a^b \delta_c^d + F_B^\lambda \vec{\lambda}_a^b \cdot \vec{\lambda}_c^d \right)\end{aligned}$$

can also be written as single trace terms.