LECTURE 1: INTRODUCTION TO LATTICE GAUGE THEORY

1) GLUONIC ACTION IN CONTINUUM

$$S = \frac{1}{2 g_0^2} \int d^4x \, \mathrm{TR}\left[F_{\mu\nu} F_{\mu\nu} \right]$$

$$T^a = T^{a+}$$

$$\mathrm{TR}\left[T^a T^b \right] = \frac{\delta^{ab}}{2}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i \left[A_\mu, A_\nu \right] \qquad ; \qquad F_{\mu\nu} = F_{\mu\nu}^a T^a$$

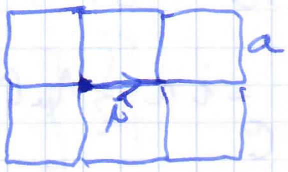
GAUGE TRANSFORMATION

$$G(x) = e^{i \Lambda(x)} \quad , \quad \Lambda(x) = \Lambda^a(x) T^a$$

$$A_\mu^G(x) = G(x) A_\mu(x) G^+(x) + i \, G(x) \partial_\mu G^+(x)$$

2) WILSON GLUONIC ACTION

- LINKS ARE FOND. GAUGE FIELD



$$U_\mu(x) = e^{-i a A_\mu(x)} \in SU(N)$$

- GAUGE TRANSFORM.

$$U_\mu^G(x) = G(x) U_\mu(x) G^+(x+\hat{\mu})$$

- PLAQUETTE

$$U_{\mu\nu}(x) = U_\mu(x) U_\nu(x+\hat{\mu}) U_\mu^+(x+\hat{\nu}) U_\nu^+(x)$$

!!

$$U_{\mu\nu}^G(x) = G(x) U_{\mu\nu}(x) G^+(x)$$

WILSON ACTION

$$S_G = \frac{\beta}{2} \sum_x \sum_{\mu,\nu} \left[1 - \frac{1}{2N_c} \mathrm{Tr}\left\{ U_{\mu\nu}(x) + U^{+}_{\mu\nu}(x) \right\} \right]$$

$$\beta = \frac{2N_c}{g_0^2}$$ (we will be interested in $N_c = 3$)

NOTE: EXACT GAUGE INVARIANCE AT FINITE LATTICE SPACING

3) CLASSICAL CONTINUUM LIMIT

BY USING CAMPBELL-HAUSDORFF FORMULA

$$e^A e^B = e^{\left\{A+B+\frac{1}{2}[A,B]\right\}+\dots}$$

$\Downarrow$

$$U_{\mu\nu}(x) = e^{-ia\left\{A_\mu(x) + A_\nu(x+\hat{\mu}) - \frac{ia}{2}\left[A_\mu(x), A_\nu(x+\hat{\mu})\right] + \dots\right\}}$$
$$\times e^{ia\left\{A_\mu(x+\nu) + A_\nu(x) + \frac{ia}{2}\left[A_\mu(x+\nu), A_\nu(x)\right] + \dots\right\}}$$

AND USE AGAIN

$$U_{\mu\nu}(x) = e^{-ia\left\{A_\mu(x) + A_\nu(x+\hat{\mu}) - \frac{ia}{2}\left[A_\mu(x), A_\nu(x+\hat{\mu})\right] - A_\mu(x+\nu) - A_\nu(x) - \frac{ia}{2}\left[A_\mu(x+\nu), A_\nu(x)\right] + \frac{ia}{2}\left[A_\mu(x) + A_\nu(x+\hat{\mu}), A_\mu(x+\nu) + A_\nu(x)\right] + \dots\right\}}$$

$\Downarrow$

$$U_{\mu\nu}(x) = \exp\left\{-ia^2\left\{ \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - i\left[A_\mu(x), A_\nu(x)\right] + O(a)\right\}\right\}$$

↳ this is an Hermitian matrix thanks to the unitarity of $U_{\mu\nu}$.

WHERE THE FORWARD DERIVATIVE IS

$$\nabla_\mu f(x) = \frac{f(x+\hat{\mu}) - f(x)}{a}$$

BY DEFINING ANALOGOUSLY TO THE CONTINUUM

$$F_{\mu\nu}(x) \equiv \nabla_\mu A_\nu(x) - \nabla_\nu A_\mu(x) - i\left[A_\mu(x), A_\nu(x)\right]$$

THEN

$$U_{\mu\nu}(x) = \exp\left\{-i a^2 \left[F_{\mu\nu}(x) + O(a)\right]\right\}$$

$$\Downarrow$$

$$S_G = \frac{\beta}{2} \sum_x \sum_{\mu,\nu} \Big[\cancel{1} - \frac{1}{2N_c} \mathrm{Tr}\Big[\cancel{1} - i a^2 \cancel{\left\{F_{\mu\nu} + O(a)\right\}}$$

$$- \frac{1}{2} a^4 F_{\mu\nu}(x) F_{\mu\nu}(x) + \cancel{1} + i a^2 \cancel{\left\{F_{\mu\nu} + O(a)\right\}}$$

$$- \frac{1}{2} a^4 F_{\mu\nu}(x) F_{\mu\nu}(x) + O(a^5)\Big]\Big]$$

By using Equations of motion becomes $O(a^6)$

SO AT THE END

$$\boxed{S_G = \frac{1}{2g_0^2} a^4 \sum_x \sum_{\mu\nu} \mathrm{Tr}\left[F_{\mu\nu}(x) F_{\mu\nu}(x)\right] + O(a)}$$

$$\Downarrow$$

$$\boxed{Z = \int \delta U \, e^{-S_G}}$$

Ok

4) FERMIONIC ACTION IN EUCLIDEAN:

$$S_F = \int d^4x \, \bar{\psi} \left\{ \gamma_\mu D_\mu + m \right\} \psi$$

$D_\mu \equiv \partial_\mu - i A_\mu \quad ; \quad D \equiv \gamma_\mu D_\mu$

5) NAIVE FERMION ACTION ON THE LATTICE

- WE CONSIDER THE FREE CASE

$$\nabla_\mu^* f(x) = \frac{f(x) - f(x-\mu)}{a}$$

AND WE USE THE SYMMETRIC DERIVATIVE

$$\frac{1}{2}\left\{ \nabla_\mu + \nabla_\mu^* \right\} \psi(x) = \frac{1}{2a}\left\{ \psi(x+\hat{\mu}) - \psi(x-\hat{\mu}) \right\}$$

THE NAIVE DISCRETIZATION IS GIVEN BY

$$S_F = a^4 \sum_x \frac{1}{2a}\left\{ \bar{\psi}(x)\gamma_\mu \psi(x+\hat{\mu}) - \bar{\psi}(x)\gamma_\mu \psi(x-\hat{\mu}) + m\,\bar{\psi}(x)\psi(x) \right\}$$

BY USING THE FOLLOWING CONVENTION FOR F.T.

$$f(x) = \int_{BZ} \frac{d^4k}{(2\pi)^4} \tilde{f}(k)\, e^{ikx} \qquad BZ^1 = \left[-\frac{\pi}{a}, \frac{\pi}{a}\right]$$

$$\tilde{f}(k) = a^4 \sum_x f(x)\, e^{-ikx}$$

$$a^4 \sum_x e^{-ix(k_1-k_2)} = (2\pi)^4 \delta_P^{(4)}(k_1-k_2)$$

WE INSERT F.T. IN S_F AND OBTAIN
(FROM NOW ON $\tilde{f}(k) \to f(k)$, IT IS CLEAR FROM ARG.)

$\to \bar{\psi}(k_1) \to \bar{\psi}(-k_1)$

$$S_F = \frac{a^4}{2a} \sum_x \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} e^{-ik_1x} e^{ik_2x} \times$$

$$\times \left\{ \bar{\psi}(k_1) \gamma_\mu e^{ik_2\hat{\mu}} \psi(k_2) - \bar{\psi}(k_1)\gamma_\mu e^{-ik_2\hat{\mu}} \psi(k_2) + m\bar{\psi}(k_1)\psi \right.$$

$$\Downarrow$$

$$S_F = \int \frac{d^4q}{(2\pi)^4} \bar{\psi}(q) \left\{ \gamma_\mu \frac{2i}{2a} \sin(q_\mu a) + m \right\} \psi(q)$$

WE DEFINE

$$\boxed{\bar{q}_\mu = \frac{1}{a}\sin(q_\mu a) \; ; \quad K(q) \equiv i\,\bar{q}_\mu \gamma_\mu + m \qquad S_F = a^4 \int_{BZ} \frac{d^4q}{(2\pi)^4} \bar{\psi}(q) K(q) \psi(q)}$$

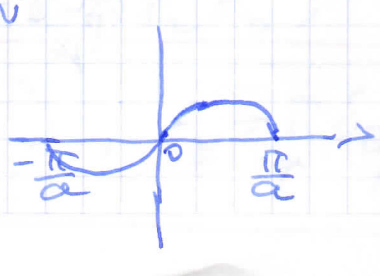
6) DOUBLING PROBLEM:

WE LOOK AT THE PROPAGATOR

$$\boxed{\langle \psi(x)\bar{\psi}(0)\rangle = \int_{BZ} \frac{d^4q}{(2\pi)^4} \frac{-i\gamma_\mu \bar{q}_\mu + m}{\sum_\mu \bar{q}_\mu^2 + m^2} e^{ikx}} \quad (6.1)$$

FOR $m=0$, IN ONE DIRECTION

$$\bar{q}_\mu^2 = 0 \iff q_\mu = 0, \pm \frac{\pi}{a}$$



THEREFORE IN 4-DIMENSIONS THE ϕ ARE

$$\begin{array}{cccc} 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ \pm\pi & \pm\pi & \pm\pi & \pm\pi \end{array} \Rightarrow 2^4 = 16 \text{ ZEROS}$$

ALL THESE ZEROS SURVIVE IN CONTINUUM LIMIT !!

7) CHIRALITY OF DOUBLERS :

LET US REWRITE (6.1) AS AN INTEGRAL OVER 16 ZONES, IN EACH DIRECTION

$$|q_\mu| < \frac{\pi}{2a} \quad , \quad \frac{\pi}{2a} \le |q_\mu| \le \frac{\pi}{a}$$

WE REWRITE

$$q_\mu = \bar{p}_\mu + \tilde{p}_\mu$$

permutations

WHERE $a\tilde{p}_\mu = (0,0,0,0), (\pi,0,0,0) + \ldots, (\pi,\pi,0,0) \ldots$

$$\boxed{\langle \psi(x)\bar{\psi}(0)\rangle = \sum_{\tilde{p}} e^{i\tilde{p}x} \int_{-\frac{\pi}{2a}}^{\frac{\pi}{2a}} \frac{d^4 p}{(2\pi)^4} \frac{-i\sum_\mu \delta_{\tilde{p}_\mu}\gamma_\mu \bar{p}_\mu + m}{\sum_\nu \bar{p}_\nu^2 + m^2} e^{i\bar{p}x}}$$

where

$$\delta_{\tilde{p}_\mu} = e^{i\tilde{p}_\mu \cdot a}$$

BY NOTICING THAT

$$\tau_{\tilde{p}} \gamma_\mu \tau_{\tilde{p}}^{-1} = \underbrace{\delta_{\tilde{p}_\mu}}\gamma_\mu$$

i.e. ~~THESE MATRICES~~ ARE RELATED TO THE ORIGINAL ON BY A SIMILARITY TRANSFORMATION ($\tilde{\tau}_\mu = \gamma_\mu \gamma_5$, $\tilde{\tau}_{\mu\nu} = \gamma_\mu \gamma_\nu$, $\tilde{\tau}_{\mu\nu\lambda} = \gamma_\mu \gamma_\nu \gamma_\lambda \gamma_5$, $\tilde{\tau}_{\mu\nu\lambda\rho} = \gamma_5$)

$$\boxed{\langle \psi(x) \bar{\psi}(0) \rangle = \sum_{\tilde{p}} e^{i\tilde{p}x} \tau_{\tilde{p}} \int_{-\frac{\pi}{2a}}^{\pi/2a} \frac{d^4p}{(2\pi)^4} \frac{-i\bar{p}_\mu \gamma_\mu + m}{\sum_\mu \bar{p}_\mu^2 + m^2} e^{ipx} \tau_{\tilde{p}}^{-1}}$$

ONE NAIVE FERMION ON THE LATTICE REPRESENT 16 CONTINUUM FERMION SPECIES ALL WITH MASS m

- LET US FOCUS ON 1 CHIRAL FERMION IN CONTINUUM → THE NAIVE LATTICE COUNTERPA[RT] IS

$$\psi_{R,L} \equiv \frac{1}{2}(1 \pm \gamma_5)\psi \; ; \; \bar{\psi}_{R,L} \equiv \frac{1}{2}\bar{\psi}(1 \mp \gamma_5)$$
$$= P_\pm \psi \qquad\qquad = \bar{\psi} P_\mp$$

THE PROP

$$\langle \psi_L(x) \bar{\psi}_L(0) \rangle = \sum_{\tilde{p}} e^{i\tilde{p}x} P_L \int_{-\frac{\pi}{2a}}^{\pi/2a} \frac{d^4p}{(2\pi)^4} \tau_{\tilde{p}} \left\{ \frac{-i\sum_\mu \gamma_\mu \bar{p}_\mu}{\sum_p \bar{p}^2} \right\} \tau_{\tilde{p}}^{-1} e^{ipx}$$

$\tau_{\tilde{p}}^{-1}\tau_{\tilde{p}}$

Now it is easy to verify ($\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$)

$$\boxed{\tau_{\tilde{p}}^{-1}(1-\gamma_5)\tau_{\tilde{p}} = (1 - \varepsilon_{\tilde{p}}\gamma_5) \, , \quad \varepsilon_{\tilde{p}} = \prod_\mu \delta_{\tilde{p}_\mu}}$$

THEREFORE

$$\boxed{\langle \psi_L(x)\bar{\psi}_L(0)\rangle = \sum_{\tilde{p}} e^{i\tilde{p}x}\, \tau_{\tilde{p}} \left\{ \int_{-\frac{\pi}{2a}}^{\pi/2a} \cdots \frac{d^4p}{(2\pi)^4}\, \frac{1}{2}\left(1-\varepsilon_{\tilde{p}}\gamma_5\right) \times \left(-i\,\frac{\varepsilon_\mu \gamma_\mu \bar{p}_\mu}{\varepsilon_\mu \bar{p}_\mu^2}\right) e^{ipx} \right\} \tau_{\tilde{p}}^{-1}}$$

BY COUNTING WE FIND 8 POLES WITH $\varepsilon_{\tilde{p}}=1$ AND 8 POLES WITH $\varepsilon_{\tilde{p}}=-1$

ONE NAIVE LATTICE CHIRAL FERMION REGULARIZE 8 LEFT-HANDED FERMIONS AND 8 RIGHT-HANDED FERMIONS !!

IT IS NOT A CHIRAL FERMION !!

8) NIELSEN - NINOMIYA THEOREM

DESIRABLE PROPERTIES OF $K(q)$

1) $D(q)$ SMOOTH FUNCTION OF q_μ WITH PERIOD $\frac{2\pi}{a}$

2) FOR $q_\mu \ll \frac{\pi}{a}$ $\quad D(q) = i\gamma_\mu q_\mu + O(aq^2)$

3) $D(q)$ INVERTIBLE AT ALL $q\neq 0$ (MOD $\frac{2\pi}{a}$)

4) $\{\gamma_5, D(q)\}=0$

SKETCH OF DEMONSTRATION

LET US ASSUME 1), 2) AND 4) FOR A CHIRAL FERMION - THEN

$$D(q) = i\, P_- \gamma_\mu F_\mu(q) \qquad F_\mu(q) \in \mathbb{R}e$$

- LET US DEVELOP AROUND A GENERIC ϕ OF F_μ

$$F_\mu(q) = M_{\mu\nu}(q_\nu - \tilde{p}_\nu) + \ldots$$

AND WE ASSUME THE GENERIC CASE WHERE $M_{\mu\nu}$ IS NON-SINGULAR (hypothesis 2+3) -

- M IS REAL MATRIX THAT CAN BE DECOMPOSED

$$M = O\,S \quad ; \quad O O^T = 1 , \qquad z^T S z > 0 \quad \|z\| \neq 0$$

- THE MATRIX O CAN BE REABSORBED INTO A REDEFINITION OF FERMION FIELDS

$$P_- \gamma_\mu O_{\mu\nu} = \tau_{\tilde{p}} \frac{(1 - \varepsilon_{\tilde{p}} \gamma_5)}{2} \gamma_\nu \tau_{\tilde{p}}^{-1} \quad ; \quad \varepsilon_{\tilde{p}} = \det O$$

- SO:

$$\boxed{D(q) = i\, \tau_{\tilde{p}} \frac{(1 - \varepsilon_{\tilde{p}} \gamma_5)}{2} \gamma_\alpha \tau_{\tilde{p}}^{-1} S_{\alpha\nu} (q_\nu - \tilde{p}_\nu)}$$

$S_{\alpha\nu}(q_\nu - \tilde{p}_\nu)$ is a positive rescaling and a rotation - **DOES NOT AFFECT MOMENTUM SPACE ORIENTATION** $\Rightarrow$ so the ϕ has chirality $\varepsilon_{\tilde{p}}$

THEOREM POINCARÉ - Hopf $\Rightarrow$ the sum of the ε_p

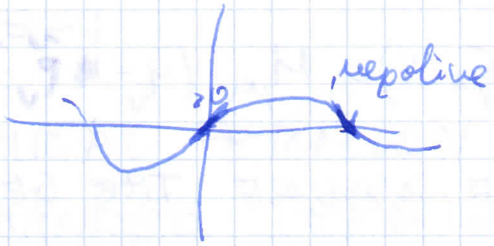
$$\sum_p \varepsilon_p = 0$$

ONE DIMENSIONAL CASE:

continuous periodic function vanishes with positive derivative as many times as with negative

$$\Downarrow$$

$$\sum_p \varepsilon_p = 0$$



IF $\sum_p \varepsilon_p = 0 \Rightarrow$ LATTICE NAIVE CHIRAL FERMION IS ACTUALLY A REGULARIZATION FOR AS MANY LEFT FERMIONS AS RIGHT ONES

HYPOTESYS 3 CANNOT BE SATISFIED!!

9) WILSON ACTION

SOLUTION: ABANDON 4) AND BREAKS CHIRALITY EXPLICITLY

$$S_F = a^4 \sum_x \bar{\psi} \left\{ D_W + m_0 \right\} \psi$$

$$D_W = \frac{1}{2} \left\{ \gamma_\mu \left(\nabla_\mu^* + \nabla_\mu \right) - \nabla_\mu^* \nabla_\mu \right\}$$

$$\nabla_\mu \psi = U_\mu(x)\, \psi(x+\hat{\mu}) - \psi(x)$$

$$\nabla_\mu^* \psi = \psi(x) - U_\mu^+(x-\hat{\mu})\, \psi(x-\hat{\mu})$$

$$K(q) = i \gamma_\mu \bar{q}_\mu + m_0 + \frac{a}{2} \hat{q}^2$$

$$\hat{q} = \frac{2}{a} \sin\left(\frac{q_\mu a}{2} \right)$$

$$\boxed{\langle \psi(x) \bar{\psi}(0) \rangle = \int \frac{d^4 q}{(2\pi)^4} \frac{-i \gamma_\mu \bar{q}_\mu + m_0 + \frac{a}{2} \hat{q}^2}{\sum_\mu \bar{q}_\mu^2 + (m_0 + \frac{a}{2} \hat{q}^2)^2}}$$

For $m=0$ and $q_\mu = \frac{\pi}{a}$, $\hat{q} = \frac{2}{a}$ => DOUBLERS HAVE MASS THAT $\to \infty$ WHEN $a \to 0$. ONLY ONE MASSLESS POLE IN CONTINUUM LIMIT.

Lecture 2: Ginsparg-Wilson fermions and chiral anomaly

1) Ginsparg-Wilson relation and Lüscher symmetry

$$\gamma_5 D + D\gamma_5 = \bar{a}\, D \gamma_5 D \quad , \quad \bar{a} = \frac{a}{1+1}$$

$$\Updownarrow$$

$$\gamma_5 D + D\hat{\gamma}_5 = 0 \quad , \quad \hat{\gamma}_5 \equiv \gamma_5 (1 - \bar{a} D)$$

Avoid NN theorem due to presence of $\hat{\gamma}_5$

- The massless fermion action

$$S = a^4 \sum_x \bar{\psi}(x) [D\psi](x)$$

under the Lüscher symmetry

$$\psi' = \psi + i\, \varepsilon_A^0 \hat{\gamma}_5 \psi \Rightarrow \delta\psi = \psi - \psi' = -i\, \varepsilon_A^0 \hat{\gamma}_5 \psi$$

$$\bar{\psi}' = \bar{\psi} + i\, \varepsilon_A^0 \bar{\psi} \gamma_5 \Rightarrow \delta\bar{\psi} = \bar{\psi} - \bar{\psi}' = -i\, \varepsilon_A^0 \bar{\psi} \gamma_5$$

is invariant

$$\delta S = -a^4\, i\, \varepsilon_A^0 \sum_x \bar{\psi}(x) \left\{ [\gamma_5 D + D\hat{\gamma}_5] \psi \right\}(x) = 0$$

<u>Note</u>: $\hat{\gamma}_5$ depends on background gauge field

2) TRANSFORMATION OF MEASURE

$$\psi' = e^{i \overset{\circ}{\varepsilon}_A \hat{\gamma}_5} \psi \quad , \quad \bar{\psi}' = \bar{\psi} e^{i \overset{\circ}{\varepsilon}_A \gamma_5}$$

$$d\bar{\psi}' d\psi' = \frac{1}{\det\left(e^{i \varepsilon_A \gamma_5}\right)} \frac{1}{\det\left(e^{i \varepsilon_A \hat{\gamma}_5}\right)} d\bar{\psi} d\psi$$

trace over space, spin, color

$$\Downarrow$$

$$d\bar{\psi}' d\psi' = e^{-i \overset{\circ}{\varepsilon}_A \mathrm{TR}[\hat{\gamma}_5]} d\bar{\psi} d\psi$$

- IF WE INTRODUCE THE TOP. CHARGE DENSITY

$$a^4 q(x) \equiv -\frac{\bar{a}}{2} \mathrm{tr}\left[\gamma_5 D(x,x)\right] \quad ; \quad Q \equiv a^4 \sum_x q(x)$$

$$\Downarrow$$

$$Q = \frac{1}{2} \mathrm{TR}[\hat{\gamma}_5]$$

NOTE: IN CLASSICAL CONTIN. LIMIT

$$q(x) \longrightarrow \frac{1}{32\pi^2} \varepsilon_{\mu\nu\rho\sigma} \mathrm{tr}\left[F_{\mu\nu}(x) F_{\rho\sigma}(x)\right]$$

3) SPECTRAL PROPERTIES OF GW OPERATORS:

$\bar a D = 1 - \gamma_5 \hat\gamma_5$, and $(\gamma_5 \hat\gamma_5)(\gamma_5 \hat\gamma_5)^+ = 1$

so

$$\bar a \bar\lambda = 1 - e^{i\alpha} \Rightarrow \begin{cases} \bar\lambda = 0 \\ \bar\lambda \in \text{complex} \\ \bar\lambda = \frac{2}{\bar a} \end{cases}$$

$\bar\lambda = 0$:

$$\bar\lambda = 0 \Rightarrow (\gamma_5 D + D\gamma_5)|u_\lambda\rangle = \bar a D \gamma_5 D |u_\lambda\rangle \Rightarrow D\gamma_5 |u_\lambda\rangle = 0$$

$\Rightarrow$ we can choose

$$\gamma_5 |u_\lambda\rangle = \pm |u_\lambda\rangle \qquad \lambda = 0$$

$n_\pm$ = number of 0 modes with chirality $\pm$

$\bar\lambda = \frac{2}{\bar a}$:

$$\bar\lambda = \frac{2}{\bar a} \Rightarrow (\gamma_5 D + D\gamma_5)|u_\lambda\rangle = \bar a D\gamma_5 D|u_\lambda\rangle \Rightarrow D\gamma_5|u_\lambda\rangle = \frac{2}{\bar a}\gamma_5|u_\lambda\rangle$$

again we can choose

$$\gamma_5 |u_\lambda\rangle = \pm |u_\lambda\rangle \qquad \bar\lambda = \frac{2}{\bar a}$$

$n'_\pm$ = number of $\frac{2}{\bar a}$ modes with chirality $\pm$

$\bar\lambda \in \mathbb{C}$:

$$\bar a \bar\lambda = 1 - e^{i\alpha} \Rightarrow (\gamma_5 D + D\gamma_5)|u_\lambda\rangle = \bar a D\gamma_5 D|u_\lambda\rangle \Rightarrow D\gamma_5|u_\lambda\rangle = \frac{1}{\bar a \bar\lambda - 1}\bar\lambda \gamma_5 |u_\lambda\rangle$$

$$\Rightarrow D\gamma_5 |u_\lambda\rangle = \bar\lambda^* \gamma_5 |u_\lambda\rangle$$

$$\Downarrow$$

$\bar\lambda, \lambda^* \in$ eigenspectrum, $|u_{\lambda^*}\rangle = \gamma_5 |u_\lambda\rangle$, $\langle u_\lambda|\gamma_5|u_\lambda\rangle = 0$

SINCE $\mathrm{Tr}[\gamma_5] = 0$

$$\mathrm{Tr}[\bar{a}\gamma_5 D] = \mathrm{Tr}[\gamma_5(\bar{a}D - 2)] \Rightarrow (n_+ - n_-) = -(n'_+ - n'_-)$$

AND THEREFORE

$$Q = n_+ - n_- \quad ; \quad \mathrm{Tr}[\hat{\gamma}_5] = 2(n_+ - n_-)$$

$$\boxed{d\bar{\psi}' d\psi = e^{-2i\varepsilon_A^0 Q}\, d\bar{\psi}\, d\psi \quad , \quad Q = n^+ - n^-}$$

4) NEUBERGER OPERATOR

$$D = \frac{1}{\bar{a}}\left\{1 + \gamma_5 \frac{Q}{\sqrt{Q^2}}\right\} \; ; \; Q = \gamma_5(aD_W - 1 - 1)$$

SO WE NEED TO COMPUTE THE SIGN OF HERMITEAN WILSON-DIRAC OPERATOR WITH NEGATIVE MASS $-\frac{(1+1)}{a}$.

<u>NOTE</u>: OPERATOR IS NOT ULTRALOCAL, BUT IS LOCAL WITH EXPONENTIALLY DECAYING TAILS

$$\left\| \frac{1}{\sqrt{Q^2}}(x,y) \right\| \sim e^{-\frac{c}{a}\|x-y\|_1}$$

4) CHIRAL MULTIPLETS AND SYMMETRY

$$\hat{P}_{\pm} = \frac{1}{2}\left(1 \pm \hat{\gamma}_5\right) \quad ; \quad P_{\pm} = \frac{1}{2}\left(1 \pm \gamma_5\right)$$

$$\psi_{R,L} \equiv \hat{P}_{\pm}\,\psi \quad , \quad \bar{\psi}_{R,L} = \bar{\psi}\, P_{\mp}$$

TRANSFORMATION OF CHIRAL GROUP $U(1)_L \otimes U(1)_R$

$$\psi_L' = V_L\,\psi_L \quad ; \quad \psi_R' = V_R\,\psi_R$$
$$\bar{\psi}_L' = \bar{\psi}_L\, V_L^{+} \quad ; \quad \bar{\psi}_R' = \bar{\psi}_R\, V_R^{+}$$

AND BILINEARS WITH CORRECT CHIRAL PROPERTIES

$$O_\Gamma \equiv \bar{\psi}\,\Gamma\,\tilde{\psi} \quad , \quad \tilde{\psi} \equiv \left(1 - \frac{\bar{a}D}{2}\right)\psi$$

VECTOR SUBGROUP: $V_L = V_R = e^{i\varepsilon_V^0}$

AXIAL TRANSF.: $V_L^{+} = V_R = e^{i\varepsilon_A^0}$

5) ACTION FOR MASSIVE FERMIONS

$$S = a^4 \sum_x \left\{ \bar{\psi} D \psi + \bar{\psi}_R M^{+} \psi_L + \bar{\psi}_L M \psi_R \right\}$$

- All terms invariant under $U(1)_V$
- $\bar{\psi} D \psi$ INVARIANT UNDER $U(1)_A$
- $\bar{\psi}_R M^{+} \psi_L$ invariant if
- $\bar{\psi}_L M \psi_R$ " "

$$\boxed{\begin{aligned} M^{+} &\to M^{+\prime} = V_R M^{+} V_L^{+} \\ M &\to M' = V_L M V_R^{+} \end{aligned}}$$

$\hookrightarrow$ SPURION TRANSFORMATION

6) ϑ AND M DEPENDENCE OF THE FREE-ENERGY

$$S = S_G - i\vartheta Q + S_F$$

$$e^{-F(\vartheta, M, M^+)} = \frac{\int \delta U\, d\bar\psi d\psi\, e^{-S_G - S_F}\, e^{i\vartheta Q}}{\int \delta U\, d\bar\psi d\psi\, e^{-S_G - S_F}} = \langle e^{i\vartheta Q} \rangle = \frac{Z(\theta)}{Z(0)}$$

By the anomalous transformation

$$\varepsilon_A^0 = -\frac{\vartheta}{2} \Rightarrow V_L^+ = V_R = e^{-i\frac{\vartheta}{2}}$$

then

$$d\bar\psi d\psi = e^{-i\vartheta Q}\, d\bar\psi' d\psi'$$

$$M' = e^{i\vartheta} M$$

$$M^{+\prime} = e^{-i\vartheta} M^+$$

$$\int \delta U\, d\bar\psi d\psi\, e^{-S_G - S_F(M, M^+)}\, e^{i\vartheta Q} = \int \delta U\, d\bar\psi' d\psi'\, e^{-S_G - S_F(M', M^{+\prime})}$$

$$\Downarrow$$

$$\boxed{F(\vartheta, M, M^+) = F(0, V_L M V_R^+, V_R M^+ V_L^+) = F(0, e^{i\vartheta} M, e^{-i\vartheta} M^+)} \quad (6.1)$$

7) AWI FOR TOPOLOGICAL SUSCEPTIBILITY

THE CUMULANTS OF CHARGE DISTRIBUTION

$$C_n \equiv (-1)^{n+1} \frac{1}{V} \frac{d^{2n}}{d\theta^{2n}} F(\theta)\Big|_{\theta=0} = \frac{a^{8n}}{V} \sum_{x_1 \cdots x_{2n}} \langle q(x_1) \cdots q(x_{2n}) \rangle_c$$

ARE INTEGRATED CORREL. FUNCTIONS OF $q(x)$

FOR $n=1$

$$\chi \equiv \frac{1}{V} \frac{d^2}{d\theta^2} F(\theta)\Big|_{\theta=0}$$

By using (6.1) l.h.s.

$$\chi = a^4 \sum_x \langle q(x) q(0) \rangle \leftarrow \langle q(x) q(0) \rangle \underset{x \to 0}{\sim} \frac{1}{(x^2)^4} \text{ SO}$$

HAS SHORT DISTANCE SINGULARITIES

By using (6.1) r.h.s.

$$\frac{d^2}{d\theta^2} F(\theta)\Big|_{\substack{\theta=0 \\ M=M^+}} = -a^4 \sum_x \langle \bar\psi_L M \psi_R + \bar\psi_R M^+ \psi_L \rangle +$$

$$+ a^8 \sum_{x,y} \langle (\bar\psi_L M \psi_R - \bar\psi_R M^+ \psi_L)(x) (\bar\psi_L M \psi_R - \bar\psi_R M^+ \psi_L)(y) \rangle$$

- By USING TRANSL. INVARIANCE, DIVIDING BY V, AND BY DEFINING

$$M = M^+ = m \,, \quad S^0 = \bar\psi_L \psi_R + \bar\psi_R \psi_L \,, \quad P^0 = \bar\psi_L \psi_R - \bar\psi_R \psi_L$$

Eq. (6.1) BECOMES

$$\boxed{a^4 \sum_x \langle q(x) q(0) \rangle = m^2 a^4 \sum_x \langle P^0(x) P^0(0) \rangle - m \langle S^0 \rangle}$$

NOTE: NOT VERY USEFUL AS IT STANDS SINCE ON R.H.S. THERE ARE ILL DEFINED QUANTITIES!

- $\langle S^0 \rangle$ U.V. DIVERGENT FOR $m \neq 0$
- $\sum_x \langle P^0(x) P^0(0) \rangle$ INTEGRATED CORR. WITH POTENTIAL CONTACT TERMS NON INTEGRABLE
- $\sum_x \langle q(x) q(0) \rangle$ SAME AS BEFORE

NOTE: χ HAS A CHANCE TO BE DEFINED ONLY IF ON R.H.S. THERE ARE CANCELLATIONS

8) CHIRAL MULTIPLETS AND SIMMETRY WITH MANY FLAVOURS

$\psi = (\psi_1, \dots, \psi_{N_f})$, $\psi_{R,L} = \hat{P}_\pm \psi$, $\bar{\psi}_{R,L} = \bar{\psi} P_\mp$

$\bar{\psi} = - - -$

TRANSFORMATION UNDER CHIRAL GROUP $U(N_f)_L \otimes U(N_f)_R$

$\psi'_L = V_L \psi_L$; $\psi'_R = V_R \psi_R$

$\bar{\psi}'_L = \bar{\psi}_L V_L^{+}$; $\bar{\psi}'_R = \bar{\psi}_R V_R^{+}$

WITH

$V_L = e^{i\varepsilon_L^0} e^{i\varepsilon_L^a T^a}$; $V_R = e^{i\varepsilon_R^0} e^{i\varepsilon_R^a T^a}$

$T^a = \frac{\sigma^a}{2}$ $N_f = 2$, $T^a = \frac{\lambda^a}{2}$ $N_f = 3$

VECTOR SUBGROUP:

$$V_L = V_R = e^{i\varepsilon_V^0} e^{i\varepsilon_V^a T^a} \Rightarrow \varepsilon_L^0 = \varepsilon_R^0 = \varepsilon_V^0$$
$$\varepsilon_L^a = \varepsilon_R^a = \varepsilon_V^a$$

AXIAL TRANSFORM:

$$V_L^+ = V_R = e^{i\varepsilon_A^0} e^{i\varepsilon_A^a T^a} \Rightarrow -\varepsilon_L^0 = \varepsilon_R^0 = \varepsilon_A^0$$
$$-\varepsilon_L^a = \varepsilon_R^a = \varepsilon_A^a$$

MASS TERM:

$$M^{+'} = V_R M^+ V_L^+ \quad ; \quad M' = V_L M V_R^+$$

9) θ AND M DEPENDENCE OF $F(\theta)$ WITH MANY FLAVOURS

$$S = S_G - i\theta Q + S_F$$

$$S_F = a^4 \sum_x \{ \bar\psi D \psi + \bar\psi_R M^+ \psi_L + \bar\psi_L M \psi_R \}$$

AGAIN WE DEFINE

$$e^{-F(\theta, M, M^+)} \equiv \langle e^{i\theta Q} \rangle$$

AS BEFORE WE TAKE THE TRANSF.

$$\varepsilon_A^0 = -\frac{\theta}{2N_f} \Rightarrow V_L^+ = V_R = e^{-i\frac{\theta}{N_f}}$$

AND THEREFORE:

$$\boxed{F(\theta, M, M^+) = F(0, e^{i\frac{\theta}{N_f}} M, e^{-i\frac{\theta}{N_f}} M^+)} \quad (9.$$

10) SINGLET AWI WITH SEVERAL FLAVOURS

FOLLOWING EXACTLY THE VERY SAME STEPS AS FOR $N_f = 1$

$$\boxed{a^4 \sum_x \langle q(x)\, q(0) \rangle = -\frac{m}{N_f^2} \langle S^0 \rangle + \left(\frac{m}{N_f}\right)^2 a^4 \sum_x \langle P^0(x)\, P^0(0) \rangle} \tag{10.1}$$

WHERE

$$M = M^+ = m I \quad , \quad S^0 = \bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L \quad , \quad P^0 = \bar{\psi}_L \psi_R - \bar{\psi}_R \psi_L$$

NOTE : AS FOR $N_f = 1$ R.H.S. AND L.H.S. ARE ILL DEFINED ~~AS THEY STAND~~ UNLESS INTERESTING CANCELLATION HAPPEN ON R.H.S

11) NON-SINGLET AWIs :

FROM POINT 8

$$V_L^{+} = V_R = 1 + i\,\varepsilon_A^a T^a + \dots$$

$$\psi_L' = (1 - i\,\varepsilon_A^a T^a + \dots)\,\psi_L \quad , \quad \psi_R' = (1 + i\,\varepsilon_A^a T^a)\,\psi_R$$

$$\Downarrow$$

$$\psi' = \psi + i\,\varepsilon_A^a T^a \hat{\gamma}_5 \psi \quad , \quad \bar{\psi}' = \bar{\psi} + i\,\bar{\psi}\gamma_5 T^a \varepsilon_A^a$$

ANALOGOUSLY TO THE SINGLET CASE

$$\delta S_F = -a^4\, i\,\varepsilon_A^a \sum_x \bar{\psi}(x)\, T^a \left\{ \cancel{[\gamma_5 D + D\hat{\gamma}_5]}^{\,GW} \psi \right\}(x) \;+$$

$$+\, a^4 (+i\,\varepsilon_A^a) \sum_x \bar{\psi}_R \{T^a, M^+\}\, \psi_L \;+$$

$$+\, a^4 (-i\,\varepsilon_A^a) \sum_x \bar{\psi}_L \{T^a, M\}\, \psi_R$$

$$\Downarrow$$

$$\delta S_F = -a^4\, i\,\varepsilon_A^a \sum_x \left[-\bar{\psi}_R \{T^a, M^+\} \psi_L + \bar{\psi}_L \{T^a, M\} \psi_R \right]$$

FOR A GENERIC OPERATOR ($\nu = 0$ IN THIS CASE)

$$\langle O \rangle \equiv \frac{1}{Z} \int \delta U\, d\bar{\psi}\, d\psi\; e^{-S_G - S_F}\, O$$

BY REMEMBERING THAT

$$d\bar{\psi}'\, d\psi' = d\bar{\psi}\, d\psi$$

$$\langle O \rangle = \frac{1}{Z} \int \delta U\, d\bar{\psi}'\, d\psi'\; e^{-S_G}\, e^{-S_F - \delta S_F}\, (O + \delta O)$$

$$\Downarrow$$

$$\langle \delta S_F \, O \rangle = \langle \delta O \rangle$$

LET US TAKE $O = P^b$

$$P^b \equiv \bar{\psi}_L T^b \psi_R - \bar{\psi}_R T^b \psi_L$$

IT IS EASY TO SHOW

$$\delta P^b = -i \, \varepsilon_A^a \left[\bar{\psi}_L \{T^a, T^b\} \psi_R + \bar{\psi}_R \{T^a, T^b\} \psi_L \right]$$

IF WE NOW TAKE

$$M = M^+ = m \mathbb{1}$$

THE WIS READS

$$\boxed{2m \, a^4 \sum_x \langle P^a(x) \, P^b(0) \rangle = \frac{\delta^{ab}}{N_f} \langle S^0 \rangle} \qquad (11.1)$$

12) COMBINING SINGLET AND NON SINGLET AWI's !

BY COMPUTING THE WICK CONTRACTIONS

$$\langle P^0(x)\, P^0(0)\rangle = -N_f \langle \bigcirc \rangle + N_f \langle \text{D}\ \text{D} \rangle$$

$$\langle P^a(x)\, P^a(0)\rangle = -\frac{1}{2} \langle \bigcirc \rangle$$

THEREFORE IN EQ. (10.1) WE CAN REPLACE THE CONDENSATE BY $\langle P^a P^a \rangle$ AND OBTAIN

$$\boxed{a^4 \sum_x \langle q(x)\, q(0)\rangle = m^2 a^4 \sum_x \langle P_{ls}(x)\, P_{sl}(0)\rangle}$$

$l \neq s$, NO SUMMATION OVER REPEATED INDICES

$$P_{il}(x) = \bar{\psi}_{iL}\, \psi_{lR} - \bar{\psi}_{lR}\, \psi_{iL}$$

(13.1)

<u>NOTE</u>: R.H.S. STILL ILL DEFINED UNLESS THERE ARE CANCELLATIONS

Lecture 3: Witten-Veneziano Relation

1) Density chains

- We assume again

$$M = M^{+} = m\,\mathbb{1}$$

and we do a non-singlet ~~axial transf.~~

$$\delta S_F = -i\,\varepsilon_A^a\, 2 m a^4 \sum_x P^a(x)$$

- Let us take the operator

$$S^b \equiv \bar{\psi}_L T^b \psi_R + \bar{\psi}_R T^b \psi_L$$

$$\Downarrow$$

$$\delta S^b = -i\,\varepsilon_A^a \left[\bar{\psi}_L \{T^a, T^b\} \psi_R - \bar{\psi}_R \{T^a, T^b\} \psi_L \right]$$

and then the operator

$$O = S^b(x_2)\, P_{33}(x_4) \qquad b = 1, 2, 3$$

and we choose $\varepsilon_A^a \neq 0$ for $a = 1, 2, 3$

$$\{T^a, T^b\} = \frac{\delta^{ab}}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Therefore in this case

$$\delta P_{33} = 0 \,, \qquad \delta S^b = -i\,\frac{\varepsilon_A^a}{2} \left[P_{11} + P_{22} \right]$$

BY USING THE GENERAL RELATION

$$\langle \delta \xi_{F} O \rangle = \langle \delta O \rangle$$

$$\cdot\, i\, \varepsilon_A^a\, 2m\, a^4 \sum_{x_1} \langle P^a(x_1)\, \xi^b(x_2)\, P_{33}(x_4) \rangle =$$

$$= - i \frac{\varepsilon_A^a}{2} \langle \left\{ P_{11}(x_2) + P_{22}(x_2) \right\} P_{33}(x_4) \rangle$$

Now, BY WICK CONTRACTING $P^a \xi^b$ ON R.H.S.

$$P^a(x_1)\, \xi^b(x_2) = - \underbrace{\mathrm{Tr}\left[T^a T^b\right]}_{\frac{1}{2}}\, a$$

$$\Downarrow$$

$$\boxed{m\, a^4 \sum_{x_1} \langle P_{12}(x_1)\, \xi_{21}(x_2)\, P_{33}(x_4) \rangle = \langle P_{11}(x_2)\, P_{33}(x_4) \rangle} \quad (1.1)$$

BY INSERTING IN (3.1) OF LECTURE 2

$$\boxed{a^4 \sum_{x} \langle q(x)\, q(0) \rangle = m^3 a^8 \sum_{x_1, x_2} \langle P_{12}(x_1)\, \xi_{21}(x_2)\, P_{33}(x_4) \rangle}$$

<u>Note</u>: THIS STARTS TO BE INTERESTING DUE TO THE FLAVOUR STRUCTURE

- IF $N_f = 5$ WE CAN ITERATE THE ARGUMENT IN A STRAIGHTFORWARD WAY, AND GET

$$a^4 \sum_x \langle q(x) q(0) \rangle = m^5 a^{16} \sum_{x_1, x_2, x_3, x_4} \langle P_{12}(x_1)\, S_{23}(x_2)\, S_{31}(x_3) \times P_{45}(x_4)\, S_{54}(0) \rangle \tag{1.2}$$

THIS IS THE FORMULA WE ARE INTERESTED IN!!!

2) ~~DERIVATION~~ INTERPRETATION OF FORMULA FROM SPECTRAL PROPERTIES OF D:

- WE START FROM

$$a^4 \sum_x \langle q(x) q(0) \rangle = m^2 a^4 \sum_x \langle P_{11}(x)\, P_{22}(0) \rangle$$

- BY REMEMBERING THAT

$$P_{11} = \bar{\psi}_1 \gamma_5 \left(1 - \frac{\bar{a} D}{2}\right) \psi_1$$

$$\Downarrow$$

$$P_{11}(x) = -\,\mathrm{tr}\left[\gamma_5 \left\{ \left(1 - \frac{\bar{a} D}{2}\right) D_m^{-1} \right\}(x, x) \right]$$

$$D_m \equiv D + m\left(1 - \frac{\bar{a} D}{2}\right)$$

D and D_m HAVE SAME EIGENVECTORS

$\lambda \neq 0$ DO NOT CONTRIBUTE to $\sum_x \hat{P}_{11}(x)$, $\langle u_\lambda | \gamma_5 | u_\lambda \rangle = 0$

$\lambda = \frac{2}{a} \Rightarrow \lambda_m = \frac{2}{a} \Rightarrow$ DO NOT CONTRIBUTE

$\lambda = 0 \Rightarrow \lambda_m = m$ contribute

$$\Downarrow$$

$$a^4 \sum_x \hat{P}_{11}(x) = -\frac{1}{m}(n_+ - n_-)$$

THIS IS THE FORMULA WE ARE INTERESTED IN !!!

$$\Downarrow$$

$$\boxed{m^2 a^4 \sum_x \langle P_{11}(x) P_{22}(0) \rangle = m^2 \frac{1}{m^2} \frac{\langle (n_+ - n_-)^2 \rangle}{V}}$$

c.v.d.

IF WE HAVE MORE DENSITIES

$$a^8 \sum_{x_1, x_2} \langle P_{12}(x_1) S_{21}(x_2) \rangle = -a^8 \sum_{x_1 x_2} \mathrm{Tr}\left[\gamma_5 \left(1 - \frac{aD}{2}\right) D_m^{-1} \times \right.$$

$$\left. \times \left(1 - \frac{aD}{2}\right) D_m^{-1} \right]$$

SINCE IT HOLDS

$$\mathrm{Tr}\left[\gamma_5 f(D)\right] = \left\{ f(0) - f\left(\frac{2}{a}\right) \right\} (n_+ - n_-)$$

then

$$a^4 \sum_{x_1 x_2} P_{12}(x_1) S_{21}(x_2) = -\left\{ \frac{1}{m^2} - 0 \right\} (n_+ - n_-)$$

$$\Downarrow$$

$$\boxed{m^3 a^8 \sum_{x_1 x_2} \langle P_{12}(x_1) S_{21}(x_2) P_{33}(x_1) \rangle = \cancel{m}^3 \frac{1}{\cancel{m}^3} \frac{(n_+ - n_-)^2}{V} = Q^2}$$

end

3) OPE AND FINITENESS OF χ:

BY TAKING (1.2)

$$a^4 \sum_x \langle q(x) q(0) \rangle = m^5 a^{16} \sum_{x_1 - x_4} \langle P_{12}(x_1) S_{23}(x_2) S_{31}(x_3) \times$$

$$\times P_{45}(x_4) S_{54}(0) \rangle$$

ON R.H.S. THE PRODUCTS

$$m \cdot P_{ij} \quad , \quad m \cdot S_{ij}$$

- DO NOT NEED RENORMALIZATION WHEN INSERTED AT PHYSICAL DISTANCE SINCE

$$\hat{m}_R = Z_m m \quad , \quad \hat{P}_{ij,R} = Z_P P_{ij} \; ; \quad \hat{S}_{ij,R} = Z_S S_{ij}$$

AND NON-SINGLET WIS GUARANTEE

$$Z_P = \frac{1}{Z_m} \quad , \quad Z_P = Z_S$$

- SINCE IS AN INTEGRATED CORRELATOR, WE NEED TO SEE IF SHORT DISTANCE SINGULARITIES ARE INTEGRABLE. IN THE CONTINUUM

$$\hat{P}_{12R}(x) \, \hat{S}_{23,R}(0) \xrightarrow[|x| \to 0]{} \frac{1}{|x|^3} \hat{P}_{13,R}(0)$$

AND THEREFORE

$$\int d^4x \; \hat{P}_{12R}(x) \, \hat{S}_{23R}(0) = \text{FINITE}$$

SINCE AT $|x| \to 0$ THE SHORT DISTANCE BEHAVIOUR IS INTEGRABLE.

- SINCE ALL P ~~an~~ AND S HAVE NON TRIVIAL FLAVOUR NUMBERS, THIS CAN BE USED FOR ALL COUPLE OF THEM, AND ALSO WHEN MORE THAN 2 CONVERGE TO THE SAME POINT. ALL SINGULARITIES ARE INTEGRABLE

- FOR INSTANCE WHEN ALL OF THEM CONVERGES TO THE SAME POINT

$$PSS\ PS \rightarrow \frac{1}{|x|^{15}} I$$

WHICH INTEGRATED OVER $d^4x_1 \dots d^4x_4$ GIVES A FINITE CONTRIBUTION.

CONCLUSION:

$$\boxed{\begin{aligned} \chi = m^5 a^{16} \sum_{x_1 \dots x_4} \langle P_{12}(x_1) S_{23}(x_2) S_{31}(x_3) \times \\ \times P_{45}(x_4) S_{54}(0) \rangle \end{aligned}} \quad (3.1)$$

IS FINITE AS IT STANDS, AND HAS A FINITE CONTINUUM LIMIT. WITH GW FERMION IT COINCIDE WITH

$$\chi = \frac{1}{V} \langle (n_+ - n_-)^2 \rangle$$

4) UNIVERSALITY OF χ:

- ONCE DEFINED AS IN (3.1), THE FINITENESS IS SHOWN ONLY BY USING NON-SINGLET WI1 AND FLAVOUR QUANTUM NUMBERS.

- WE CAN THEREFORE WRITE ~~THE~~ VERY SAME FORMULA, BUT WITH WILSON-FERMIONS

- BY JUST REPLACING $\frac{1}{Z_W} \to Z_P$ WE CAN PROVE THAT (3.1) WITH WILSON FERMIONS HAS NO SHORT-DISTANCE SINGULARITIES

$\Downarrow$

$$\chi = m^5 a^{16} \sum_{x_1 \dots x_4} \langle PSS \quad PS \rangle$$

IS A UNIVERSAL DEFINITION. ALL CONTINUUM LIMITS TEND TO THE SAME VALUES.

5) FINITINESS FOR $N_f < 5$:

- WE CAN INTRODUCE A MULTIPLET OF VALENCE QUARKS, WITH FERMION ACTION GIVEN BY

$$S_F = a^4 \sum_x \left\{ \sum_{i=1}^{2N_v} \bar{\psi}_i D_m \psi_i + \sum_{k=1}^{N_v} |D_m \phi_k|^2 \right\}$$

$\psi_i, \bar{\psi}_i$ ARE THE VALENCE QUARK, ϕ_k ARE THE ASSOCIATED PSEUDOFERMION FIELDS

- WITH SMALL MODIFICATIOS ALL GOES AS BEFORE

6) WITTEN - VENEZIANO RELATION:

- FROM PREVIOUS POINTS WE KNOW

$$\chi \begin{cases} \to \text{FINITE} \\ \to \text{UNAMBIGOUSLY DEFINED} \end{cases}$$

- POINT OF WITTEN:

(6.1) $\boxed{\chi^{YM} \neq 0}$ NO REASON WHY $= 0$

- WITH FERMIONS IF χ HAS A REGULAR EXPANSION IN N_f/N_c THEN

(6.2) $\boxed{\chi = \chi^{YM} + O\left(\frac{N_f}{N_c}\right)}$

IN THE CHIRAL LIMIT THE LOCAL WI SAYS

$$\langle \partial_\mu A^0_\mu(x)\, q(0)\rangle = 2N_f \langle q(x)\, q(0)\rangle$$

~~POSTULATING~~ ASSUMING THE ABSENCE OF MASSLESS PARTICLES IN THIS CHANNEL

$$\int d^4x \,\langle \partial_\mu A^0_\mu(x)\, q(0)\rangle = 0 \Rightarrow \boxed{\chi\big|_{m=0} = 0 \quad \forall \frac{N_f}{N_c}} \quad (6.3)$$

~~NOTE: BY USING THE STANDARD DEFINITIONS~~

WITTEN QUESTION: HOW IT IS POSSIBLE TO RECONCILE (6.1), (6.2) AND (6.3)?

SOLUTION: (6.2) MUST BE WRONG.

BY USING THE KÄLLEN-LEHMANN REPRESENTATION

$$\chi(p) \equiv \int d^4x\, e^{ipx} \langle q(x)\, q(0)\rangle$$

All contributions except the one of η'

$$\boxed{\chi(p) = C_0\Lambda^4 + C_1\Lambda^2 p^2 + C_2 (p^2)^2 - \frac{R^2}{p^2 + m^2_{\eta'}} + (p^2)^3 \int_{M^2}^{\infty} \frac{\rho(t)}{(t+p^2)\, t^3}\, dt} \quad (6.4)$$

THEN $\forall m$

$$\chi \equiv \lim_{p\to 0} \chi(p) = C_0\Lambda^4 - \frac{R^2}{m^2_{\eta'}} \quad (6.5)$$

IS UNAMBIGUOUS THANKS TO ALL SHOWN BEFORE

BY USING (6.3)

$$\lim_{m\to 0} \chi = 0 \Rightarrow \boxed{\frac{R^2}{m^2_{\eta'}}\bigg|_{m=0} = C_0\Lambda^4\bigg|_{m=0}} \quad (6.6)$$

BY ASSUMING THE SMOOTHNESS (IN $\frac{N_f}{N_c}$) OF $\chi(p)$ at $p \neq 0$

$$\lim_{\frac{N_f}{N_c} \to 0} \chi(p) = \chi^{YM}(p) \qquad \forall m$$

BY USING 6.4

$$\chi^{YM} = \lim_{p \to 0} \chi^{YM} p = \lim_{p \to 0} \lim_{\frac{N_f}{N_c} \to 0} \chi(p) = \lim_{\frac{N_f}{N_c} \to 0} c_0 \Lambda^4$$

$$\Downarrow$$

$$\boxed{\chi^{YM} = \lim_{\frac{N_f}{N_c} \to 0} c_0 \Lambda^4} \qquad \forall m \quad \text{eq. 2.4 n.} \,\cancel{?}$$

SINCE R.H.S. IS VALID $\forall m$ AND IN PARTICULAR FOR $m = 0$

$$\boxed{\lim_{\frac{N_f}{N_c} = 0} \lim_{m \to 0} \frac{R^2}{m_{\eta'}^2} = \chi^{YM}}$$

WITTEN-VENEZIANO FORMULA FOR THE MASS

COMPUTATION OF R:

By USING THE STANDARD CONVENTION FOR $F_{\eta'}$

$$\langle 0 | A_0(x) | \eta' \rangle = i \sqrt{2N_f}\, F_{\eta'} M_{\eta'} e^{-M_{\eta'} x_0}$$

$$\partial_0 \bar{A}_0(x_0) = 2N_f\, \bar{q}(x_0)$$

$$\Downarrow$$

$$\langle 0 | q(0) | \eta' \rangle = -i \frac{F_{\eta'} M_{\eta'}^2}{\sqrt{2N_f}} \Rightarrow R^2 = \frac{F_{\eta'}^2 m_{\eta'}^4}{2N_f}$$

AND THEREFORE

$$\boxed{\lim_{\frac{N_f}{N_c} \to 0} \lim_{m \to 0} \frac{F_{\eta'}^2 m_{\eta'}^2}{2N_f} = \chi^{YM}}$$

NOTE:

1) $\cancel{}\ \chi^{YM}$ IS $O(1)$ IN $\frac{1}{N_C}$, $F_{\eta'}^2 \propto N_C$ =>

=> $m_{\eta'}^2 = O\left(\frac{N_f}{N_C}\right)$ ← this is the answer to Witten question, there is 1 particle with a mass $\frac{N_f}{N_C}$, so we need to resum an ∞ series of diagrams and we get $O(1)$

2) when $N_C \to \infty$, or $\frac{N_f}{N_C} \to 0$, $m_{\eta'}^2$ BECOMES A GOLDSTONE BOSON

3) IN THIS LIMIT $U(1)_A$ IS RESTORED

WITTEN 79: "WE CANNOT ASK WHETHER THE FORMULA IS CORRECT, BECAUSE IT INVOLVES χ^{YM}, WHICH WE CAN NEITHER MEASURE NOR CALCULATE.

LECTURE 4: SPONTANEOUS SYMMETRY BREAKING AND BANKS-CASHER RELATION

1) LOCAL NON-SINGLET AWI:

$$\boxed{\langle \partial_\mu A^a_\mu(x)\, P^b(0)\rangle = 2m\,\langle P^a(x)\, P^b(0)\rangle - \frac{\delta^{(4)}(x)}{N_f}\,\delta^{ab}\,\langle S^0\rangle}$$

WHERE

$$\delta^{(4)}(x) = \frac{1}{a^4}\,\delta_{x,0}$$

<u>NOTE:</u> $Z_S\langle S^0\rangle$ FINITE ONLY IN THE CHIRAL LIMIT

$$S^0 = \bar\psi_L\psi_R + \bar\psi_R\psi_L \leftarrow \text{UNDER } SU(N_f)_L \otimes SU(N_f)_R$$

$$(\bar f_L, f_R) + (f_L, \bar f_R)$$

SO CANNOT MIX WITH $\hat I$ (SINGLET) BUT CAN MIX WITH M^+ and M

$$\hat S^0_R = Z_S\left\{S^0 + c_0\frac{M}{a^2} + c_0\frac{M^+}{a^2} + \ldots\right\}$$

AND THEREFORE OUT OF THE CHIRAL LIMIT IS NOT DEFINED

2) SPONTANEOUS SYMMETRY BREAKING AND GOLSTONE BOSONS:

- LET US TAKE THE $m \to 0$ LIMIT AND $a \to 0$, ALL OPERATORS IN THIS SECTION ARE INTENDED TO BE RENORMALIZED.

$$\langle \partial_\mu A^a_\mu(x) P^b(0) \rangle = -\frac{\delta^{(4)}(x)}{N_f} \langle S^0 \rangle \qquad (2.1)$$

FOR $x \neq 0$ LORENTZ INVARIANCE IMPLIES

$$\langle A^a_\mu(x) P^b(0) \rangle = \delta^{ab} x_\mu f(x^2)$$

AND BY INSERTING IN PREVIOUS EQUATION

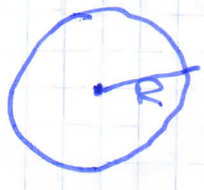
$$\sum_\mu \partial_\mu \{ x_\mu f(x^2) \} = 0 \Rightarrow 4 f(x^2) + x_\mu \partial_\mu f(x^2) = 0 \Rightarrow$$

$$\Rightarrow f(x^2) = \frac{1}{(x^2)^2}$$

$$\Downarrow$$

$$\langle A^a_\mu(x) P^b(0) \rangle = \delta^{ab} K \frac{x_\mu}{(x^2)^2} \qquad (2.2)$$

- WE CAN THEN INTEGRATE (2.1) ON A 4-SPHERE INCLUDING 0



$$\int_R \partial_\mu \langle A^a_\mu(x) P^b(0) \rangle d^4x = -\frac{1}{N_f} \delta^{ab} \langle S^0 \rangle$$

$$\Downarrow$$

$$\int_{|x|=R} d\sigma_\mu(x) \langle A^a_\mu(x) P^b(0) \rangle = -\frac{\delta^{ab}}{N_f} \langle S^0 \rangle$$

WE CAN NOW USE (2.2) [WE ARE ON-SHELL]

$$\delta^{ab} K \int_{|x|=R} d\sigma_\mu(x) \frac{x_\mu}{(x^2)^2} = -\frac{\delta^{ab}}{N_f} \langle S^0 \rangle$$

$$\delta^{ab} K \Omega_4 \cancel{|x|^3} \cancel{\frac{x_\mu}{|x|}} \cancel{\frac{x_\mu}{|x|^4}} = -\frac{\delta^{ab}}{N_f} \langle S^0 \rangle$$

By remembering that

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} \quad , \quad \Gamma(2) = 1 \Rightarrow \Omega_4 = 2\pi^2$$

$$\Downarrow$$

$$\boxed{K = -\frac{1}{N_f} \frac{\langle S^0 \rangle}{2\pi^2}}$$

$$\Downarrow$$

$$\boxed{\langle A^a_\mu(x) P^b(0) \rangle = -\delta^{ab} \frac{x_\mu}{(x^2)^2} \frac{\langle S^0 \rangle}{2\pi^2 N_f}} \quad (2.3)$$

Note: CURRENT-DENSITY CORRELATOR IS LONG RANGE IF $\langle S^0 \rangle \neq 0$, SUPPRESSED POWER-LIKE

Note: THE ENERGY SPECTRUM OF THE THEORY DOES NOT HAVE A GAP, MASSLESS PARTICLES WITH QUANTUM NUMBERS OF A^a_μ P^b.

LET US PROJECT ON $\vec{0}$ MOMENTUM. IF

$$\langle \bar{A}_0^a(x_0) P^b(0) \rangle \equiv \int d^3x \, \langle A_0^a(x) P^b(0) \rangle$$

WE GET

$$\langle \bar{A}_0^a(x_0) P^b(0) \rangle = -\delta^{ab} \frac{\langle S^0 \rangle}{2\pi^2 N_f} x_0 \int d^3x \frac{1}{(x_0^2 + |\vec{x}|^2)^2}$$

BY REMEMBERING THAT

$$\int d^3x \frac{1}{(x_0^2 + |\vec{x}|^2)^2} = \frac{\pi^2}{x_0}$$

$\Downarrow$

$$\boxed{\langle \bar{A}_0^a(x_0) P^b(0) \rangle = -\frac{\delta^{ab}}{2N_f} \langle S^0 \rangle \qquad x_0 \neq 0} \quad 2.u$$

<u>NOTE:</u> CONSTANT IN EUCLIDEAN TIME, $N_f^2 - 1$ STATES WITH $E = 0$!!!

3) GOLDSTONE BOSONS (PIONS) MATRIX ELEMENTS

BY NORMALIZING THE STATES AS USUAL

$$\langle \pi^a(p) | \pi^b(p') \rangle = (2\pi)^3 \, 2p^0 \, \delta^{(3)}(\vec{p} - \vec{p}') \, \delta^{ab}$$

$$d\mu(p) = \frac{d^3p}{(2\pi)^3 \, 2p^0}$$

BY REMEMBERING THAT

$$O(t) = e^{tH} \, O(0) \, e^{-Ht}$$

$$\langle \pi^b(p) | \bar{\psi} \gamma_5 T^a \psi | 0 \rangle = -i \, \delta^{ab} \, G_\pi \, e^{-E_\pi(p) x_0} \, e^{-i \vec{p} \cdot \vec{x}}$$

$$\langle 0 | A^a_\mu(x) | \pi^b(p) \rangle = i \, \delta^{ab} F_\pi \, p_\mu \, e^{-E_\pi(p) x_0} \, e^{i \vec{p} \cdot \vec{x}}$$

THAN

$$\langle \bar{A}^a_0(x_0) \, P^b(0) \rangle = \sum_c \int \frac{d^3p}{(2\pi)^3 \, 2p^0} \int d^3x \, \langle 0 | A^a_0(x) | \pi^c \rangle \langle \pi^c | P^b(0) | 0 \rangle$$

$$= \sum_c \delta^{ac} \delta^{cb} \int d^3x \int \frac{d^3p}{(2\pi)^3 \, 2p^0} \left(+ G_\pi F_\pi \, p_0 \right) e^{i \vec{p} \cdot \vec{x}} \, e^{-E_\pi(p) x_0}$$

$$= \delta^{ab} \left(+ \frac{G_\pi F_\pi}{2} \right) e^{-E_\pi(\vec{0}) x_0}$$

$$\Downarrow$$

$$\boxed{\langle \bar{A}^a_0(x_0) \, P^b(0) \rangle \xrightarrow[x_0 \to \infty]{} \frac{\delta^{ab}}{2} \, G_\pi F_\pi}$$

BY COMPARING WITH (2.4)

$$\boxed{G_\pi F_\pi = -\frac{\langle S^0 \rangle}{N_f} \qquad m=0} \qquad (3.1)$$

NOTES:

$\langle 0|P|\pi\rangle \neq$ ONLY IF π IS A (PSEUDO)SCALAR

$\langle 0|A_\mu|\pi\rangle \propto P_\mu$ BECAUSE π IS A (PSEUDO)SCALAR

NOTE:

$\langle S^0\rangle \neq 0$ IF $G_\pi \neq 0$ AND $F_\pi \neq 0$ AND π IS A GOLDSTONE BOSON

4) GELL-MANN-OAKES-RENNER (GMOR) RELATION:

- LET US MOVE TO $m \neq 0$ AND $x_0 \neq 0$

$$\partial_0 \langle \bar{A}_0^a(x_0) P^b(0)\rangle = 2m \langle \bar{P}^a(x_0) P^b(0)\rangle$$

WE SATURATE WITH PION WHICH DOMINATES FOR $x_0 \to \infty$

$$\partial_0 \left\{ \cancel{\delta^{ab}} \frac{G_\pi F_\pi}{2} e^{-M_\pi x_0} \right\} = 2m \left\{ -\frac{G_\pi^2}{2M_\pi} e^{-M_\pi x_0} \cancel{\delta^{ab}} \right\}$$

$$-M_\pi F_\pi \cancel{e^{-M_\pi x_0}} = -2m \frac{G_\pi}{M_\pi} \cancel{e^{-M_\pi x_0}}$$

$$\Downarrow$$

$$\boxed{M_\pi^2 F_\pi = 2m G_\pi}$$

THEREFORE IN THE CHIRAL LIMIT

$$\lim_{m\to 0} \frac{M_\pi^2 F_\pi}{2m} = \lim_{m\to 0} G_\pi$$

WE CAN USE (3.1)

$$\boxed{\lim_{m\to 0} \frac{M_\pi^2 F_\pi^2}{2m} = \lim_{m\to 0} - \frac{\langle S^0 \rangle}{N_f}}$$

GMOR RELATION

NOTE: • NUMERICALLY WE CAN WORK AT FINITE VOLUME, WHERE WE NEED $m \neq 0$ TO HAVE A NON ZERO CONDENSATE

• BUT $m \neq 0$ THE CONDENSATE IS DIVERGENT!! VERY BAD NUMERICALLY

• WE NEED TO FIND A WAY AROUND TO EXTRACT THE CONDENSATE AT $m \neq 0$

4) BANKS - CASHER RELATION:

$$-\frac{\langle S^0 \rangle}{N_B} = \langle Tr\left[\left(1-\frac{\bar{a} D}{2}\right)\frac{1}{D_m}\right]\rangle \frac{1}{V}$$

- BY REMEMBERING THAT

$$D_m = D + m\left(1 - \bar{a}\frac{D}{2}\right) \quad ; \quad D_m^+ D_m = DD^+ + m^2\left[1 - \left(\frac{\bar{a}}{2}\right)^2 DD^+\right]$$

IT IS EASY TO SHOW THAT (WI 11.1)

$$Tr\left[\left(1-\frac{\bar{a} D}{2}\right)\frac{1}{D_m}\right] = m\, Tr\left[\frac{1-\left(\frac{\bar{a}}{2}\right)^2 D^+D}{D^+D + m^2\left\{1-\left(\frac{\bar{a}}{2}\right)^2 D^+D\right\}}\right]$$

AND THEREFORE

$$\boxed{-\frac{\langle S^0 \rangle}{N_B} = \frac{m}{V}\langle Tr\left[\frac{1-\left(\frac{\bar{a}}{2}\right)^2 D^+D}{D^+D + m^2\left\{1-\left(\frac{\bar{a}}{2}\right)^2 D^+D\right\}}\right]\rangle}$$

NOTE: • $\frac{2}{a}$ MODES CLEARLY DO NOT CONTRIBUTE

NOTE: ϕ MODES GIVE CONTRIBUTIONS THAT VANISH IN THE INFINITE VOLUME LIMIT

NON-ZERO MODES:

$$DD^+ \to \left[\frac{2}{\bar{a}} \sin\left(\frac{\alpha_i}{2}\right)\right]^2 \quad ; \quad D \to \frac{1}{a}\left\{1 - e^{i\alpha}\right\}$$

We can define:

$$\lambda_i \equiv \frac{2}{\bar{a}} \frac{\sin\left(\frac{\alpha_i}{2}\right)}{\cos\left(\frac{\alpha_i}{2}\right)} \qquad \alpha_i \in [-\pi, \pi]$$

THEN

$$-\frac{\langle S^0 \rangle}{N_f} = +\frac{m}{V} \sum_i \left\langle \frac{1}{\lambda_i^2 + m^2} \right\rangle$$

WE DEFINE THE SPECTRAL DENSITY AS

$$\rho(\lambda) \equiv \frac{1}{V} \sum_i \langle \delta(\lambda - \lambda_i) \rangle$$

AND THEREFORE

$$-\frac{\langle S^0 \rangle}{N_f} = +2m \int_0^\infty \frac{\rho(\lambda)}{\lambda^2 + m^2} \, d\lambda$$

WHERE WE HAVE USED CHIRAL SYMMETRY ~~BEC~~ FOR $\rho(-\lambda) = \rho(\lambda)$.

BANKS - CASHER RELATION

- WE CAN AGAIN CHANGE VARIABLE

$$x = \frac{\lambda}{m} \Rightarrow -\frac{\langle S^0 \rangle}{N_f} = +2 \int_0^\infty \frac{1}{x^2+1} \rho(m x) \, dx$$

We are interested in the limit $m \to 0$.

$$-\frac{\langle S^0 \rangle}{N_f} = +2 \int_0^\Lambda \frac{1}{x^2+1} \rho(m x) \, dx + \int_\Lambda^\infty \frac{1}{x^2+1} \rho(m x) \, dx$$

$$\Downarrow$$

$$\lim_{m \to 0} -\frac{\langle S^0 \rangle}{N_f} = +2 \int_0^\Lambda \frac{1}{x^2+1} \lim_{m \to 0} \rho(m x) \, dx + \lim_{m \to 0} \int_\Lambda^\infty \frac{1}{x^2+1} \rho(m x) \, dx$$

$$\Downarrow$$

$$\boxed{-\frac{\langle S^0 \rangle}{N_f}\bigg|_{m=0} = +2 \rho(0) \int_0^\Lambda \frac{1}{x^2+1} \, dx + \lim_{m \to 0} \int_\Lambda^\infty \frac{1}{x^2+1} \rho(m x) \, dx}$$

SINCE WE CAN TAKE Λ AS LARGE AS WE WISH BUT FINITE

$$\rho(m x) = c\, m^3 x^3 + \cdots$$

$$\Downarrow$$

$$\lim_{m\to 0} -\frac{\langle S^0\rangle}{N_f} = +2\rho(0)\int_0^{\Lambda}\frac{dx}{x^2+1} + \underbrace{\lim_{m\to 0} c\, m^3 \int_{\Lambda}^{\infty}\frac{x^3 dx}{x^2+1}}_{\text{divergent but goes to 0 in chiral limit}}$$

$$\Downarrow$$

Now I can take back $\Lambda \to \infty$

$$-\frac{\langle S^0\rangle}{N_f}\Big|_{m=0} = +2\rho(0)\, \mathrm{Arctg}(x)\Big|_0^{\infty}$$

$$\Downarrow$$

$$\boxed{-\frac{\langle S^0\rangle}{N_f}\Big|_{m=0} = \pi\rho(0)}$$

BANKS-CASHER RELATION

NOTE: CAN BE READ IN EITHER DIRECTIONS! A NON-ZERO SPECTRAL DENSITY AT THE ORIGIN IMPLIES THAT $\langle S^0\rangle \neq 0$ AND VICEVERSA

5) RENORMALIZATION OF SPECTRAL DENSITY:

QUESTION: IS THE SPECTRAL DENSITY A RENORMALIZ... QUANTITY WITH A UNIVERSAL CONTINUUM LIMIT?

QUESTION: IF YES, THIS IS ALSO TRUE FOR $m \neq 0$?

TO THIS AIM LET US DEFINE THE SPECTRAL SUMS

$$\sigma_k(m, \mu) = 2V \int_0^\infty \frac{1}{(\lambda^2 + \mu^2)^k} \rho(\lambda, m)\, d\lambda$$

BY INTRODUCING VALENCE QUARKS AS FOR TOP. SUSC., IT IS STRAIGHTFORWARD TO SHOW THAT

$$\sigma_k(m, \mu) = - a^{8k} \sum_{x_1 \dots x_{2k}} \langle P_{12}(x_1) P_{23}(x_2) \dots P_{2k\,1}(x_{2k}) \rangle$$

AGAIN BY NOTICING THAT

$$P_{12}(x) P_{23}(0) \sim \frac{1}{|x|^3} S_{13}(0)$$

IT IS POSSIBLE TO PROVE THAT ALL SHORT DISTANCE SINGULARITIES ARE INTEGRABLE IF $k \geq 3$

$\Downarrow$

$$\boxed{\sigma_k^R(m, \mu) = Z_P^{2k} \sigma_k(m, \mu) \qquad k \geq 3}$$

6) FROM SPECTRAL SUMS TO SPECTRAL DENSITY:

BY NOTICING THAT

$$\int_{\mu_0^2}^{\mu_1^2} \frac{1}{(\lambda^2+\mu^2)^k}\, d\mu^2 = \frac{1}{1-k}\left[\frac{1}{(\lambda^2+\mu_1^2)^{k-1}} - \frac{1}{(\lambda^2+\mu_0^2)^{k-1}}\right]$$

BY ITERATING A SECOND TIME

$$\int_{\mu_0^2}^{\mu_2^2} d\mu_1^2 \int_{\mu_0^2}^{\mu_1^2} \frac{1}{(\lambda^2+\mu^2)^k}\, d\mu^2 = \frac{1}{(1-k)(2-k)}\left[\frac{1}{(\lambda^2+\mu_2^2)^{k-2}} - \frac{1}{(\lambda^2+\mu_0^2)^{k-2}}\right]$$

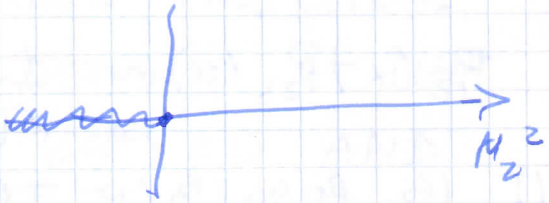
$$- \frac{(\mu_2^2-\mu_0^2)}{1-k}\,\frac{1}{(\lambda^2+\mu_0^2)^{k-1}}$$

SO FOR $k=3$

$$\int_{\mu_0^2}^{\mu_2^2} d\mu_1^2 \int_{\mu_0^2}^{\mu_1^2} \sigma_3(m,\mu)\, d\mu^2 = \frac{2V}{2}\int_0^\infty \frac{\rho(\lambda,m)}{(\lambda^2+\mu_2^2)}\, d\lambda +$$

$$+ \beta(\mu_0^2,\mu_2^2)$$

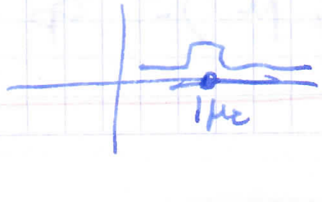
THE R.H.S. AS A FUNCTION OF μ_2^2 HAS A CUT FOR $\mu_2^2<0$.

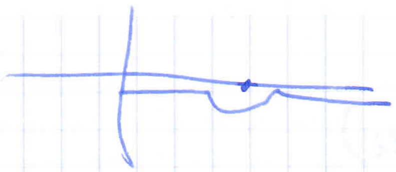


IN THIS CAS

$$\int_0^\infty \frac{\rho(\lambda,m)}{(\lambda^2+\mu_2^2)^2} = \int_0^\infty \frac{\rho(\lambda,m)}{(\lambda-|\mu_2|)(\lambda+|\mu_2|)}\, d\lambda$$

We can do the integral



OR 

THE DIFFERENCE OF THE TWO GLUES S —

7) RENORMALIZATION OF SPECTRAL DENSITY.

$$\sigma_n^R = z_p^{2k}\, 2V \int_0^{\delta} \frac{\rho(\lambda, m)}{(\lambda^2+\mu^2)^k}\, d\lambda \qquad z_p = \frac{1}{z_m}$$

$$= 2V \int_0^{\delta} \frac{\rho(\lambda, m_R \cdot z_p)}{\left[(\lambda/z_p)^2 + \mu_R^2\right]^k}\, d\lambda$$

BY DEFINING $\lambda_R = \frac{\lambda}{z_p}$

$$\sigma_n^R = 2V \int_0^{\delta} \frac{z_p\, \rho(\lambda_R z_p, m_R z_p)}{(\lambda_R^2 + \mu_R^2)^n}\, d\lambda_R$$

BY USING PREVIOUS FORMULAE WITH μ_R

$$\boxed{\rho_R(\lambda_R, m_R) = z_p\, \rho(\lambda, m) \qquad \lambda_R = \frac{\lambda}{z_p} \quad m_R = \frac{m}{z_p}}$$

- THE SPECTRAL DENSITY IS RENORMALIZABLE AND HAS A UNIVERSAL VALUE IN CONTINUUM LIMIT FOR $m \geq 0$ —

8) MODE NUMBER

$$\nu(m,\Lambda) \equiv V \int_{-\Lambda}^{\Lambda} d\lambda \; \rho(\lambda, m)$$

NOTE: IT IS THE AVERAGE NUMBER OF MODES IN THE INTERVAL $[-\Lambda, \Lambda]$.

$$\nu(m,\Lambda) = V \int_{-\Lambda}^{\Lambda} d\lambda \; \frac{1}{Z_P} \cdot \rho_R(\lambda_R, m_R)$$

$$\parallel$$

$$\nu(m,\Lambda) = V \int_{-\Lambda^R}^{\Lambda^R} d\lambda_R \; \rho_R(\lambda_R, m_R) \qquad \Lambda^R = \frac{\Lambda}{Z_P}$$

$$\parallel$$

$$\nu_R(m_R, \Lambda_R)$$

$$\boxed{\nu(m,\Lambda) = \nu_R(m_R, \Lambda_R)}$$

- THE MODE NUMBER IS A RENORMALIZATION GROUP INV. QUANTITY FOR $m \neq 0$
- IT IS ULTRAVIOLET FINITE AS IT STANDS AND HAS A UNIVERSAL MEANING.