The Standard Model as effective field theory

with fundamental scale $\Lambda_{UV}^2 \gg 1\,\mathrm{TeV}$

$$\mathcal{L}_{SM} = \mathcal{L}_{kin} + gA_\mu \bar{F}\gamma_\mu F + Y_{ij}\bar{F}_i H F_j + \lambda(H^\dagger H)^2$$

d=4

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d=4

$$+ \quad \frac{b_{ij}}{\Lambda_{UV}} L_i L_j H H$$

$$+ \quad \frac{c_{ijkl}}{\Lambda_{UV}^2} \bar{F}_i F_j \bar{F}_k F_\ell \; + \; \frac{c_{ij}}{\Lambda_{UV}} \bar{F}_i \sigma_{\mu\nu} F_j G^{\mu\nu} + \ldots$$

$$+ \quad \ldots$$

d>4

$$\mathcal{L}_{SM} \;=\; \mathcal{L}_{kin} + gA_\mu \bar{F}\gamma_\mu F + Y_{ij}\bar{F}_i H F_j + \lambda(H^\dagger H)^2$$

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$$+ \quad \ldots$$

d>4

$\Lambda_{UV} \gg$ TeV (pointlike limit) nicely accounts for ‘what we see’

$$+\theta\,\tilde{G}_{\mu\nu}\tilde{G}^{\mu\nu}$$

d=4

$$\mathcal{L}_{SM} \;=\; \mathcal{L}_{kin} + gA_\mu \bar{F}\gamma_\mu F + Y_{ij}\bar{F}_i H F_j + \lambda(H^\dagger H)^2$$

d=4

$$
\begin{aligned}
&+ \quad \frac{b_{ij}}{\Lambda_{UV}} L_i L_j H H \\
&+ \quad \frac{c_{ijkl}}{\Lambda_{UV}^2}\, \bar{F}_i F_j \bar{F}_k F_\ell \;+\; \frac{c_{ij}}{\Lambda_{UV}} \bar{F}_i \sigma_{\mu\nu} F_j G^{\mu\nu} + \ldots \\
&+ \quad \ldots
\end{aligned}
$$

d>4

$\Lambda_{UV} \gg$ TeV (pointlike limit) nicely accounts for ‘what we see’

$$+ c\Lambda_{UV}^2 \, H^\dagger H$$ d=2

$$+ \theta \, \tilde{G}_{\mu\nu} \tilde{G}^{\mu\nu}$$ d=4

$$\mathcal{L}_{SM} \; = \; \mathcal{L}_{kin} + g A_\mu \bar{F} \gamma_\mu F + Y_{ij} \bar{F}_i H F_j + \lambda (H^\dagger H)^2$$ d=4

$$\begin{aligned} & + \quad \frac{b_{ij}}{\Lambda_{UV}} L_i L_j H H \\ & + \quad \frac{c_{ijkl}}{\Lambda_{UV}^2} \bar{F}_i F_j \bar{F}_k F_\ell \; + \; \frac{c_{ij}}{\Lambda_{UV}} \bar{F}_i \sigma_{\mu\nu} F_j G^{\mu\nu} + \ldots \\ & + \quad \ldots \end{aligned}$$ d>4

$\Lambda_{UV} \gg$ TeV (pointlike limit) nicely accounts for 'what we see' the three problems

$$+\Lambda_{UV}^4 \sqrt{g}$$ d=0

$$+c\Lambda_{UV}^2\, H^\dagger H$$ d=2

$$+\theta\, \tilde{G}_{\mu\nu}\tilde{G}^{\mu\nu}$$ d=4

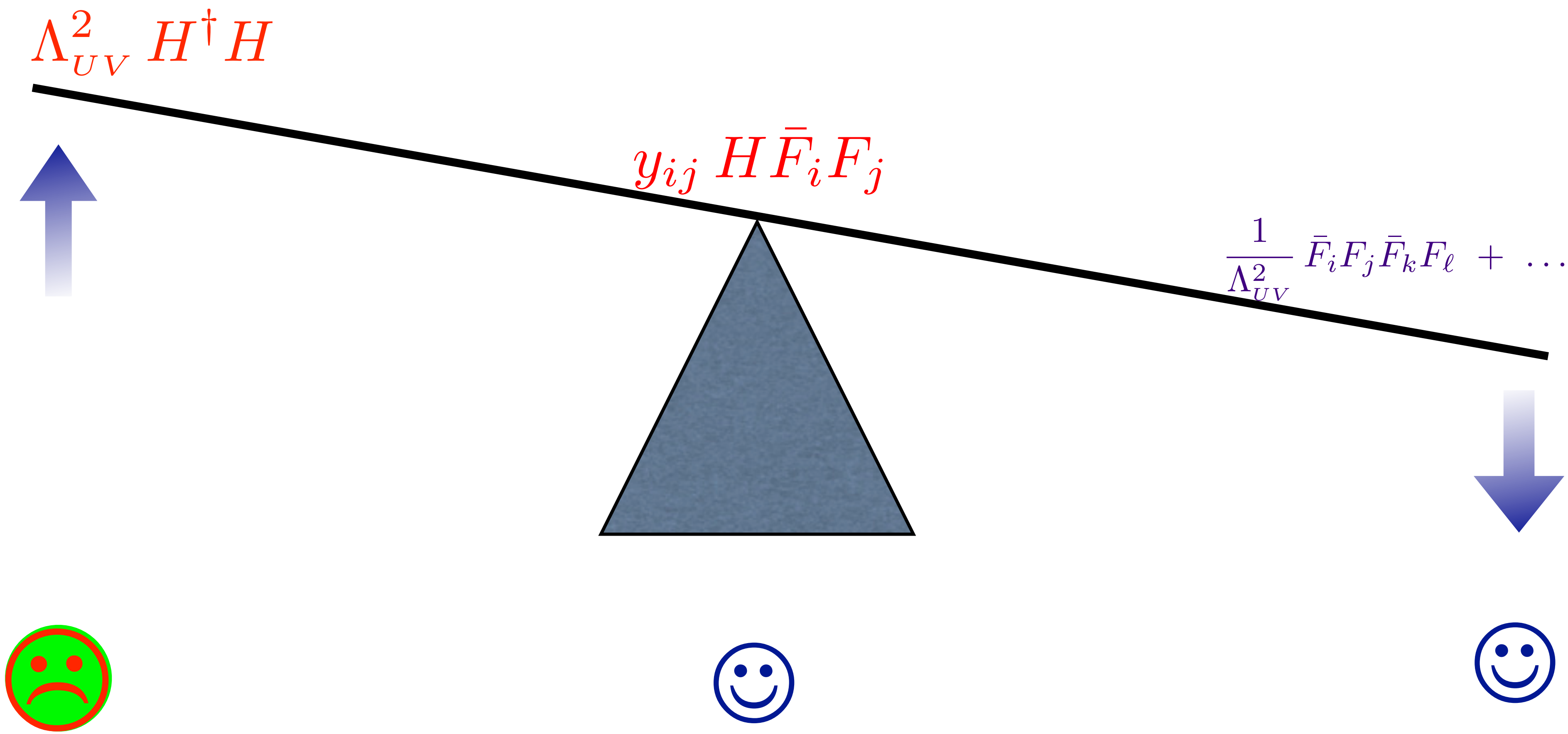
$$\mathcal{L}_{SM} \;=\; \mathcal{L}_{kin} + gA_\mu \bar{F}\gamma_\mu F + Y_{ij}\bar{F}_i H F_j + \lambda(H^\dagger H)^2$$ d=4

$$+ \quad \frac{b_{ij}}{\Lambda_{UV}} L_i L_j H H$$
$$+ \quad \frac{c_{ijkl}}{\Lambda_{UV}^2}\bar{F}_i F_j \bar{F}_k F_\ell \;+\; \frac{c_{ij}}{\Lambda_{UV}}\bar{F}_i\sigma_{\mu\nu}F_j G^{\mu\nu} + \ldots$$
$$+ \quad \ldots$$ d>4

$\Lambda_{UV} \gg$ TeV (pointlike limit) nicely accounts for 'what we see'

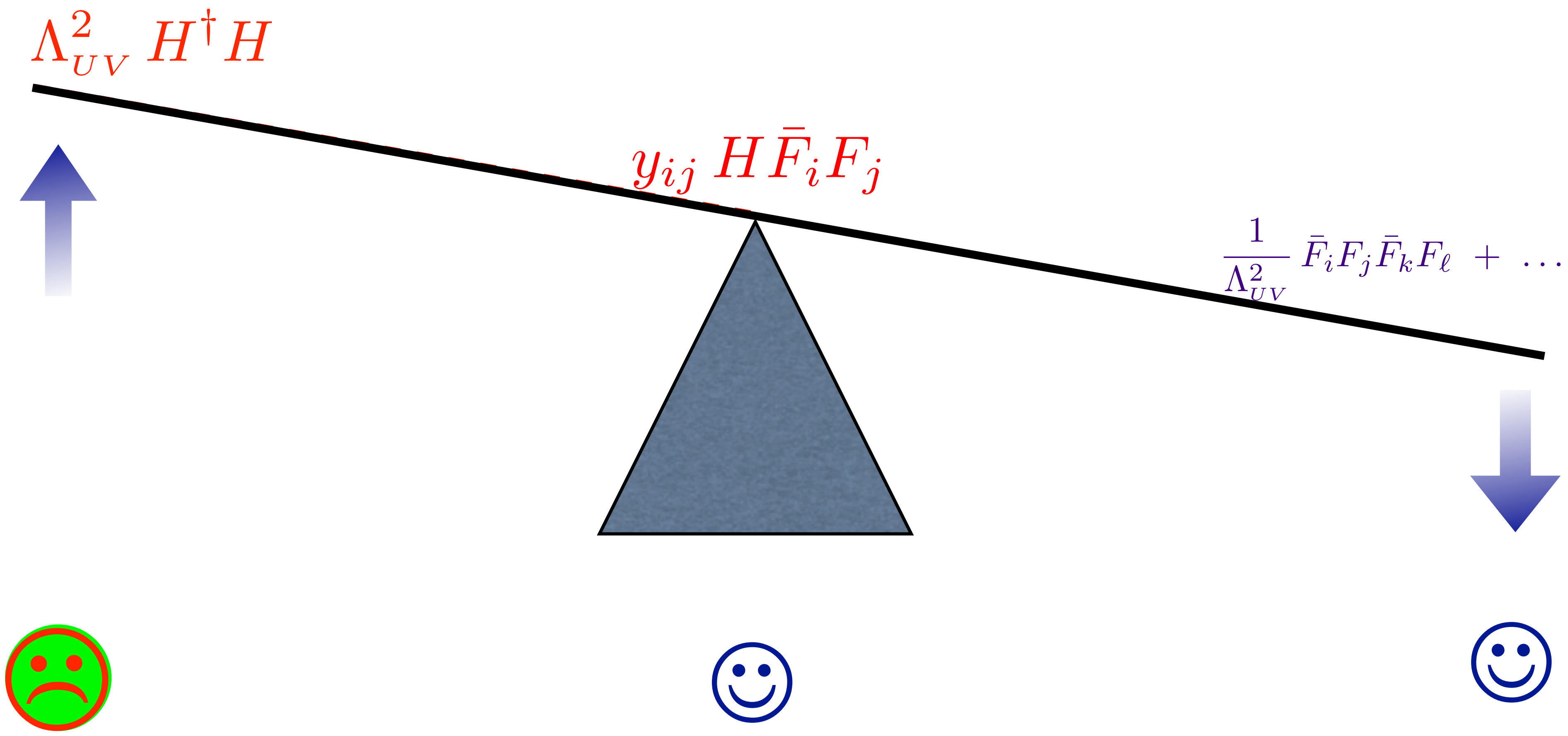
Hierarchy see-saw

Standard Model up to some $\Lambda_{UV}^2 \gg 1\,\mathrm{TeV}$



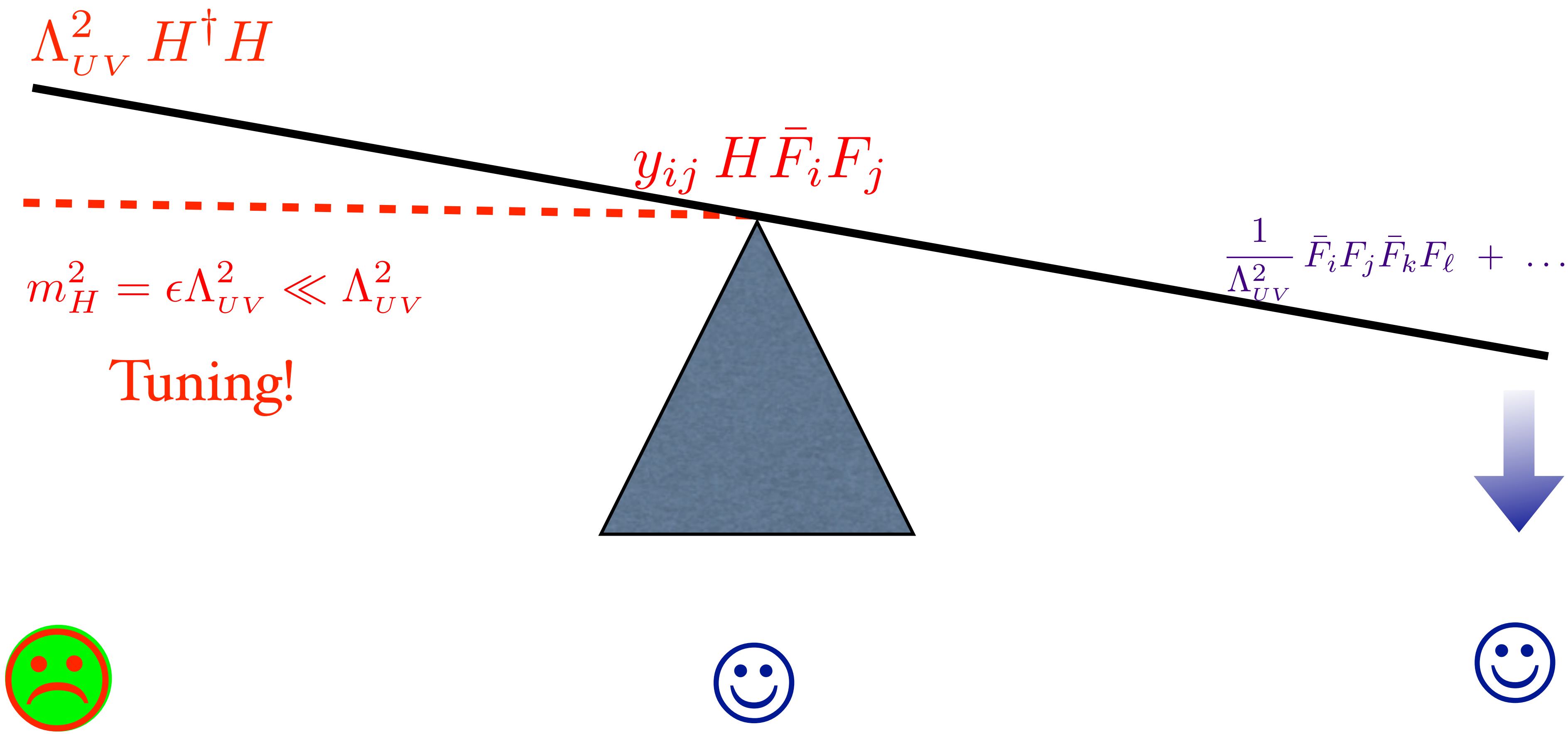
Hierarchy see-saw

Standard Model up to some $\Lambda_{UV}^2 \gg 1\,\mathrm{TeV}$



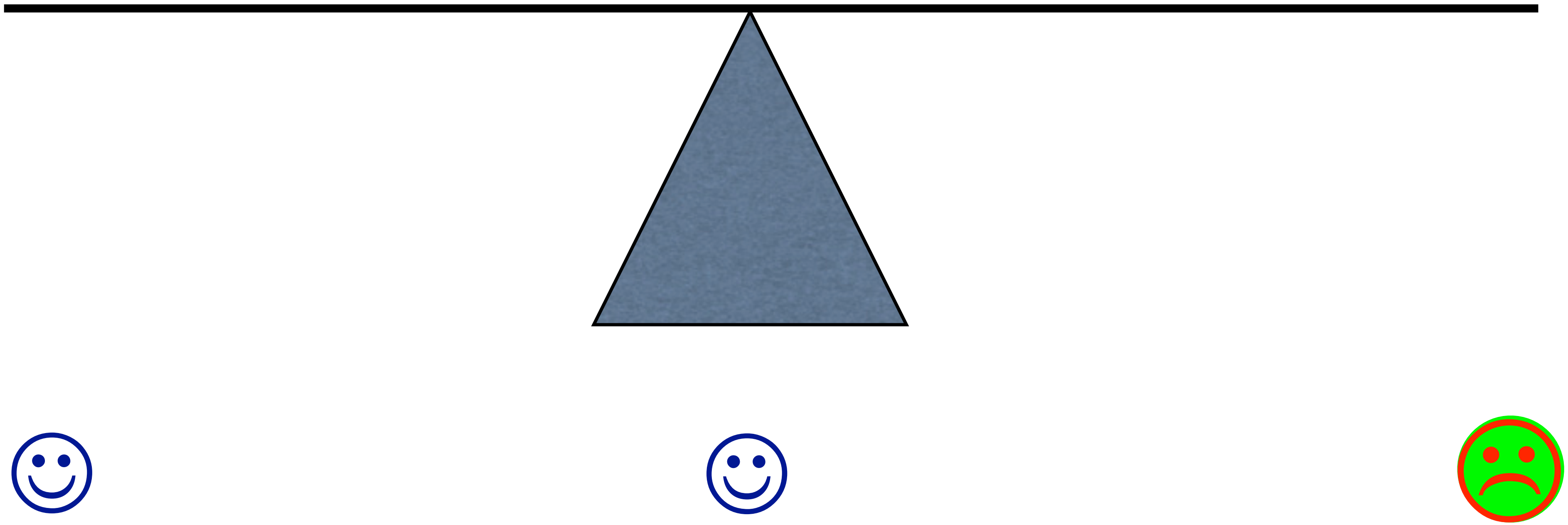
Hierarchy see-saw

Standard Model up to some $\Lambda^2_{UV} \gg 1\,\mathrm{TeV}$



Natural SM : $\quad \Lambda_{UV}^2 \lesssim 1\,\mathrm{TeV}$

$\Lambda_{UV}^2 \, H^\dagger H \qquad y_{ij} \, H \bar{F}_i F_j \qquad \frac{1}{\Lambda_{UV}^2} \bar{F}_i F_j \bar{F}_k F_\ell \; + \; \ldots$



The two possible microphysics scenarios

I. The SM is the correct description up to $\Lambda_{UV} \gg TeV$

- B, L and Flavor: beautifully in accord with observation
- Hierarchy remains a mystery, probably hinting that the question was not correctly posed
    - anthropic principle
    - failure of effective field theory ideology (UV/IR connection)

II. The SM is not the correct description already at $\Lambda_{UV} \sim 1\,\mathrm{TeV}$

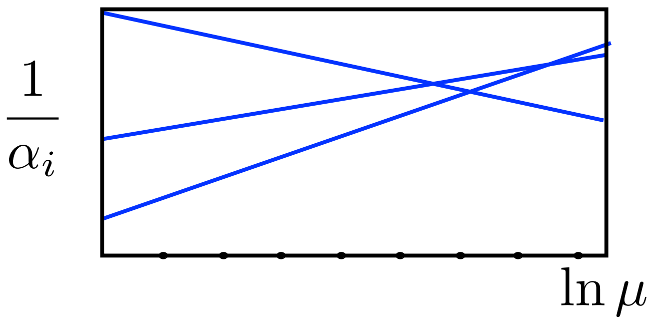
- In the correct theory the hierarchy problem does not even arise (naturalness)
- What about B, L and Flavor? In all models not nearly as nice as in SM

A high scale scenario

- $\mathcal{L}^{d=4}$ experimental success (some 2- 3-σ glitches here and there)

- Θ-QCD and Dark Matter ➔ high scale axion $\boxed{f_a \sim 10^{12}\ \mathrm{GeV}}$

- gauge couplings ready to unify around $\boxed{10^{15} \lesssim M \lesssim M_{Planck}}$



- neutrino masses $\frac{\ell\ell HH}{\Lambda}$ ➔ $m_\nu \sim \frac{v^2}{\Lambda}$ $\boxed{\Lambda \sim 10^{14}\ \mathrm{GeV}}$

- RG-evolution of SM couplings, including λ_h remarkably do not require lower scales

1. Marginality

Λ_{UV}

the fixed point theory does not possess scalar operators with dimension strictly less than 4

Λ_{IR}

$$\mathcal{L}_{\rm mass} = c\,\Lambda_{UV}^{\epsilon}\,\mathcal{O}_{4-\epsilon}$$

$$\Lambda_{IR}^{\epsilon} = c\,\Lambda_{UV}^{\epsilon} \qquad\qquad \Lambda_{IR} = c^{1/\epsilon}\,\Lambda_{UV}$$

algebraically small c and ϵ is enough to produce hierarchy

see Strassler arXiv:hep-th/0309122

Ex: Yang-Mills, TechniColor, Randall-Sundrum model

2. Symmetry

Λ_{UV}

$$\mathcal{L}_{\rm mass} = \epsilon\, \Lambda_{UV}^2\, \mathcal{O}_2$$

small parameter protected by symmetry

Λ_{IR}

$$\Lambda_{IR} = \sqrt{\epsilon}\, \Lambda_{UV}$$

- ϵ must be *hierarchically* small
- how does this smallness originate?

Ex: QCD, Supersymmetry

3. Sequestering

Λ_{UV}

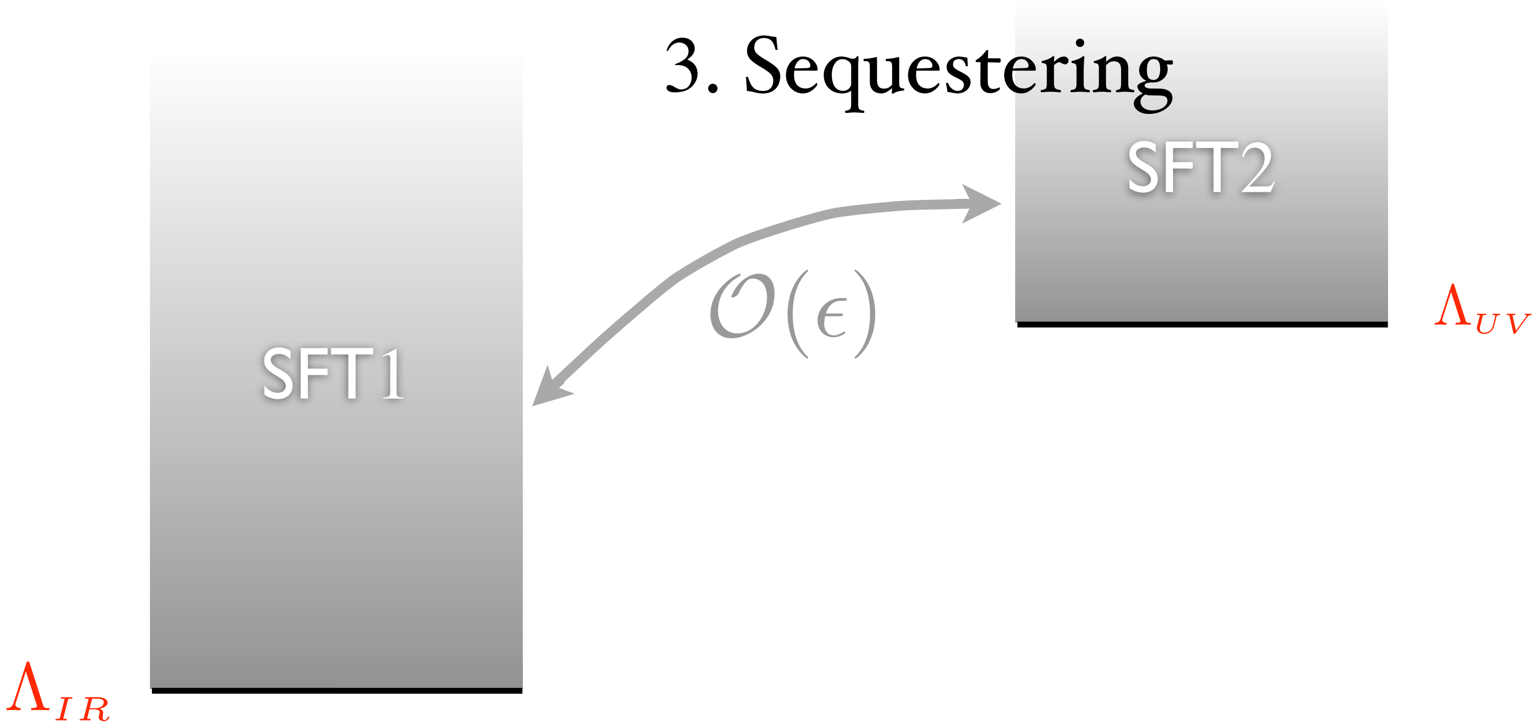
Λ_{IR}

3. Sequestering

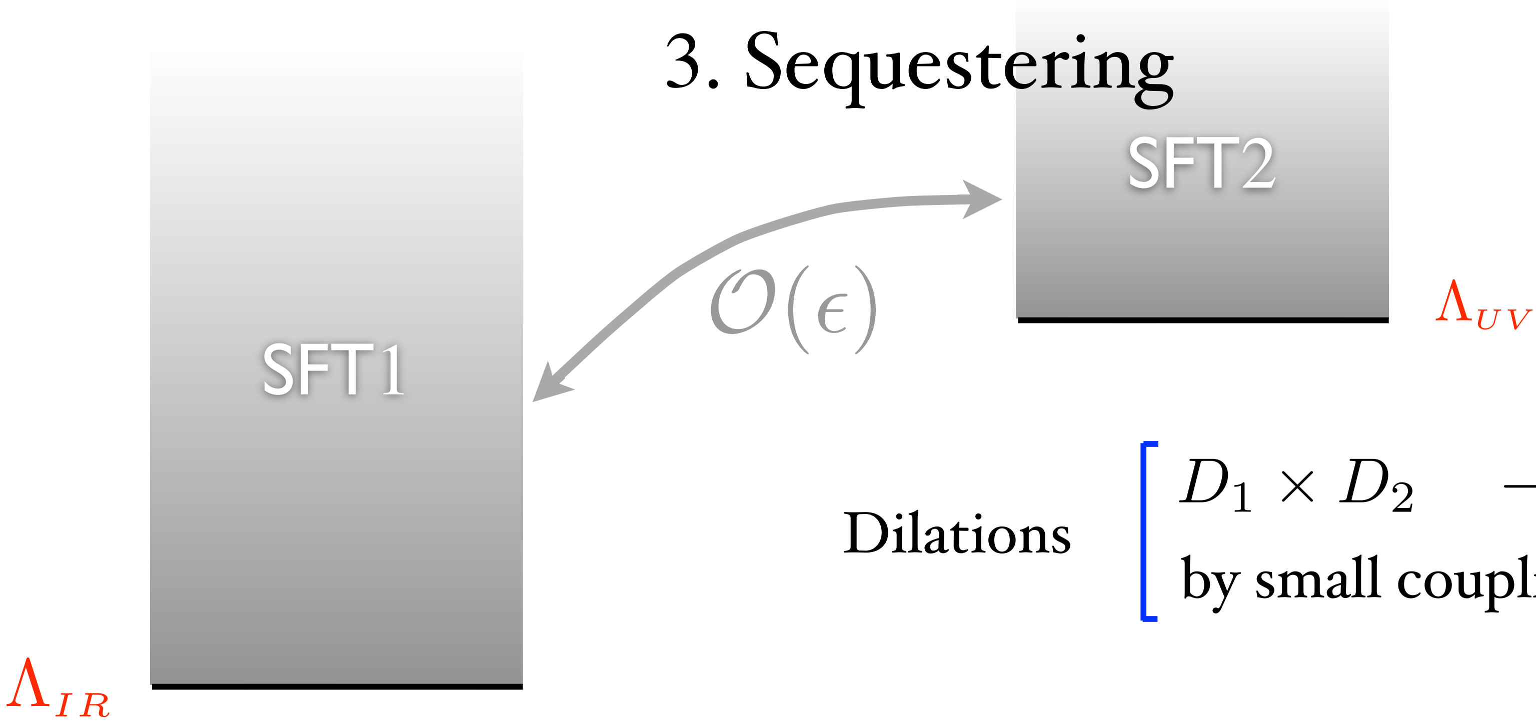
Λ_{UV}

Λ_{IR}

3. Sequestering



3. Sequestering



Dilations $\left[\begin{array}{l} D_1 \times D_2 \quad \rightarrow \quad D_{1+2} \\ \text{by small coupling} \quad \epsilon \end{array}\right.$

$\Lambda_{IR} \sim \epsilon\,\Lambda_{UV}$ technically natural

- SFT2 = 'UV completion of gravity' $\quad \epsilon = \Lambda_{UV}/M_P$
- not clearly compatible with basic principles
- but imagine we find a gorgeous candidate for SFT1?

Ex. a-gravity by a-strumia

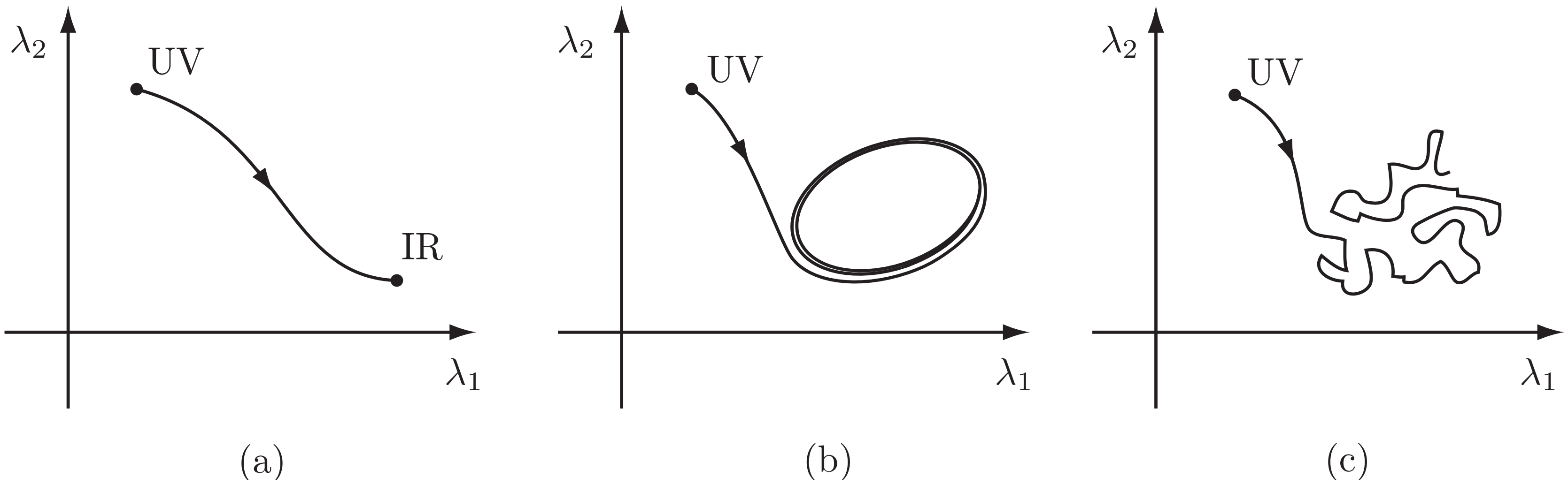
Salvio, Strumia 2012

Lecture IV

Constraining the structure of RG flows in 4D

- Irreversibility of CFT-to-CFT RG flows: a-theorem
- Ruling out non-CFT asymptotics in perturbation theory

conceivable RG flows



but all known examples asymptote to a CFT fixed point

- free (QED, massless QCD)
- strongly coupled (Supersymmetry)
- trivial (real QCD)

In particular: there are no known SFT asymptotics !

Scale Invariance versus Conformal Invariance

Wess 1960, Polchinski 1988

SFT $\quad S^\mu = T^\mu{}_\nu \, x^\nu + V^\mu \quad\quad V^\mu \neq \partial_\nu L^{\nu\mu}$

- SFT examples, if any, necessarily entail quantum effects

Callan, Coleman, Jackiw 1970

$V_\mu \equiv$ genuine non-conserved current with scaling dimension exactly equal to 3 even including quantum effects

- Often, there simply doesn't exist a candidate for

Ex.: axial current in massless (S)QCD excluded by parity selection rule

Exploring the structure of QFT by turning on an external metric

- Irreversibility of CFT-to-CFT RG flows: a-theorem
- Ruling out non-CFT asymptotics in perturbation theory

RG flow describes the change of the dynamics under a dilation

$\equiv$ change of the action under a dilation

Whenever we have some explicitly broken symmetry it proves useful to

- formally restore it by promoting couplings to sources transforming non trivially
- gauge it by adding the suitable gauge field

We shall play various related games

A. $\eta_{\mu\nu}, \quad \lambda_i \quad \longrightarrow \quad g_{\mu\nu}(x), \quad \lambda_i(x)$ + Weyl symmetry

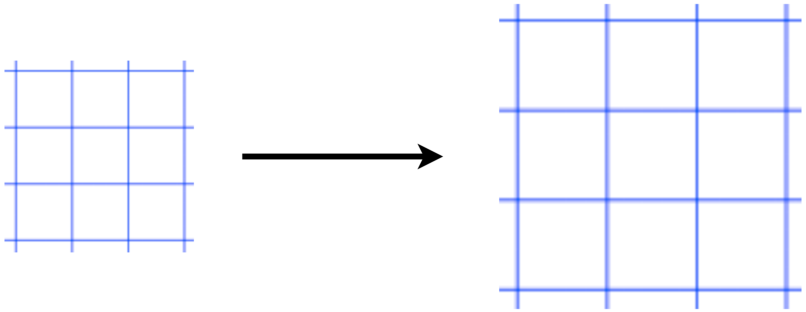
B. $\lambda_i = \text{const}$

$$\eta_{\mu\nu} \quad \longrightarrow \quad e^{-2\tau}\eta_{\mu\nu}$$

$e^{-\tau} \equiv \Omega \equiv 1 + \varphi$ background dilaton field

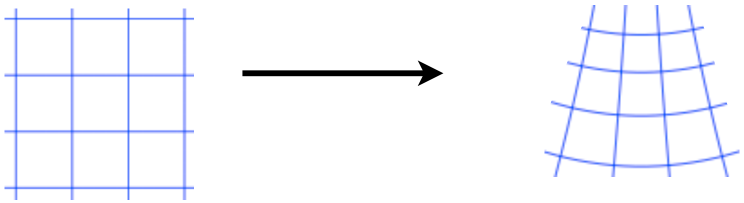
Geometric picture

dilations $\quad x^\mu \to \tilde{x}^\mu = k x^\mu$



$$(d\tilde{x})^2 = k^2 (dx)^2$$

conformal $\quad x^\mu \to \tilde{x}^\mu = \frac{x^\mu + b^\mu x^2}{1 + b^2 x^2 + 2b\cdot x}$



$$(d\tilde{x})^2 = \frac{1}{(1 + b^2 x^2 + 2b\cdot x)^2}\,(dx)^2$$

QFT in a gravitational background

Weyl Symmetry

$$g_{\mu\nu}(x) \to e^{-2\sigma(x)} g_{\mu\nu}$$
$$\Phi_a(x) \to e^{-k_a \sigma(x)} \Phi_a(x)$$

O(D,2) = subgroup of Weyl x Diffs that leaves $\eta_{\mu\nu}$ invariant

$S[g,\Phi]$ Weyl invariant $\Rightarrow$ $S[\eta,\Phi]$ Conformal invariant

Converse is also true (at classical level)

Ex.: free massless scalar field

$$\mathcal{L}_{flat} = \frac{1}{2}(\partial\varphi)^2$$

$$T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - \frac{\eta_{\mu\nu}}{2}(\partial\varphi)^2$$

$$T^\mu_\mu = -(\partial\varphi)^2 \neq 0$$

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$$\Theta_{\mu\nu} = T_{\mu\nu} - \frac{1}{6}(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box)\varphi^2$$

$$\Theta^\mu_\mu = 0$$

improvement

Ex.: free massless scalar field

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improvement

$$\mathcal{L}_{curved} = \sqrt{g}\frac{1}{2}\left[(\partial\varphi)^2 + \frac{1}{6}R\varphi^2\right] = \frac{1}{6}\sqrt{\hat{g}}R(\hat{g})$$

$$\hat{g}_{\mu\nu} \equiv \varphi^2 g_{\mu\nu}$$

$$\varphi \to e^\sigma\varphi$$
$$g_{\mu\nu} \to e^{-2\sigma}g_{\mu\nu}$$

Weyl symmetry manifest

$$\text{Weyl symm:} \quad \int \sigma(x) \left(2g^{\mu\nu} \frac{\delta S}{\delta g^{\mu\nu}(x)} + k_a \Phi_a \frac{\delta S}{\delta \Phi_a(x)} \right) = 0$$

$\sigma(x)$ arbitrary $\Downarrow$ Φ_a on-shell

$$T^\mu_\mu \equiv g^{\mu\nu} \frac{\delta S}{\delta g^{\mu\nu}(x)} = 0$$

With only global Weyl, $\boldsymbol{\sigma}$ = constant, we would instead deduce

$$\int T^\mu_\mu = 0 \qquad \longrightarrow \qquad T^\mu_\mu = \partial^\mu V_\mu$$

QFT in gravity background

✦ quantum effective action $$e^{iW[g_{\mu\nu}]} = \int D[\Phi] e^{iS[g,\Phi]}$$

- need regulation
- diff invariant
- finite by adding suitable local counterterms

In general the introduction of a regulator in curved background breaks explicitly Weyl invariance even when flat space theory is conformally invariant

$$\delta_\sigma \equiv \int d^4x \, 2\sigma(x) \, g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}(x)}$$

In ordinary QFT $\quad \delta_\sigma W =$ non-local

In CFT $\quad \delta_\sigma W = \int \sigma(x) \sqrt{g} \, \mathcal{A}(x) =$ Weyl Anomaly (local!)

also written as $\quad \langle T \rangle \equiv \langle T^\mu_\mu \rangle = \mathcal{A}(x)$

Christensen, Duff '74

The structure of the Weyl anomaly in a CFT

$\mathcal{A}(x)$ is a scalar function of the metric

in general

$$\mathcal{A}(x) = aE_4 - bR^2 - cW^2 - d\Box R$$

$$\begin{aligned} E_4 &= R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2 \\ W^2 &= R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 2R^{\mu\nu}R_{\mu\nu} + \frac{1}{3}R^2 \end{aligned}$$

$$\int \sigma(x)\sqrt{g}\left(-d\Box R + e\Lambda^2 R + f\Lambda^4\right) = \delta_\sigma \int (-1)\sqrt{g}\left(\frac{d}{12}R^2 + \frac{e}{2}\Lambda^2 R + \frac{f}{4}\Lambda^4\right)$$

the last three terms can be written as variation of local functional

they can be eliminated by a choice of counterterms

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the last three terms can be written as variation of local functional

they can be eliminated by a choice of counterterms

Wess-Zumino consistency condition

$$\delta_\sigma W \;=\; \int \sigma(x)\sqrt{g}\,\mathcal{A}(x)$$

Weyl symmetry is abelian

$$[\delta_{\sigma_2}, \delta_{\sigma_1}]\, W = \delta_{\sigma_2}\left(\int d^4x_1 \sigma_1 \sqrt{g}\mathcal{A}\right) - \delta_{\sigma_1}\left(\int d^4x_2 \sigma_2 \sqrt{g}\mathcal{A}\right) = 0$$

$$\mathcal{A}(x) \;=\; aE_4 - bR^2 - cW^2 \qquad \longrightarrow \qquad \boxed{aE_4 - cW^2}$$

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$$\mathcal{A}(x) \;=\; aE_4 - \cancel{bR^2} - cW^2 \quad \longrightarrow \quad \boxed{aE_4 - cW^2}$$

Easy to check using

$$
\begin{aligned}
\delta\sqrt{g} &= -4\sigma\sqrt{g} \\
\delta\Box &= 2\sigma\Box - 2\nabla_\mu\sigma\nabla^\mu \\
\delta R &= 2\sigma R + 6\Box\sigma \\
\delta E_4 &= 4\sigma E_4 - 8G^{\mu\nu}\nabla_\mu\nabla_\nu\sigma \\
\delta W^2 &= 4\sigma W^2 \\
\delta G_{\mu\nu} &= 2\left(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box\right)\sigma
\end{aligned}
$$

✦ In general CFT

$$\langle T_{\mu\nu}(x) T_{\rho\sigma}(0)\rangle = \frac{c}{x^8} I_{\mu\nu\rho\sigma}(x)$$

$$\langle T_{\mu\nu}(x) T_{\rho\sigma}(0) T_{\gamma\delta}(y)\rangle = c\, C_{\mu\nu\rho\sigma\gamma\delta}(x,y) + a\, A_{\mu\nu\rho\sigma\gamma\delta}(x,y)$$

Stanev '88
Osborn, Petkou '94

✦ In a free field theory one has

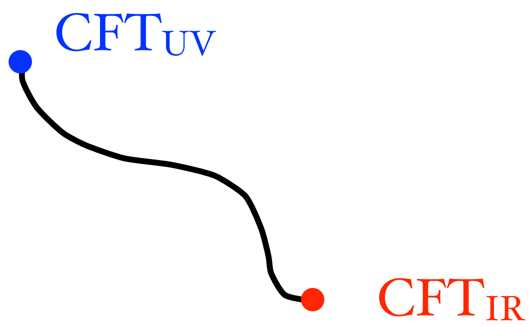
Christensen, Duff '79

$$a = \frac{1}{5760\pi^2}\left(n_s + \frac{11}{2} n_f + 62 n_v\right)$$

$$c = \frac{1}{5760\pi^2}\left(3n_s + 9n_f + 36n_v\right)$$

both *a* and *c* are a weighted measure of the number of degrees of freedom

Cardy's Conjecture (1988) : a decreases monotonically along the RG flow



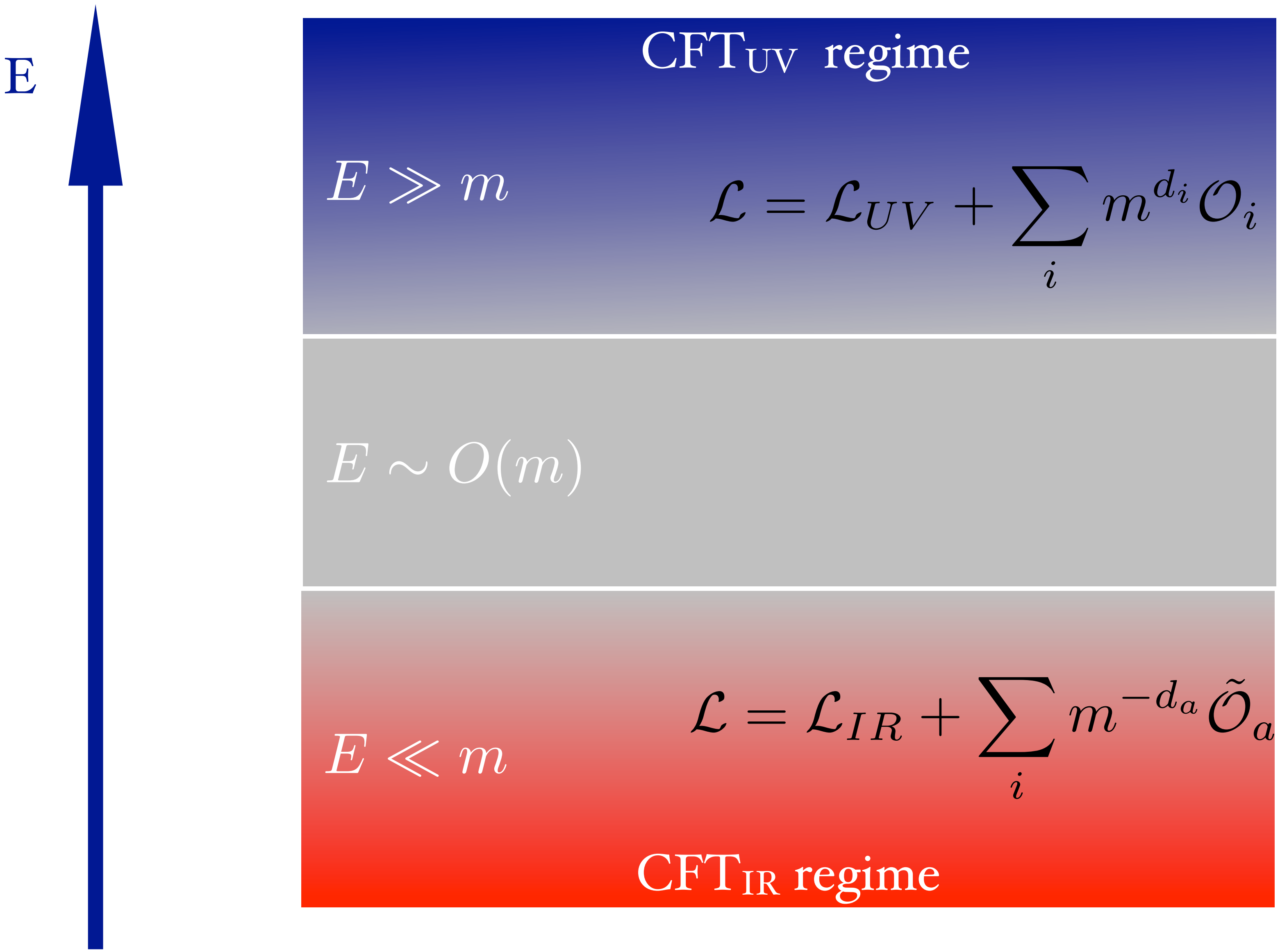
$$\boxed{a_{UV} > a_{IR}}$$

In 2D there was already Zamolodchikov's c-theorem (1986) stating the monotonicity of the unique coefficient in the 2D Weyl anomaly

$$\delta_\sigma W = \int d^2x \sqrt{g}\, c\, R(g) \equiv \int d^2x \sqrt{g}\, c\, E_2(g)$$

Proof of Cardy's Conjecture: the a-theorem

Komargodski and Schwimmer 2011



Consider now putting this system in an extermal metric

$$\mathcal{L} = \mathcal{L}_{UV} + \sum_i m^{d_i} \mathcal{O}_i + \Delta\mathcal{L}_{UV}(g)$$

$E \gg m$

$E \sim O(m)$

$E \ll m$

$$\mathcal{L} = \mathcal{L}_{IR} + \sum_a m^{-d_a} \tilde{\mathcal{O}}_a + \Delta\mathcal{L}_{IR}(g)$$

$\Delta\mathcal{L}_{UV} \equiv$ metric (curvature) dependent counterterms needed to define a renormalized quantum action W[g]

$\Delta\mathcal{L}_{IR} \equiv$ metric dependent terms associated with positive powers of m

The general structure of $\Delta\mathcal{L}_{UV}$ and $\Delta\mathcal{L}_{IR}$ is the same:

local scalar functions of dimension ≤ 4

In the classification of these terms it is crucial to consider what happens for the case of a conformally flat background metric

$$\hat{g}_{\mu\nu} = \Omega(x)^2 \, \eta_{\mu\nu}$$

Counterterms

$d > 4$ in sensible theories

I. $\sqrt{\hat{g}} R \, \mathcal{O}$ $\quad d_{\mathcal{O}} \leq 2$ $\quad \sqrt{\hat{g}} \nabla_\mu R \, J^\mu$ $\quad \sqrt{\hat{g}} R_{\mu\nu} \, J^{\mu\nu}$

II. $\sqrt{\hat{g}}$ $\quad \sqrt{\hat{g}} R$ $\quad \sqrt{\hat{g}} R^2$ $\quad \sqrt{\hat{g}} E_4$ $\quad \sqrt{\hat{g}} W^2$

Counterterms

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I. $\sqrt{\hat{g}} R \, \mathcal{O}$ $\quad d_{\mathcal{O}} \leq 2$ $\quad \cancel{\sqrt{\hat{g}} \nabla_\mu R \, J^\mu \quad \sqrt{\hat{g}} R_{\mu\nu} \, J^{\mu\nu}}$

irrelevant

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irrelevant

$$\sqrt{\hat{g}} R\, \mathcal{O} \;\downarrow\; \Omega \Box \Omega\, \mathcal{O}$$

II.

$$\sqrt{\hat{g}} \;\downarrow\; \Omega^4 \qquad \sqrt{\hat{g}} R \;\downarrow\; \Omega \Box \Omega \qquad \sqrt{\hat{g}} R^2 \;\downarrow\; \Omega^{-2} (\Box \Omega)^2 \qquad \sqrt{\hat{g}} E_4 \;\downarrow\; 0 \qquad \sqrt{\hat{g}} W^2 \;\downarrow\; 0$$

Counterterms

$d > 4$ in sensible theories

I. $\sqrt{\hat{g}} R\, \mathcal{O}$ $\quad d_{\mathcal{O}} \leq 2$ $\quad \cancel{\sqrt{\hat{g}} \nabla_\mu R\, J^\mu \quad \sqrt{\hat{g}} R_{\mu\nu}\, J^{\mu\nu}}$

irrelevant

$$\sqrt{\hat{g}} R\, \mathcal{O} \;\downarrow\; \Omega \Box \Omega\, \mathcal{O}$$

II.

$$\sqrt{\hat{g}} \;\downarrow\; \Omega^4 \qquad \sqrt{\hat{g}} R \;\downarrow\; \Omega \Box \Omega \qquad \sqrt{\hat{g}} R^2 \;\downarrow\; \Omega^{-2} (\Box \Omega)^2 \qquad \sqrt{\hat{g}} E_4 \;\downarrow\; 0 \qquad \sqrt{\hat{g}} W^2 \;\downarrow\; 0$$

On shell dilaton : $\quad \Box \Omega = 0$

Counterterms

$d > 4$ in sensible theories

I. $\cancel{\sqrt{\hat{g}} R\, \mathcal{O}}$ $\quad d_{\mathcal{O}} \leq 2$ $\quad \cancel{\sqrt{\hat{g}} \nabla_\mu R\, J^\mu \quad \sqrt{\hat{g}} R_{\mu\nu}\, J^{\mu\nu}}$

irrelevant

$$\downarrow$$

$$\Omega \Box \Omega\, \mathcal{O}$$

II. $\sqrt{\hat{g}}$ $\quad \cancel{\sqrt{\hat{g}} R}$ $\quad \cancel{\sqrt{\hat{g}} R^2}$ $\quad \cancel{\sqrt{\hat{g}} E_4}$ $\quad \cancel{\sqrt{\hat{g}} W^2}$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$\Omega^4 \qquad \Omega \Box \Omega \qquad \Omega^{-2} (\Box \Omega)^2 \qquad 0 \qquad 0$$

On shell dilaton : $\quad \Box \Omega = 0$

Counterterms

$d > 4$ in sensible theories

I. $\cancel{\sqrt{\hat{g}} R\, \mathcal{O}}$ $\quad d_{\mathcal{O}} \leq 2$ $\quad \cancel{\sqrt{\hat{g}} \nabla_\mu R\, J^\mu \quad \sqrt{\hat{g}} R_{\mu\nu}\, J^{\mu\nu}}$

$\downarrow$

$\Omega \Box \Omega\, \mathcal{O}$

irrelevant

II. $\sqrt{\hat{g}}$ $\quad \cancel{\sqrt{\hat{g}} R}$ $\quad \cancel{\sqrt{\hat{g}} R^2}$ $\quad \cancel{\sqrt{\hat{g}} E_4}$ $\quad \cancel{\sqrt{\hat{g}} W^2}$

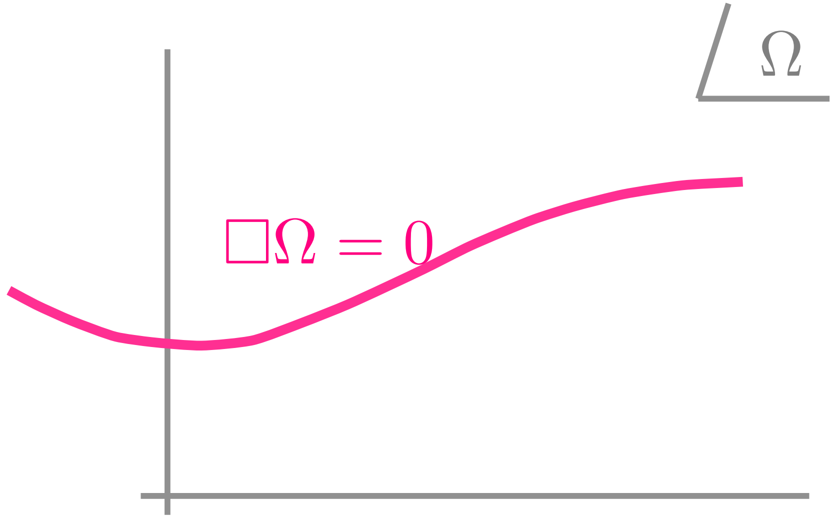
$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$

$\Omega^4 \quad \Omega \Box \Omega \quad \Omega^{-2} (\Box \Omega)^2 \quad 0 \quad 0$

On shell dilaton : $\quad \Box \Omega = 0$

$$\boxed{W\Big|_{\Omega\, :\, \Box\Omega = 0} \quad = \quad W[\Omega, \lambda_{QFT}, \Lambda_{cc}]}$$

schematically



consider from here on $\quad W[\Omega] \equiv W[\Omega]\Big|_{\Box\Omega=0}$

$$\frac{\delta}{\delta\Omega(x_1)}\cdots\frac{\delta}{\delta\Omega(x_n)}W\Big|_{\Omega=1} = \mathcal{M}(x_1,\ldots,x_n)$$

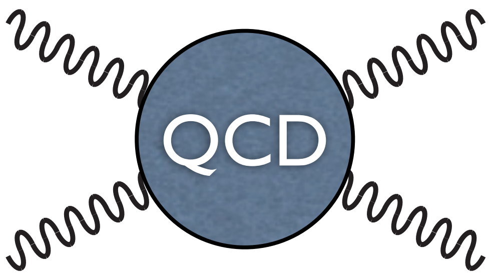
$\mathcal{M}$ can be interpreted as the n-dilaton scattering amplitude

QCD analogy

Effective QCD action in background photon field : $W[A_\mu]$

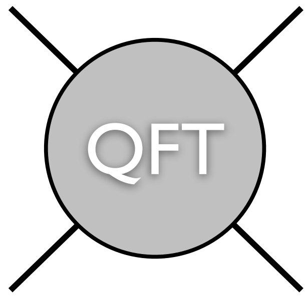
$$\frac{\delta}{\delta A_{\mu_1}(x_1)} \cdots \frac{\delta}{\delta A_{\mu_n}(x_n)} W \quad = \quad \text{QCD mediated n-photon amplitude}$$

Ex. $$\frac{\delta}{\delta A_{\mu_1}(x_1)} \cdots \frac{\delta}{\delta A_{\mu_4}(x_4)} W \quad =$$



light-by-light
scattering

The analogue QFT mediated dilaton-by-dilaton scattering



affords a remarkable insight into the structure of our QFT

Like for all on-shell n-point dilaton amplitudes the only renormalization needed to define this amplitude concerns a constant term associated with the cosmological constant

$$\mathcal{M} \equiv \mathcal{M}(\lambda_{QFT}, \Lambda_{cc})$$

4-point amplitude

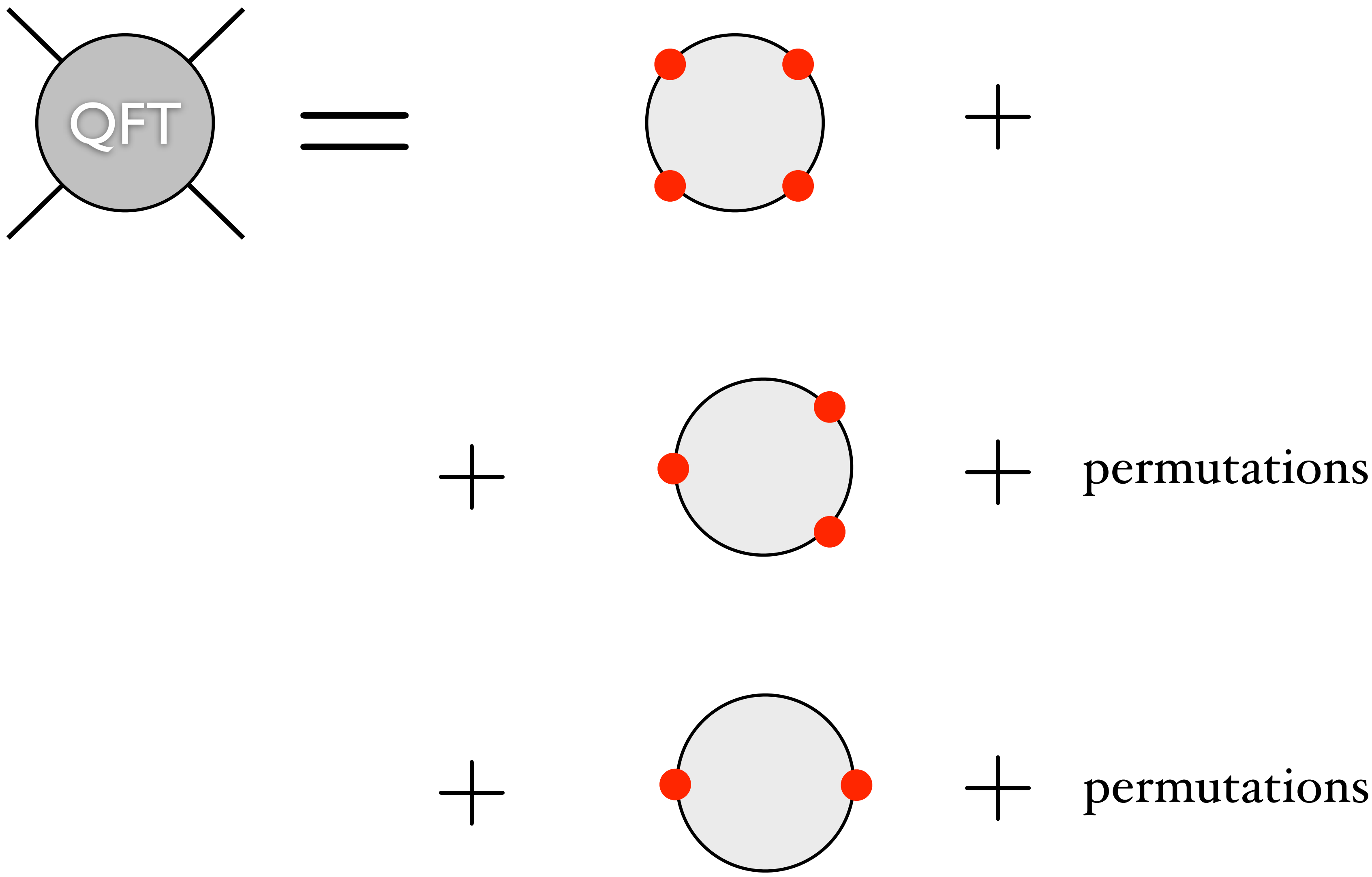
$$\hat{g}_{\mu\nu} = \Omega(x)^2 \eta_{\mu\nu}$$

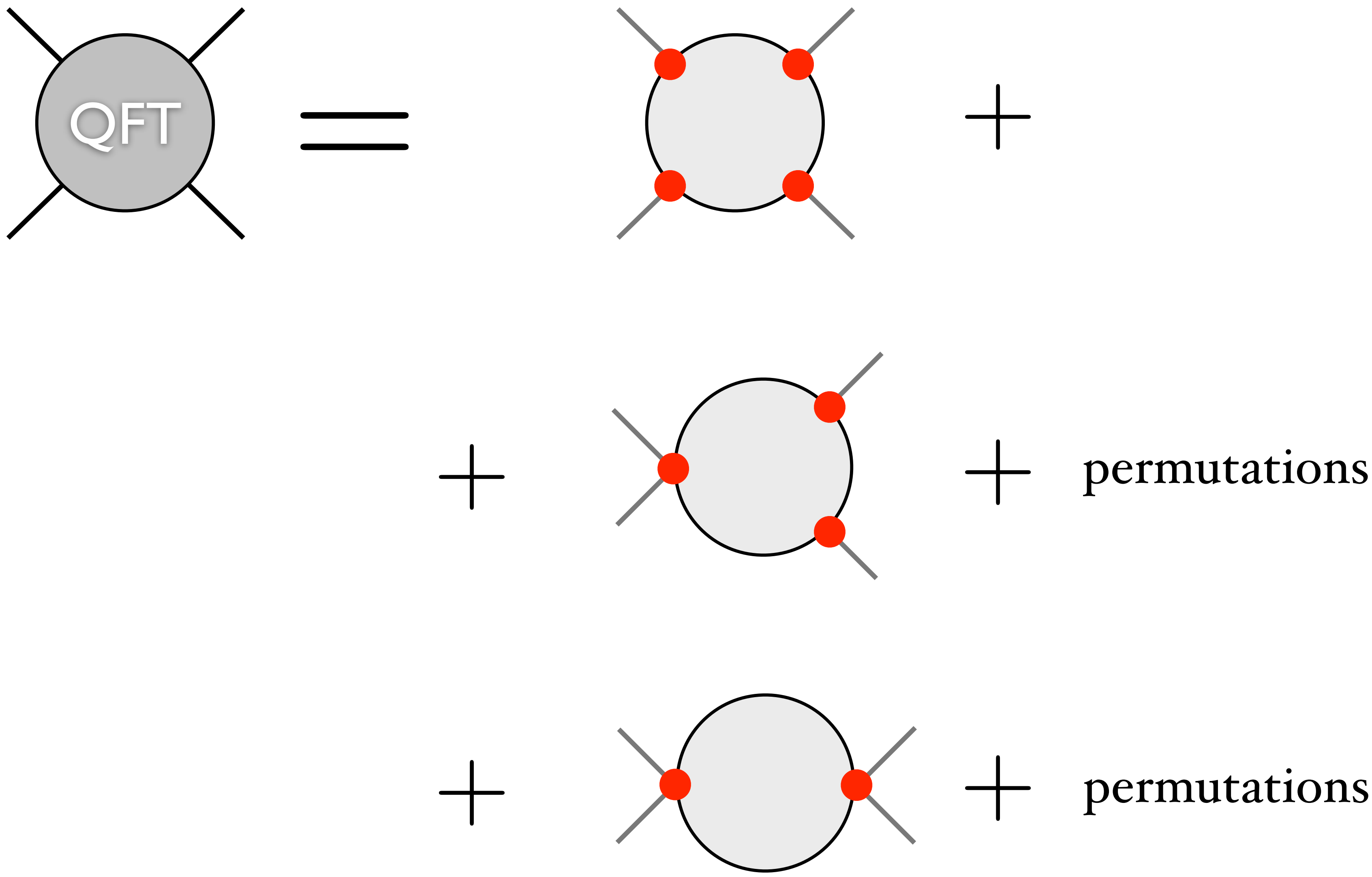
$$\frac{\delta}{\delta\Omega} = \frac{1}{\Omega}\frac{\delta}{\delta \ln \Omega} \qquad\qquad -\frac{\delta}{\delta \ln \Omega} W = T^{\mu}_{\mu} \equiv T$$

$$\mathcal{M}(p_1,\ldots,p_4) = \frac{\delta^4 W}{\delta\Omega(p_1)\cdots\delta\Omega(p_4)} = \langle T(p_1)T(p_2)T(p_3)T(p_4)\rangle + \langle T(p_1+p_2)T(p_3)T(p_4)\rangle + \text{permutations}$$
$$+ \langle T(p_1+p_2)T(p_3+p_4)\rangle + \text{permutations}$$
$$+ \langle T(p_1+p_2+p_3)T(p_4)\rangle + \text{permutations}$$

$$p_1^2 = p_2^2 = p_3^2 = p_4^2 = 0$$

$$\mathcal{M}(p_1,\ldots,p_4) \equiv (2\pi)^4 \delta^4(p_1+p_2+p_3+p_4)\, A(s,t)$$


QFT
permutations
permutations


QFT
permutations
permutations

In a CFT $\quad W[\Omega]$ is local and fully determined by the Weyl anomaly up to cosmological constant term

$$\xrightarrow{\text{on shell} \quad \Omega} \qquad \Delta W = \frac{\Lambda_{cc}}{4!}\,\Omega^4$$

neglecting momentarily CC term

$$\boxed{W_{CFT}[\Omega^2 g_{\mu\nu}] = W_{CFT}[g_{\mu\nu}] - S_{\rm WZ}[g_{\mu\nu},\Omega; a, c]}$$

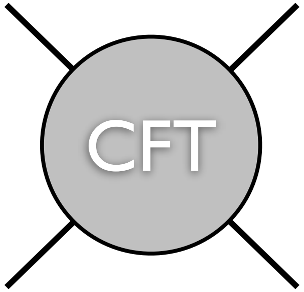
Cappelli, Coste 1988
Tomboulis 1990
Schwimmer, Theisen '11

$$S_{\rm WZ}[g_{\mu\nu},\Omega; a, c] = \int d^4x\sqrt{-g}\,\Big\{a\Big[\ln\Omega\,E_4(g) - 4\big(R^{\mu\nu}(g) - \frac{1}{2}g^{\mu\nu}R(g)\big)\Omega^{-2}\partial_\mu\Omega\partial_\nu\Omega - 4\Omega^{-3}(\partial\Omega)^2\Box\Omega + 2\Omega^{-4}(\partial\Omega)^4\Big] - c\,\ln\Omega\,W^2(g)\Big\}.$$

$$g_{\mu\nu} = \eta_{\mu\nu}$$

$$W_{CFT}[\Omega] \quad \longrightarrow \quad -2\,a\;\Omega^{-4}(\partial\Omega)^2(\partial\Omega)^2 + \frac{\Lambda}{4!}\,\Omega^4$$

$$\Box\Omega = 0$$



$$\boxed{A(s,t) \;=\; -4\,a\,\left[s^2 + t^2 + (s+t)^2\right] \;+\Lambda}$$

this basic result leads to a simple proof of the a-theorem

Komargodski, Schwimmer '11


$\mathrm{CFT}_{\mathrm{UV}}$
$\mathrm{CFT}_{\mathrm{IR}}$

$\mathrm{CFT}_{\mathrm{UV}}$

$$\mathcal{L} = \mathcal{L}_{UV} + \sum_i c_i m^{4-d_i} \mathcal{O}_i$$

$$d_i < 4$$

$$\mathcal{L} = \mathcal{L}_{IR} + \sum_a b_a \frac{1}{m^{d_a - 4}} \tilde{\mathcal{O}}_a$$

$$d_a > 4$$

$\mathrm{CFT}_{\mathrm{IR}}$

$\mathrm{CFT_{UV}}$

$$\mathcal{L} = \mathcal{L}_{UV} + \sum_i c_i m^{4-d_i} \mathcal{O}_i$$

$$d_i < 4$$

$$A(s,0) = -8a_{UV}\, s^2 \left[1 + \left(\frac{m}{\sqrt{s}}\right)^{\#}\right] + \Lambda_{cc}^{UV}$$

$$\mathcal{L} = \mathcal{L}_{IR} + \sum_a b_a \frac{1}{m^{d_a-4}} \tilde{\mathcal{O}}_a$$

$$d_a > 4$$

$$A(s,0) = -8a_{IR}\, s^2 \left[1 + \left(\frac{\sqrt{s}}{m}\right)^{\#}\right] + \Lambda_{cc}^{IR}$$

$\mathrm{CFT_{IR}}$

Can relate UV to IR via dispersive argument

Using

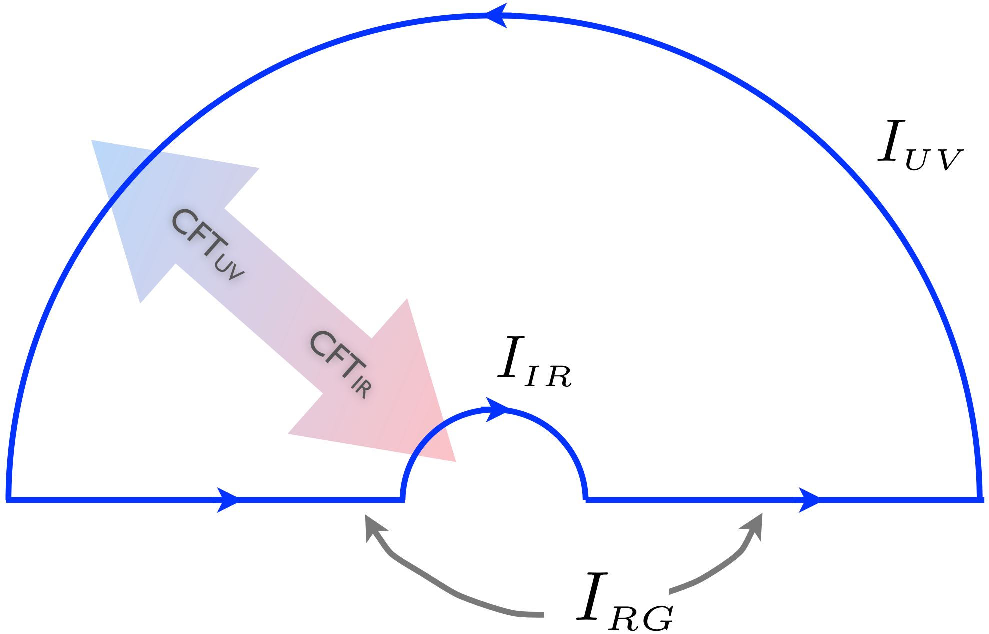
- $A(s) \equiv A(s,0)$ is analitic with cut on real s axis

•crossing $A(s) = A(-s)$ $\qquad (t=0,\ s \leftrightarrow u \quad \equiv \quad s \leftrightarrow -s)$

•'reality' $A^*(s) = A(s^*)$

•optical theorem

$$-i[A(s+i\epsilon) - A(-s+i\epsilon)]$$
$$= \ \mathrm{Im}A(s+i\epsilon) \ = \ s\,\sigma(\Omega\Omega \to \mathrm{QFT})$$



$$\frac{1}{2\pi i}\oint_C \frac{A(s,0)}{s^3}ds = 0$$

$$I_{IR} + I_{UV} + I_{RG} = 0$$

$$I_{IR} = 4\,a_{IR}$$

CC term drops !

$$I_{UV} = -4\,a_{UV}$$

$$I_{RG} = \frac{1}{\pi}\int \frac{\mathrm{Im}\,A}{s^3} = \frac{1}{\pi}\int \frac{\sigma(\Omega\Omega \to \mathrm{QFT})}{s^2} > 0$$

$$\boxed{a_{IR} = a_{UV} - \frac{1}{4} I_{RG} < a_{UV}}$$

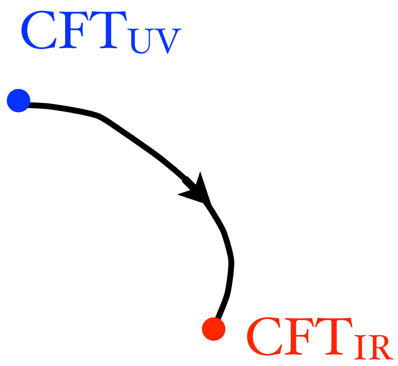
- $I_{RG} = a_{UV} - a_{IR}$ is nicely finite in CFT-to-CFT flows

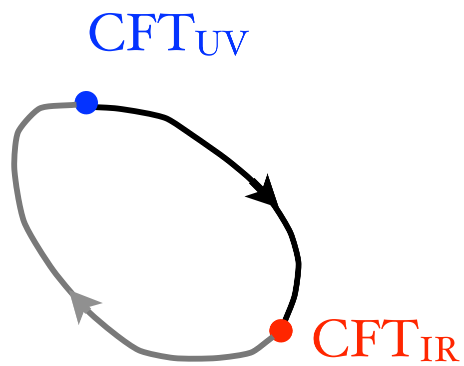
- can check directly that convergence of I_{RG} in both UV and IR corresponds to convergence of RG flow to a CFT

- It had to be so, since $d^2 A/ds^2$ is finite; just a function of the renormalized QFT couplings

- Finiteness of I_{RG} $\Rightarrow$ constraint on QFT asymptotics a-theorem implies the deep notion of irreversibility of RG flow

given





is impossible

however I do not know of insightful applications in particle physics

Does a-theorem constrains phases of N_C,N_F QCD ?

$$a \propto N_S + 11N_F + 62N_V$$

UV: quarks and gluons $\qquad a_{UV} \propto 11N_F + 62(N_C^2 - 1)$

IR: assume chiral symmetry breaking vacuum and mass gap

$N_F^2 - 1$ NG-bosons $\qquad a_{IR} \propto N_F^2 - 1$

easy to check that $\qquad a_{IR} < a_{UV}$

for any asymptotically free choice of N_F and N_C

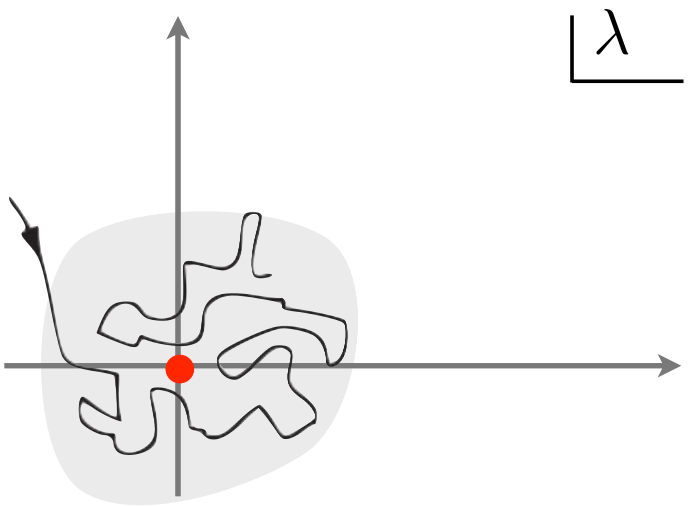
Lecture III

Constraining

- RG asymptotics in weakly coupled deformations of CFTs
- SFT asymptotics

Goal: study RG flows (perturbatively) near CFT fixed point

Ex: free field theory with small marginal couplings



$$\beta_I = b_{IJK}\lambda_J\lambda_K + \ldots$$

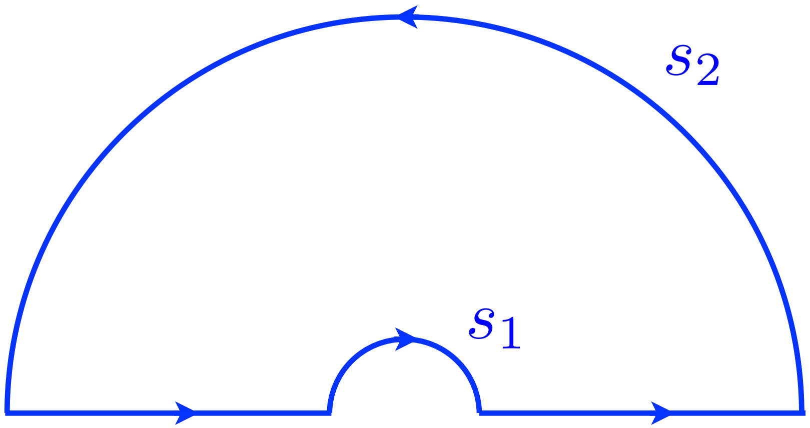
$$|\lambda_I| \ll 1$$

basic idea: $A(s)$ is finite, modulo CC term

more precisely: all UV divergences encountered in its computation must get reabsorbed in the running QFT couplings

$$A(s) = \alpha(s) s^2 \qquad\qquad \alpha(s) \equiv \alpha(\lambda(s))$$

$$\alpha(s) = -8a$$ in CFT limit



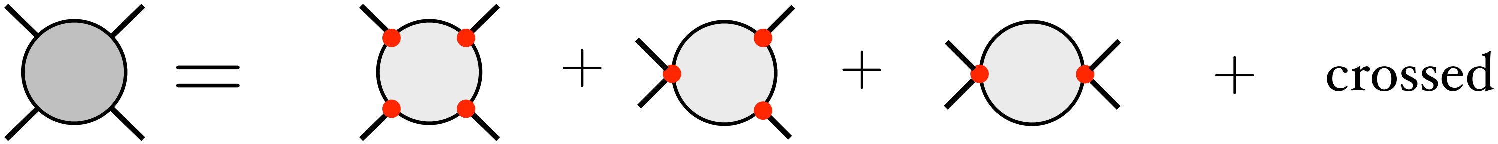
$$\bar{\alpha}(s) \equiv \frac{1}{\pi}\int_0^{\pi} d\theta\, \alpha(se^{i\theta})$$

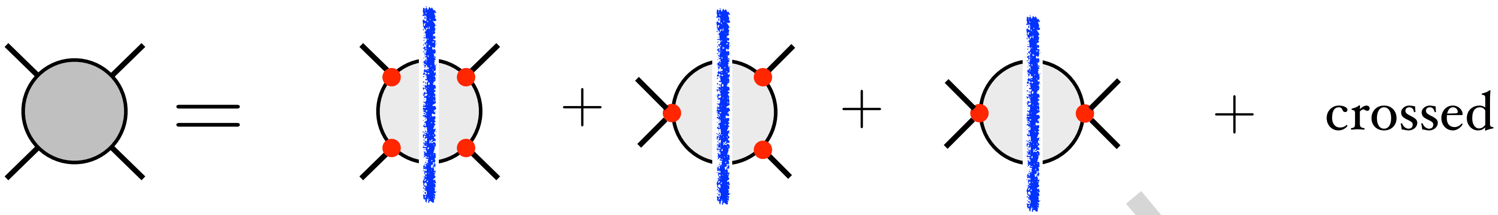
$$\bar{\alpha}(s_2) - \bar{\alpha}(s_1) = \frac{2}{\pi}\int_{s_1}^{s_2} \frac{ds}{s}\, \mathrm{Im}\, \alpha(s)$$

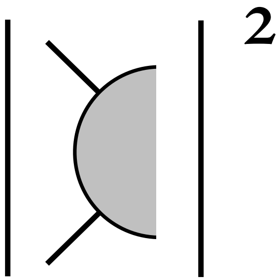
$\mathrm{Im}\,\alpha(s) \longrightarrow\ \geq 0$ by unitarity

absence of divergences $\Rightarrow$ $\lim_{s \to \pm\infty} \mathrm{Im}\, \alpha(s) = 0$

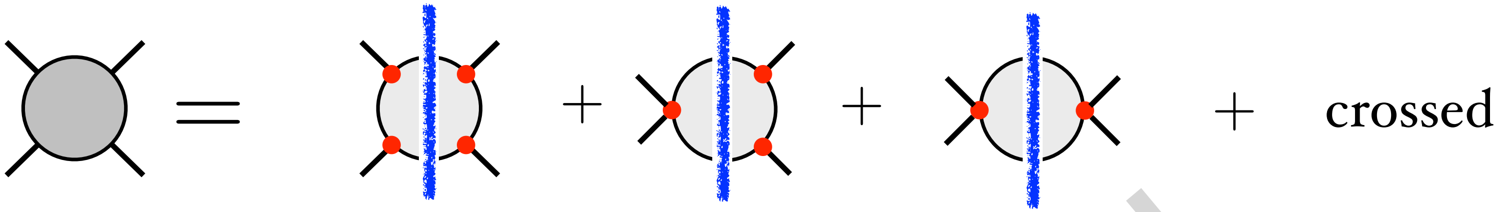
quickly drawing conclusions


crossed

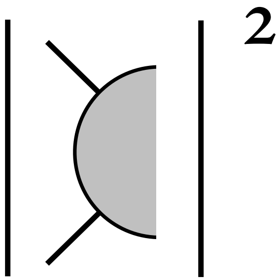


$\mathrm{Im}\,\alpha(s) =$ 

$$= \frac{1}{s^2} \sum_{\Psi} \left| \langle \Psi | \, T(p_1) T(p_2) + T(p_1 + p_2) \, | 0 \rangle \right|^2$$

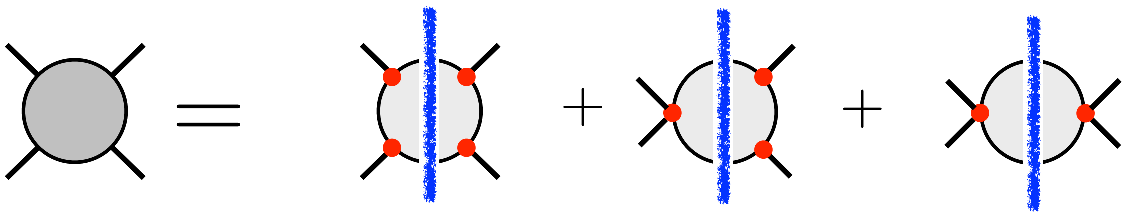


$$\mathrm{Im}\,\alpha(s) =$$

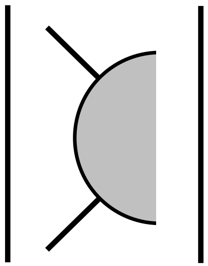


$$= \frac{1}{s^2}\sum_{\Psi}\left|\langle\Psi|\,T(p_1)T(p_2)+T(p_1+p_2)\,|0\rangle\right|^2$$

$$= \sum_I \beta_I \mathcal{O}_I$$



$+$ crossed

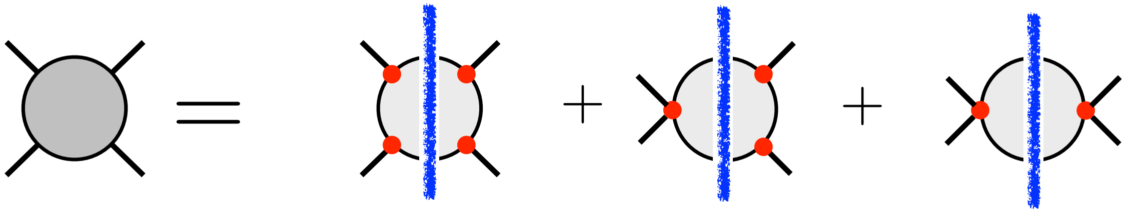
$\mathrm{Im}\,\alpha(s) =$  ^{2} $= \dfrac{1}{s^2}\sum_{\Psi}\left|\langle\Psi|\,T(p_1)T(p_2) + T(p_1+p_2)\,|0\rangle\right|^2$

subleading if

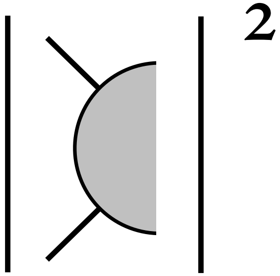
$\lambda_I,\ \beta_J \ll 1$

$= \sum_I \beta_I \mathcal{O}_I$

qualification needed !



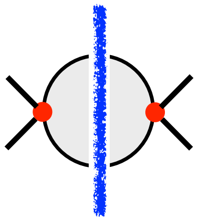
$+$ crossed

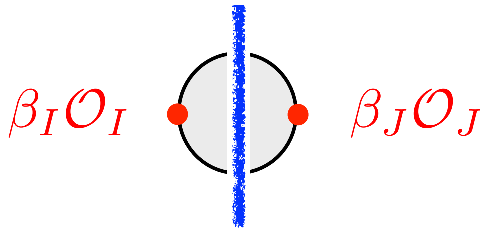
$\mathrm{Im}\,\alpha(s) =$  $= \frac{1}{s^2}\sum_{\Psi} \left| \langle\Psi|\, T(p_1)T(p_2) + T(p_1+p_2)\, |0\rangle \right|^2$

subleading if $\lambda_I,\ \beta_J \ll 1$

$= \sum_I \beta_I \mathcal{O}_I$

qualification needed !

$\mathrm{Im}\,\alpha$ is dominated by  like in 2D proof !!



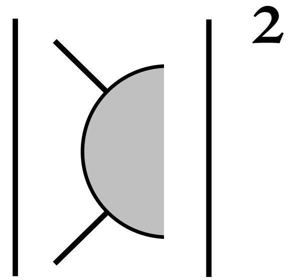
$$\mathrm{Im}\,\alpha = \sum_{IJ} \beta_I \beta_J \left[\underbrace{\frac{\mathrm{Im}\,\langle \mathcal{O}_I \mathcal{O}_J \rangle}{s^2}}_{C_{IJ}} + O(\lambda) \right]$$

C_{IJ} positive definite by unitarity

$\int \frac{ds}{s}\,\mathrm{Im}\,\alpha$ finite $\Longleftrightarrow$ $\beta_I \to 0$ asymptotically

The theory necessarily asymptotes a CFT !

Non perturbative argument contra 4D SFTs

$\mathrm{Im}\, a(s)$ =  = $C \;=\; \mathrm{const}$

absence of divergences

$$C \;=\; \frac{1}{s^2} \sum_{\Psi} \Big|\langle\Psi|\, T(p_1)T(p_2) + T(p_1+p_2)\,|0\rangle\Big|^2 \;=\; 0$$

by unitarity $\boxed{\mathrm{T}\left\{T(p_1)T(p_2)\right\} + T(p_1+p_2) \;=\; 0}$

- p_1 et p_2 are not arbitrary: $p_1^2 \;=\; p_2^2 \;=\; 0$

cannot yet directly infer $\mathrm{T}\left\{T(x_1)T(x_2)\right\} + \delta^4(x_1-x_2)\,T(x_1) \;=\; 0$

and conclude T is trivial

- yet the matrix elements should be very peculiar

$$\langle\Psi|\,T(p_1)T(p_2) + T(p_1+p_2)\,|0\rangle \;=\; 0$$

$$\ell \;=\; 0,1,2,\ldots \qquad\qquad \ell \;=\; 0$$

$$\langle\Psi,\ell\geq 1|\,T(p_1)T(p_2)\,|0\rangle \;=\; 0$$

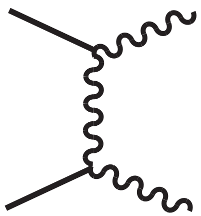
The importance of Unitarity

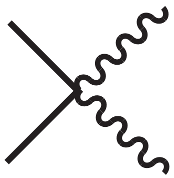
- Non-unitary SFT: massless vector without gauge invariance

$$S = \int d^4x \sqrt{-\hat{g}} \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{h}{2} (\nabla_\mu A^\mu)^2 \right)$$

Coleman, Jackiw 1971
Riva, Cardy 2005

virial current $\quad V^\mu = h\, A_\nu F^{\mu\nu}$





partial cross section $\neq 0$ $\quad \langle \Psi | \, T(p_1) T(p_2) + T(p_1 + p_2) \, |0\rangle \neq 0$

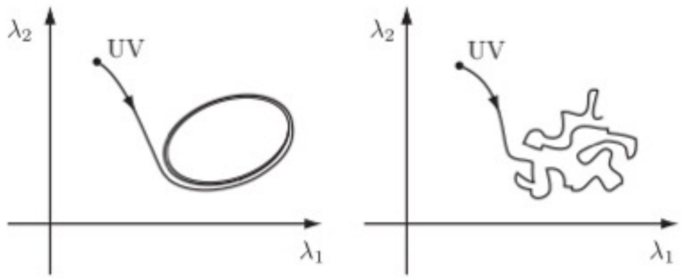
total cross section $= 0$ $\quad \sum_\Psi \left| \langle \Psi | \, T(p_1) T(p_2) + T(p_1 + p_2) \, |0\rangle \right|^2 = 0$

Summary

- Finiteness of RG flow of dilaton scattering amplitude
- Unitarity

Powerful constraint on RG-flow

✦ Perturbative theories
✦ *Small* deformations of strongly coupled CFTs



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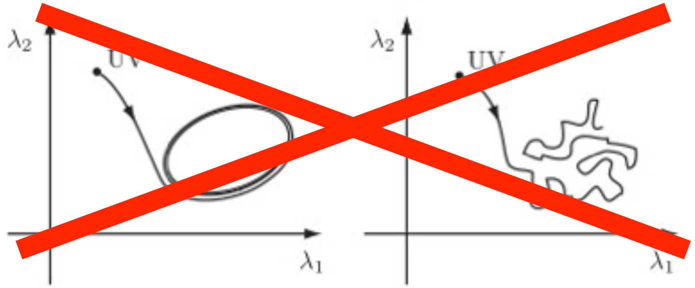

the only possible asymptotics are CFTs