Focus Week
"Lattice Models: Exact Results and Combinatorics"
GGI, Florence, May 18-22

Abstracts of talks

Smirnov's parafermionic observable, at and away from criticality
Vincent Beffara *(Ecole Normale Supérieure, Lyon)*

Smirnov's parafermionic observable, together with the theory of discrete holomorphicity, was instrumental in proving the conformal invariance of the interfaces of the Ising model at its critical point; it exhibits a flavor of integrability at the critical temperature for all the random-cluster models, and in a few cases (namely the Ising model itself and the RC models for $q>4$) it can be studied directly at non-critical temperatures, where it provides precise information about the two-point function. I will present a few results in this direction, joint with H. Duminil-Copin and S. Smirnov.

Covering trees of the graph of covering trees
Philippe Biane *(CNRS-Université Paris-Est)*

The set of covering trees of a directed graph can be endowed with a natural graph structure covering the original graph. We show that the generating function of covering trees of this new graph exhibits a remarkable factorization property. This is joint work with Guillaume Chapuy.

Dimer Model: Full Asymptotic Expansion of the Partition Function
Pavel Bleher *(Indiana University-Purdue University Indianapolis)*

We obtain the full asymptotic expansion of the partition function of the 2D dimer model on the m x n lattice with free and cylindrical boundary conditions. We show that the asymptotic expansion goes over powers of $1/S$, where $S=(m+1)(n+1)$ for the free boundary conditions and $S=(m+1)n$ for the cylindrical boundary conditions. The coefficients of the asymptotic expansion are expressed in terms of the Kronecker double series, the Dedekind eta function, and the Jacobi theta functions. Our calculations use the technique of the work of Ivashkevich, Izmailian, and Hu, in which the full asymptotic expansion is developed for the periodic boundary conditions.
This is a joint work with Brad Elwood and Drazen Petrovic.

Inhomogenous Multispecies TASEP on a ring
Luigi Cantini *(Université de Cergy-Pontoise)*

In this talk I will present some results about a multispecies version of the TASEP, a model which describes the stochastic evolution of a system of particles of different species on a periodic oriented one dimensional lattice, where two neighboring particles exchange their positions with a rate which depends on their species. For some choice of these rates the Markov matrix turns out to be integrable and for the same choice the (unnormalized) stationary probability is conjectured to show remarkable positivity and combinatorial properties related to Schubert polynomials. I will discuss how integrability leads to an interesting algebraic structure underlying the problem, and allowing to prove some remarkable properties of the stationary measure and to give exact formulas for the stationary probability of some classes of configurations.

Lozenge tilings with gaps in a 90 degree wedge domain with mixed boundary conditions
Mihai Ciucu *(Indiana University, Bloomington)*

We consider a triangular gap of side two in a 90 degree angle on the triangular lattice with mixed boundary conditions: a constrained, zig-zag boundary along one side, and a free lattice line boundary along the other. We study the interaction of the gap with the corner as the rest of the angle is completely filled with lozenges. We show that the resulting correlation is governed by the product of the distances between the gap and its three images in the sides of the angle. This provides evidence for a unified way of understanding the interaction of gaps with the boundary under mixed boundary conditions, which we present as a conjecture. Our conjecture is phrased in terms of the steady state heat flow problem in a uniform block of material in which there are a finite number of heat sources and sinks. This new physical analogy is equivalent in the bulk to the electrostatic analogy we developed in previous work, but arises as the correct one for the correlation with the boundary.
The starting point for our analysis is an exact formula we prove for the number of lozenge tilings of certain trapezoidal regions with mixed boundary conditions, which is equivalent to a new, multi-parameter generalization of a classical plane partition enumeration problem (that of enumerating symmetric, self-complementary plane partitions).

Combinatorics of exclusion processes with open boundaries
Sylvie Corteel *(CNRS-Université Paris Diderot 7, Paris)*

We study the combinatorics of the two-species asymmetric simple exclusion process with open boundaries studied among others by Masaru Uchiyama. We give a combinatorial interpretation of the partition function thanks to the Matrix Ansatz and the combinatorics of lattice paths. This brings us to a general positivity conjecture on "Koornwinder moments"". We also present a generalization of the staircase tableaux and TAT tableaux of Mandelshtam and Viennot which gives a combinatorial interpretation of the stationary distribution. This is joint work with Lauren Williams (Berkeley) and also with Olya Mandelshtam (Berkeley) for the tableaux parts.

A matrix product formula for Macdonald polynomials
Jan de Gier *(The University of Melbourne)*

I will discuss how an inhomogeneous version of the multi-species asymmetric simple exclusion process gives rise to a matrix product formula for Macdonald polynomials. This is work in collaboration with Luigi Cantini and Michael Wheeler.

Phase separation, interfaces and wetting in two dimensions.
Gesualdo Delfino *(SISSA, Trieste)*

Separation of phases with finite correlation length and the same free energy is a common phenomenon. For long time the lattice integrability of the Ising model in two dimensions has provided the only exact result for the scaling limit of the order parameter profile. It has been recently realized that field theory yields a series of exact results for the different universality classes in two dimensions, including order parameter profiles, interface structure and wetting properties, both in the bulk and at boundaries.
This is joint work with J. Viti (*Phase separation and interface structure in two dimensions from field theory*, J. Stat. Mech. (2012) P10009) and with A. Squarcini (*Exact theory of intermediate phases in two dimensions,* Annals of Physics 342 (2014) 171, and *Phase separation in a wedge. Exact results*, PRL 113 (2014) 066101

Physics and combinatorics of the octahedron equation: from cluster algebras to arctic curves
Philippe Di Francesco *(Univ. of Illinois at Urbana-Champaign)*

The octahedron equation is a non-linear system of recursion relations in discrete 2+1 dimensions, that first appeared in the context of generalized Heisenberg quantum spin chains (T-system). Some solutions of this equation also have a purely combinatorial interpretation as generalized Coxeter-Conway Frieze patterns.
This same equation plays a central role in the seemingly unrelated purely combinatorial problem of enumeration of domino tilings of a plane square-shaped domain called the Aztec diamond. For large size, the tiling configurations display some drastic change of typical behavior between the corners of the domain, where it freezes in certain tiling patterns, and a central region in the bulk, with disorder. In the continuum limit of large size and small mesh, the separation of phases is along the so-called "arctic circle" inscribed inside the square domain.
In this talk, we shall show that the octahedron equation is part of a combinatorial structure called cluster algebra, and how its exact solution may be rephrased in terms of non-intersecting lattice paths, and eventually domino tilings. Using the octahedron equation explicitly, we will proceed and determine various arctic curves depending on initial data. As predicted by Kenyon and Okounkov, we find in general new "facet" phases of the dimer model within some connected components of these curves.
(Joint works with R. Kedem and R. Soto Garrido).

Low-temperature spectrum of correlation lengths of the XXZ chain in the anti-ferromagnetic massive regime
Frank Göhmann *(Bergische Universität Wuppertal)*

We consider the spectrum of correlation lengths of the spin ½ XXZ chain in the antiferromagnetic massive regime. These are given as ratios of eigenvalues of the quantum transfer matrix of the model. In the low-temperature limit we obtain the expressions for the eigenvalues as functions of a finite number of parameters which satisfy finite sets of `higher-level Bethe Ansatz equations'.
If we send the temperature T to zero, these equations behave differently for zero and non-zero magnetic field h. If h is zero the situation is much like in the case of the usual transfer matrix. Non-trivial higher-level Bethe Ansatz equations remain which determine certain complex excitation parameters as functions of hole parameters which are free on a line segment in the complex plane. If h is non-zero, on the other hand, a remarkable restructuring occurs, and all parameters which enter the description of the quantum transfer matrix eigenvalues can be interpreted entirely in terms of particles and holes which are freely located on two curves when T goes to zero. This will be important for the calculation of correlation functions within a form-factor approach.

Integrability, Solvability and Enumeration.
Tony Guttmann *(The University of Melbourne)*

There are a number of seminal two-dimensional lattice models that are integrable, but have only been partially solved, in the sense that only some properties are fully known (e.g. the two-dimensional Ising model, where the free-energy is known, but not the susceptibility). Alternatively, critical properties are known for some lattices but not others. For example, the critical point of the self-avoiding walk model is known rigorously for the honeycomb lattice, but not for other lattices. Similarly for the q-state Potts model and both bond and site percolation. The critical manifold of the former is known only for some lattices, likewise the percolation threshold is known only for some lattices.
A range of numerical procedures exist, based on exact enumeration, or other numerical work, such

as calculating the eigenvalues of transfer matrices, which, when combined with various structural invariants seem to give exact results in those cases that are known to be exact, but can be used to give increasingly precise estimates in those cases which are not exactly known. Reasons for this partial success are not well understood. In this talk I will describe four such procedures, and demonstrate their performance, and speculate on their partial success.

Random matrices and Aztec diamonds
Kurt Johansson *(KTH, Stockholm)*

Random tilings, or dimer models on bipartite graphs, give rise to determinantal point processes. Natural scaling limits of these processes give rise to limiting processes and distributions of the same type as we see in random matrix theory and also in random growth models in the KPZ universality class, e.g the Airy process. I will give some background on this and in particular discuss it in connection with the two-periodic Aztec diamond, which is recent joint work with Sunil Chhita.

Fusion products and *q*-Whittaker functions
Rinat Kedem *(University of Illinois at Urbana-Champaign)*

The solution of the generalized Heisenberg model via Bethe ansatz comes with generalized fermionic combinatorics for the spectrum. Using this to compute linearized partition functions relates this model to conformal blocks of WZW theories and graded finite-dimensional representations of affine algebras, which are tensor products. In the simplest example, these are affine Demazure modules, and the limit as the size of the system becomes infinite are the well-known level-1 representations associated with the CFT. In this talk, I concentrate on the finite system, and explain why the linearized partition functions in the level-1 case are Whittaker functions for the quantum (finite) algebra, which satisfy the quantum difference Toda equation.

Boundary correlation functions with a hidden quantum group
Kalle Kytölä *(Aalto University)*

In the scaling limit, critical lattice model correlation functions satisfy partial differential equations of conformal field theories. In this talk, we present a method of constructing boundary correlation functions using a hidden quantum group. Solutions with desired asymptotic behavior are found by solving a problem in representations of the quantum group. Two applications to random curves will be considered: multiple SLE pure partition functions and boundary visit amplitudes of chordal SLEs.
The talk is based on joint work with Eveliina Peltola ([arXiv:1408.1384] and in preparation), and with Niko Jokela and Matti Järvinen ([arXiv:1311.2297]).

Generalized Cauchy determinant and Schur Pfaffian, and their applications
Soichi Okada *(Nagoya University)*

The Cauchy determinant and the Schur Pfaffian, together with their generalizations, play a fundamental role in combinatorics and representation theory. For example, such determinant/Pfaffian identities are used to evaluate Izergin-Korepin-type determinants/Pfaffians in enumeration problems of alternating sign matrices. In this talk, we present yet other generalizations of the Cauchy determinant and the Schur Pfaffian, and give their applications to symmetric function identities.

Lozenge tilings and other lattice models from the viewpoint of symmetric functions
Greta Panova *(University of Pennsylvania, Philadelphia)*

What do lozenge tilings (a.k.a. plane partitions, dimer covers of the hexagonal lattice), alternating sign matrices (or the six-vertex model) and the dense loop model have in common? For one, their limiting behavior can be studied with the help of some "asymptotic" algebraic combinatorics.
We develop methods to analyze normalized symmetric functions (Schur functions and more general Lie group characters), as the indexing partition converges to a limiting profile. We apply this analysis together with some combinatorial interpretations to study the limiting behavior of the integrable models listed above. In particular, we show that the positions of horizontal lozenges near a vertical flat boundary are distributed like the eigenvalues of GUE matrices, and this holds for a wide class of domains (including such with free boundary). These methods can also be used to establish the existence of limit shapes also for free boundary domains. We discover Gaussian distribution for some observables of the Alternating Sign Matrices, leading again to GUE eigenvalues for the positions of 1s near the border (result of V. Gorin). We also find the asymptotics for the [conjectural] expected value of the mean total current between two adjacent points in the dense loop model. Based on joint work with Vadim Gorin.
We will also discuss limit behavior under non-uniform (q^volume) distributions, based on ongoing work with Jonathan Novak.

Towards combinatorics of elliptic lattice models
Hjalmar Rosengren *(Chalmers University of Technology and University of Gothenburg)*

Much of the recent progress on combinatorics of solvable lattice models is concerned with the six-vertex model and XXZ spin chain. For the more general eight-vertex model and XYZ spin chain, which are naturally parametrized by elliptic functions,
progress has been slower. Various quantities related to the models are described in terms of polynomials with positive integer coefficients, but a combinatorial explanation for this phenomenon is still lacking. On the other hand, new intriguing
features appear in the elliptic case, such as relations to the Painlevé VI equation already at the finite lattice. In the present talk, I will try to give an overview of the current knowledge, discussing the work of myself and others (V. Bazhanov, V. Mangazeev, A. Razumov, Yu. Stroganov, P. Zinn-Justin).

From lattice to CFT via fermionic basis
Fedor Smirnov *(LPTHE, UPMC, Paris)*

For XXZ (six-vertex) model we construct fermionic basis of local operators. Similar construction for CFT follows from description of quotient of Verma modules by action of local integrals of motion. I shall explain that the two constructions are in correspondence.

Posets of alternating sign matrices and totally symmetric self-complementary plane partitions
Jessica Striker *(North Dakota State University)*

Alternating sign matrices and totally symmetric self-complementary plane partitions are mysteriously equinumerous combinatorial objects with interesting connections to physics. In this talk, we discuss how encoding these objects as order ideals of partially ordered sets, or posets, yields a unifying framework for investigation. We study several actions on these objects from the poset perspective, including gyration on fully-packed loops, and show some surprising periodicity and orbit-average properties. We also give a permutation-case bijection and study the poset structures on both sides of this bijection.

Discrete Holomorphicity and Perturbed Conformal Field Theory

Robert Weston *(Heriot-Watt University, Edinburgh)*

I will introduce the concept of discrete holomorphicity for functions on a 2D lattice embedded into the complex plane. I will briefly discuss the important role that functions with this property have played in proving the existence and uniqueness of the continuum limit of certain models of statistical mechanics. I will then go on to discuss a systematic procedure for constructing discretely holomorphic operators in solvable statistical mechanical models in terms of the underlying quantum group symmetry of these models. This procedure will be illustrated for the Chiral Potts model. I will show how the resulting discrete holomorphicity conditions for non-critical models leads directly to an identification of these models with a corresponding perturbed conformal field theory.