# Exercises on Inflation

1. Consider a FRW metric in conformal coordinates, $ds^2 = a(\eta)^2(-d\eta^2 + d\vec{x}^2)$. The scale factor in front of the whole metric does not affect the propagation of light rays and therefore does not affect causality. So why the condition $d^2a/dt^2 > 0$ solves the horizon problem?

2. Suppose all matter fields – photons for example – are coupled not directly to the metric $g_{\mu\nu}$, but to $h(\phi)g_{\mu\nu} \equiv \tilde{g}_{\mu\nu}$, where $\phi$ is a scalar field, evolving in time and $h$ is a given function. We are assuming to be in 'Einstein frame', i.e. that the action for $g_{\mu\nu}$ is just the standard Einstein-Hilbert action. Is it enough to have acceleration of the "effective" scale factor $\tilde{a}$ of the metric $\tilde{g}$ to solve the horizon problem? Why?

3. In a matter dominated Universe, show that $\Omega$ equals, in a Newtonian treatment, the ratio between (the absolute value of) potential and kinetic energy. The curvature problem is thus interpreted as the tuning of these two quantities.

4. In a bouncing model, the scale factor is initially contracting, then reaches a minimum (the bounce) and then starts a (decelerated) expansion. Compare the diagram $k^{-1}a$ vs $H^{-1}$ for inflation and for bouncing models. Discuss how to realize a bounce.

5. The ratio between pressure and energy density is usually called $w$: $w \equiv p/\rho$. What are the possible values of $w$ for a scalar field, with standard kinetic term and potential, which evolves in time, assuming $\rho > 0$? And if we assume a positive definite potential?

6. Prove that a coupling $f(\phi)R$ between the inflaton and the metric can be set to zero by an appropriate conformal transformation: $\tilde{g}_{ab} = \Omega^2(x)g_{ab}$.

7. Assuming instantaneous reheating after inflation at temperature $T_{\rm rh}$ show, using entropy conservation, that we need at least
$$N = 46 + \log \frac{T_{\rm rh}}{10^{10}{\rm GeV}} + \frac{1}{2}\log|\Omega_i - 1| \qquad (1)$$
e-folds of inflation to solve the flatness problem. Here $\Omega_i$ is the curvature parameter when inflation begins.

8. Consider a scalar field with action
$$S = \int d^4x\sqrt{-g}P(X,\phi) \qquad X \equiv -g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \,. \qquad (2)$$
Write the stress energy tensor. What is the condition to have inflation, $\ddot{a} > 0$, assuming that the scalar only depends on time? Models with this kind of Lagrangian are called $k$-inflation.

9. Let us verify that an action of the form
$$S = \int d^4x\sqrt{-g}F(R) \qquad (3)$$
is equivalent to GR + a scalar with standard kinetic term. Show that the action above is equivalent to
$$\int d^4x\sqrt{-g}\left[F'(A)(R-A) + F(A)\right] \qquad (4)$$

once the equation of motion for the auxiliary field $A$ is used. Now we can make a conformal transformation to demix $A$ from the metric; this will give a kinetic term to $A$. After dust has settled you should get

$$\int d^4x\sqrt{-g}\left[R - \frac{3}{2}g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - V(\sigma)\right] \tag{5}$$

with

$$V(\sigma) = \frac{A}{F'(A)} - \frac{F(A)}{F'(A)^2} \qquad \text{with } \sigma = -\log F'(A)\ . \tag{6}$$

10. An harmonic oscillator of frequency $\omega_i$ is in its vacuum state. Its frequency is instantaneously changed to $\omega_f$. Write the state of the system in terms of the new eigenstates and calculate its energy. [This integral of Hermite polynomials may be useful:

$$\int_{-\infty}^{+\infty} e^{-x^2} H_{2m}(xy)dx = \sqrt{\pi}\frac{(2m)!}{m!}(y^2-1)^m\ .] \tag{7}$$

11. An harmonic oscillator of frequency $\omega_i$ is in its first excited state and its frequency is instantaneously changed to $\omega_f \ll \omega_i$. What is its final energy? [No long calculation is needed.]

12. Photons are massless, but they are not produced during inflation. Why?

13. Calculate the equal time 2-point function of a massless scalar in a fixed de Sitter background in real space. What is the physical meaning of the IR divergence?

14. Using symmetry arguments, calculate the tilt of the spectrum of a scalar with small mass, $m^2 \ll H^2$, in a fixed de Sitter background.

15. Calculate the spectrum of a scalar with small mass, $m^2 \ll H^2$, in a fixed de Sitter background.

16. Calculate the spectrum of $\zeta$ in the spatially flat gauge.

17. Consider a model of inflation based on the potential $V = \lambda\phi^4$. Find the value of $\lambda$ required to get the right normalization of the power spectrum. Find the tilt of the scalar spectrum and the tensor to scalar ratio.

18. A Goldstone boson, like an axion, with decay constant $f_a$ lives during inflation and therefore gets quantum perturbations. Estimate the corrections you expect to the free field theory calculation from self interactions, i.e. the deviation from gaussianity. What happens if $H > 4\pi f_a$?

19. Using symmetry arguments, show that the $n$-point function of $\zeta$ in Fourier space in a generic model of inflation is of the form

$$\langle\zeta_{\vec{k}_1}\dots\zeta_{\vec{k}_n}\rangle = (2\pi)^3\delta(\sum \vec{k}_i)F(k_i)\ , \tag{8}$$

where $F$ is an homogeneous function of the $k$'s of degree $-3(n-1)$.

20. Consider a massless scalar $\varphi$ in de Sitter space with an interaction $\frac{M}{6}\varphi^3$. Calculate the 3-point function $\langle\varphi_{\vec{k}_1}\varphi_{\vec{k}_2}\varphi_{\vec{k}_3}\rangle$.

21. de Sitter is maximally symmetric, i.e. it has 10 isometries. In inflation, the correlation functions of $\zeta$ are translationally, rotationally and scale invariant. These are 7 symmetries. What happened to the other 3?

22. Calculate (numerically) $f_{\rm NL}^{\rm loc.}$ in a model of modulated reheating coming from the non-linear relation $\zeta(\Gamma)$ as a function of $\Gamma/H$, where $H$ is the Hubble parameter at the end of inflation.