Problems with Hot FRW cosmology

(Take a look at the original paper: A. Guth PRD 23, 2 (1981) 347)

Problems: <u>fine tuning</u> of initial conditions, not experimental inconsistencies

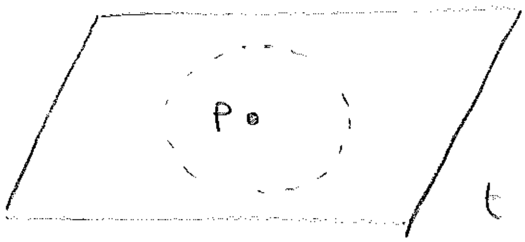
Do we have to care?

<u>Horizon problem</u>: homogeneity vs particle horizon

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right) \qquad \text{Flat FRW}$$

Matter dominance: $a(t) \propto t^{2/3}$ MD

Radiation dominance: $a(t) \propto t^{1/2}$ RD



$t = \text{const}$

Which comoving observers could communicate with P?

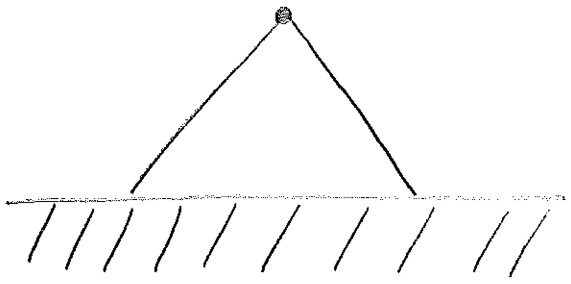
Conformal time: $\eta = \int \frac{dt}{a(t)}$ $\qquad a(t)\, d\eta = dt$

$$ds^2 = a^2(\eta)\left[-d\eta^2 + dx^2 + dy^2 + dz^2\right]$$

Conformally flat metric $\qquad$ light rays, $ds^2 = 0$, do not care about $a(\eta)$

Both in RD and MD the integral converges in the past

$t = 0$



SINGULARITY

(I used flat FRW, but anyway curvature becomes irrelevant in the past)

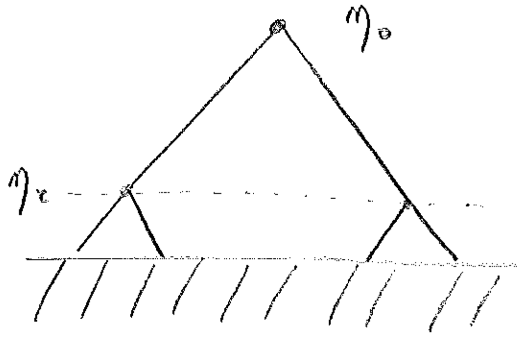
Let us calculate it:

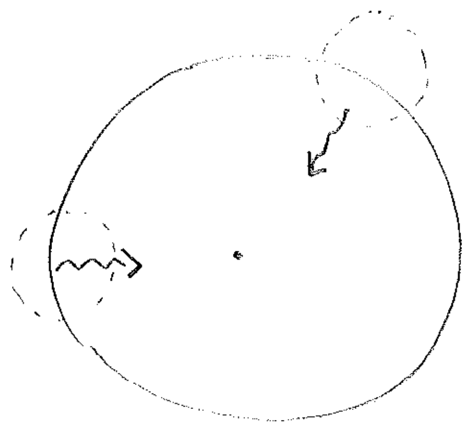
MD $\quad a = a_0 t^{2/3} \qquad H = \frac{\dot a}{a} = \frac{2}{3} t^{-1}$ Hubble rate

$$d_{HOR} = a \int_0^t \frac{d\tilde t}{a_0 \tilde t^{2/3}} = a \, \frac{3 t^{1/3}}{a_0} = 3t = 2H^{-1}$$

(Obviously if the integral converges I always get $\sim H^{-1}$
E.g. in RD I get $d_{HOR} = H^{-1}$)

Let us look at recombination. The Universe becomes neutral and photons free stream: a real causality diagram





(CMB is very clean as photons do not interact anymore, otherwise I always see things which are ~ causal contact)

Why is CMB so homogeneous? It comes from points that were never in causal contact!

How many uncorrelated spots?

$$\left(\begin{array}{llll} MD & a \propto t^{2/3} & \eta = \int \frac{dt}{a} \propto t^{1/3} & \Rightarrow a \propto \eta^2 \\ RD & a \propto t^{1/2} & \eta = \int \frac{dt}{a} \propto t^{1/2} & \Rightarrow a \propto \eta \end{array} \right)$$

$$\frac{\eta_r}{\eta_0} = \sqrt{\frac{a_r}{a_0}} \sim \frac{1}{\sqrt{1+z_{rec}}} \sim \frac{1}{\sqrt{1100}}$$

few degrees (which is obviously the CMB peak)

Even worse if we go further back: $T \sim 10^{17}$ GeV

initial conditions below Planck era

Very roughly using RD

$$\frac{\eta_{Planck}}{\eta_0} \sim \frac{a_{Planck}}{a_0} \sim \frac{10^{-4}\,eV}{10^{17}\,GeV} \sim 10^{-30}$$

(rough, I should use entropy conservation + MD phase)

Our present Universe is composed by $\sim 10^{90}$ boxes which are disconnected at the Planck era!

I have to choose carefully these 10^{90} initial conditions to give rise to our Universe, which is so homogeneous

Planck abracadabra

As we do not know anything about quantum gravity, maybe everything is solved there (locality breaks down...)

→ But at the Planck era the separation between the horizon and present Universe is maximal...

→ Inflation will be a way to solve the problems with physics which is under control

Nowadays the focus is more on the perturbations and predictions of inflation. I cannot "prove" inflation using FRW problems

Particle horizon vs Hubble radius (or "horizon" unfortunately)

$$d_{HOR} = a \int_0^t \frac{d\tilde{t}}{a(\tilde{t})}$$

we would like to make this distance longer and longer

$$d_{HOR} = a \int_0^a \frac{d\tilde{a}}{\tilde{a}\,\dot{a}(\tilde{a})}$$

The convergence of the integral in the past depends on

$$\ddot{a} \gtrless 0$$

In particular for $\ddot{a} > 0$ the integral diverges!

MD, RD confuse two physically ≠ scales

- H^{-1} Timescale of evolution. Whether I can neglect or not the expansion in a given process. Quantity defined at a given time

  $K/a \lessgtr H$ a wavelength inside / outside H behaves very differently

- d_{HOR} Distance travelled by a photon since the beginning. Global quantity

$d_{HOR} \simeq H^{-1}$ in MD, RD (or any decelerated phase)

Inflation completely separates the two concepts

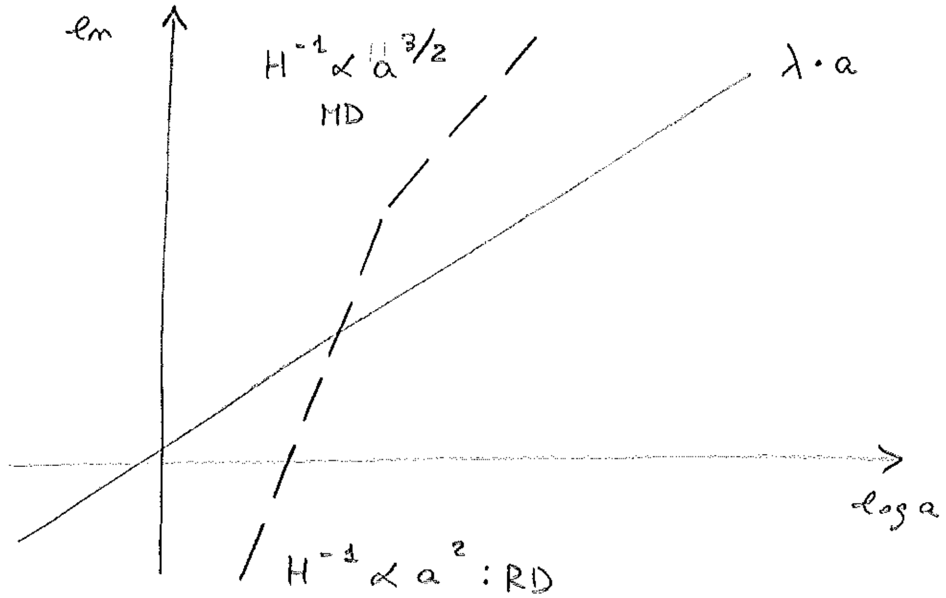
$$\boxed{\text{INFLATION} = \ddot{a} > 0}$$

With this definition we are inflating, of course I am talking about an early phase

Inside and outside Hubble (or "horizon")

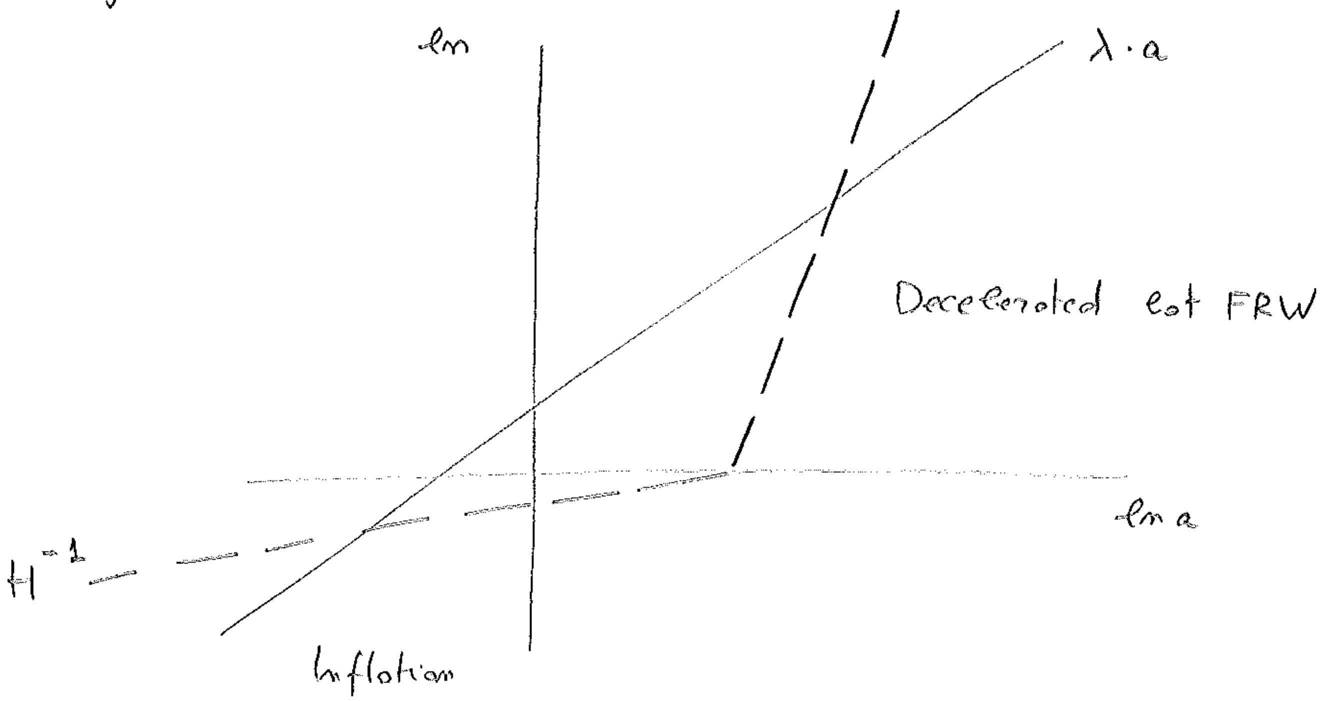
$$\frac{k}{aH} \quad \begin{array}{lll} \gg & \text{inside } H & \sim \text{Minkowski} \\ \ll & \text{outside } H & \sim \text{dominated by the expansion} \end{array}$$

$$\frac{k}{a\,\frac{\dot a}{a}} \quad \begin{array}{ll} \ddot a < 0 & \text{Modes come in} \\ \ddot a > 0 & \text{Modes go out} \end{array} \qquad \ddot a \text{ again}$$

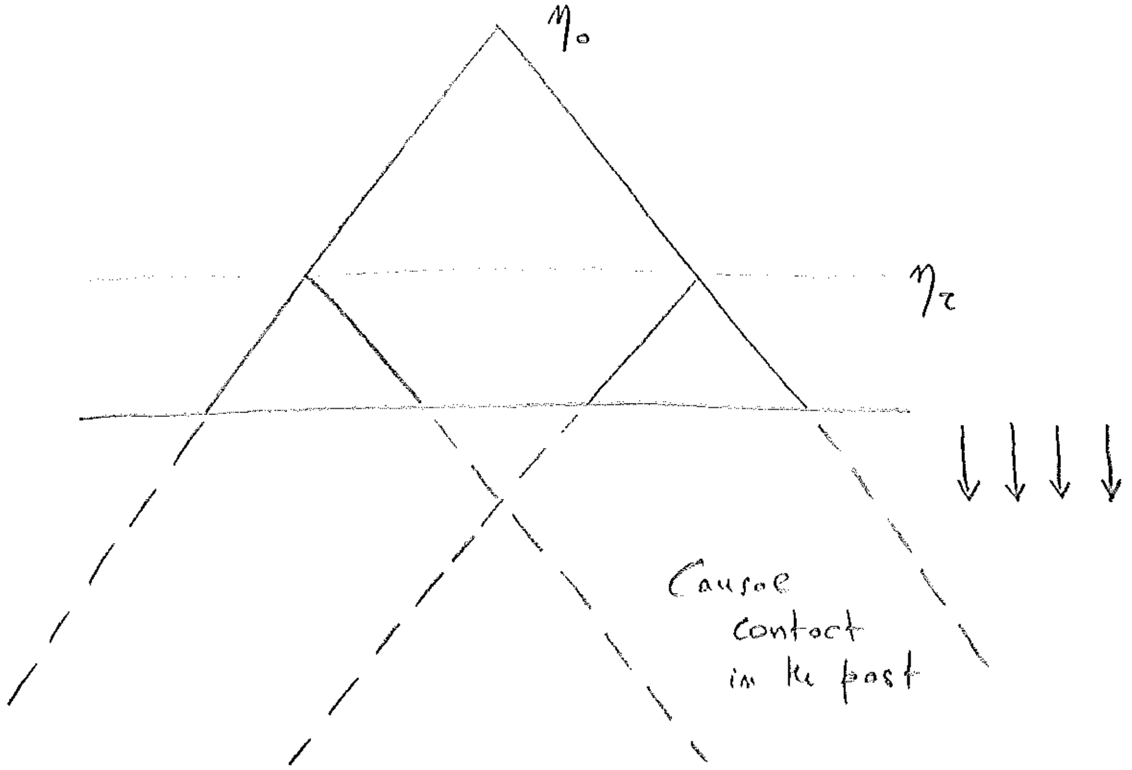


We see again the horizon problem in Fourier space : the initial condition for each mode must be set way out of H^{-1}.

Why we do not see crazy things entering all the time?



In the CMB diagram in conformal time :



Curvature problem

Friedmann equation : $H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho$

Spatial curvature enters in the Einstein equation like a source red-shifting as a^{-2}. More and more important wrt matter / radiation (before dark energy, it was the curvature that decided the fate of the Universe)

As the curvature is irrelevant nowadays, it must have been very irrelevant in the past

$$\rho_{CR} \equiv \frac{3H^2}{8\pi G} \qquad \Omega(t) = \frac{\rho}{\rho_{CR}} = \frac{k}{(aH)^2} + 1$$

k has a ≠ meaning, but anyway it is the same ratio as before

RD: $|\Omega - 1|_0 \propto \left(\frac{a_0}{a_i}\right)^2 |\Omega - 1|_i$

The density must be very close to critical at the beginning of RD

As $|\Omega - 1| \propto \frac{1}{\dot{a}^2}$ $\qquad |\Omega - 1| \longrightarrow 0$ during $\ddot{a} > 0$

During inflation $|\Omega - 1| \longrightarrow 0$. Avoiding "tuning" we expect very small $(\Omega - 1)_0$. General "prediction" of inflation

Now (2012): $|\Omega - 1|_0 \lesssim 10^{-2}$

Future : $|\Omega - 1|_0 \lesssim 10^{-3}; 10^{-4}$

(Of course one can get to 10^{-5}, which is the approximation to which we can talk of curved FRW)

(Notice curvature has a typical wavelength, so the previous analyses of exiting / entering is exactly the same as solving the curvature problem)

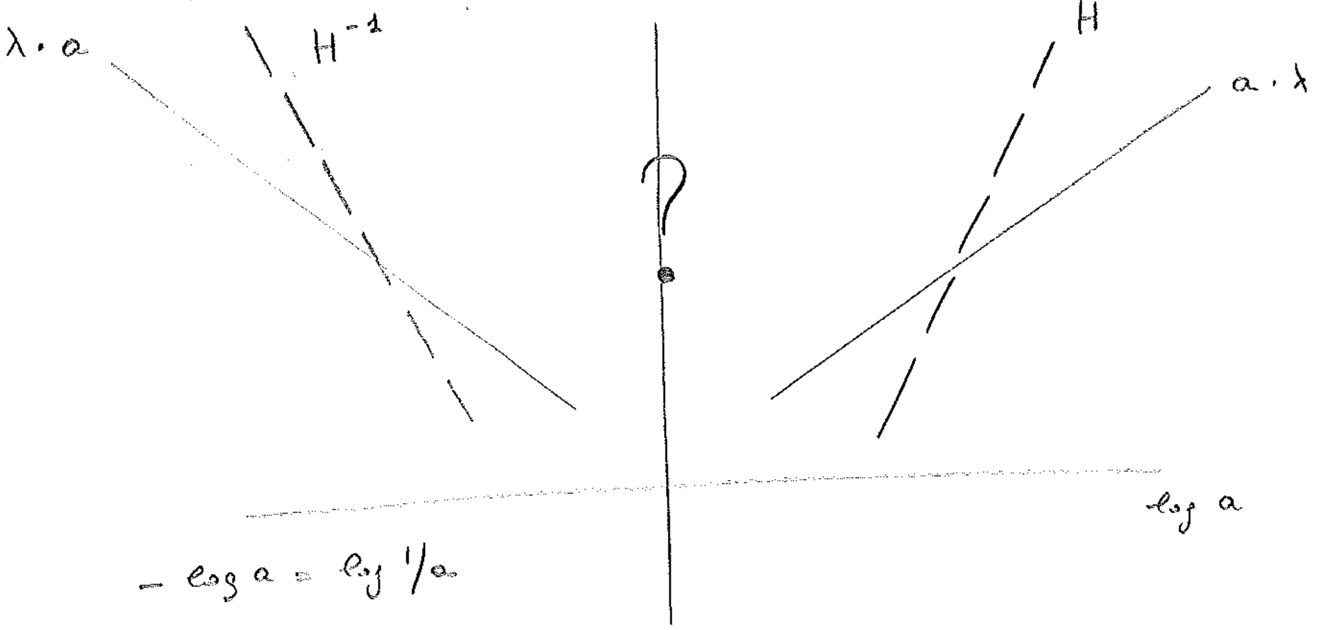
In other words: if M_P is the typical scale why an FRW (even assuming homogeneity) should last so long?

Notice that M_P exists only with $\hbar \neq 0$ ...

Bouncing models

Useful to compare inflation with bouncing models in terms of kinematics of the modes

Before the standard cosmology a phase of contraction



BOUNCE

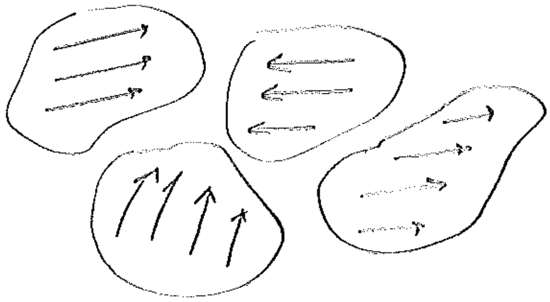
- Modes leave the horizon during the accelerated (standard) contraction
- What happens at the bounce? Naively $H \longrightarrow 0$ to flip sign and all modes come back. But the hope is that this happens in a very short time so that modes have no time to evolve

Not compelling, but a logical possibility!

Monopole problem :

another classic, but less compelling, motivation for inflation

GUT phase transition: $G \longrightarrow SM$ Scale $M \simeq 10^{16}$ GeV



By Kibble argument, I expect at least 1 monopole per Hubble volume: by causality the gauge configuration is unrelated out of H^{-1}

$$n_{MONOP.} \simeq H^3 \simeq \left(\frac{M^2}{M_P}\right)^3 \quad \text{while} \quad n_\gamma \simeq T^3 \simeq M^3$$

$$\frac{n_{MONOP}}{n_\gamma} \simeq \left(\frac{M}{M_P}\right)^3 \simeq 10^{-9}$$

Experimental limits are $< 10^{-39}$ monopole/photon !

But it could be that there is no GUT and monopole to start with ...

How to obtain $\ddot{a} > 0$

- $$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) > 0$$

"Gravity becomes repulsive" actually nothing happens to gravity, it is just a change $T_{\mu\nu}$

(This inequality violates the strong energy condition (SEC)

$$\left(T_{ab} - \frac{1}{2} g_{ab} T\right) t^a t^b \geq 0 \qquad \forall t \text{ time-like}$$

which, for comoving observers, gives $\rho + 3p \geq 0$)

- Vacuum energy has $p = -\rho$ $\quad\Lambda$ gives $\ddot{a} > 0$

Not so exotic nowadays as we are now accelerating!

The complete solution is de Sitter space: $ds^2 = -dt^2 + e^{2Ht} d\vec{x}^2$

Maximally symmetric space with isometry $SO(4;1)$

- But we want to have de Sitter with a transition to a standard hot FRW cosmology: we want de Sitter with a clock!

- Notice that $\ddot{a} > 0$ is very different from $\dot{H} > 0$

$$\dot{H} = \frac{\ddot{a}}{a} - H^2$$

The energy density is always decreasing in an expanding Universe (or at most constant)

$$\dot{H} = -4\pi G(\rho + p)$$

I would need a violation of the null energy condition (NEC): $\quad T_{\mu\nu} n^\mu n^\nu \geq 0 \qquad \forall n^\mu \text{ null}$

(Bouncing models require $\dot H > 0$ at the bounce to flip sign, violation of the NEC)

How inflation addresses these problems

One has to check in an explicit model but we can discuss few qualitative points

- Event horizon:

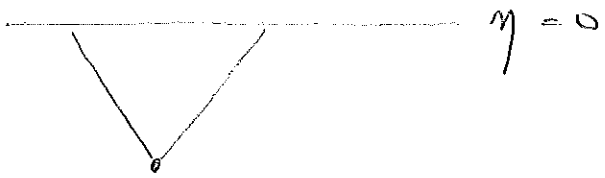
Now this integral converges in the future

$$a(t)\int \frac{da}{a\,\dot a(a)} = r_e(t)$$

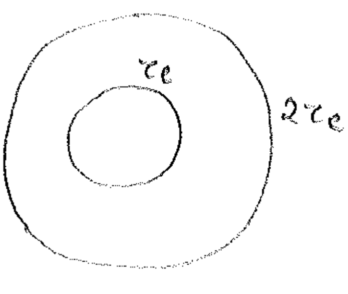
A particle can influence (if inflation lasts forever) only a limited portion of comoving coordinates

For example dS space in conformal coordinates reads

$$ds^2 = \frac{1}{H^2\eta^2}\left(-d\eta^2 + d\vec x^2\right) \qquad \eta \in (-\infty; 0)$$



This implies that if I start with a homogeneous region of radius $2r_e$, the internal ring cannot be reached by inhomogeneities outside



As $\frac{k}{aH} \to 0$ this region becomes very large compared to the Hubble radius

$\Longrightarrow$ Enough to have a good spot somewhere

- Gradients are stretched out of Hubble

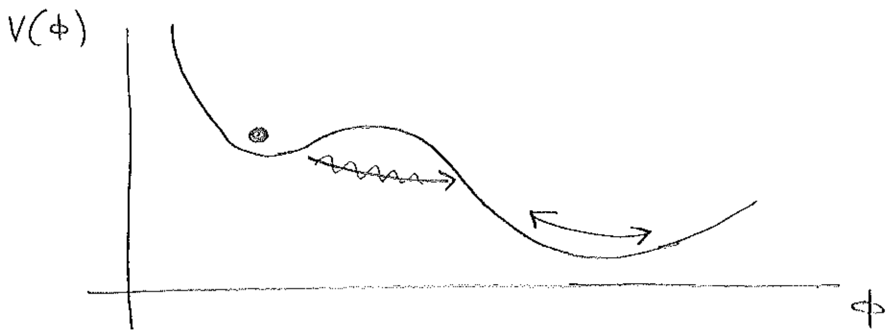
- For homogeneous (but anisotropic) cosmologies there are exact results in GR: anisotropies die off during inflation in all Bianchi models

Wald PRD 28 (1983) 2118

Kitada, Maeda PRD 45 (1992) 1416

"Cosmic no hair theorem" (?)

Old inflation   Original proposal by A. Guth '81



Metastable minimum : can I solve my problems and then tunnel?

The decay happens non-perturbatively through bubble nucleation

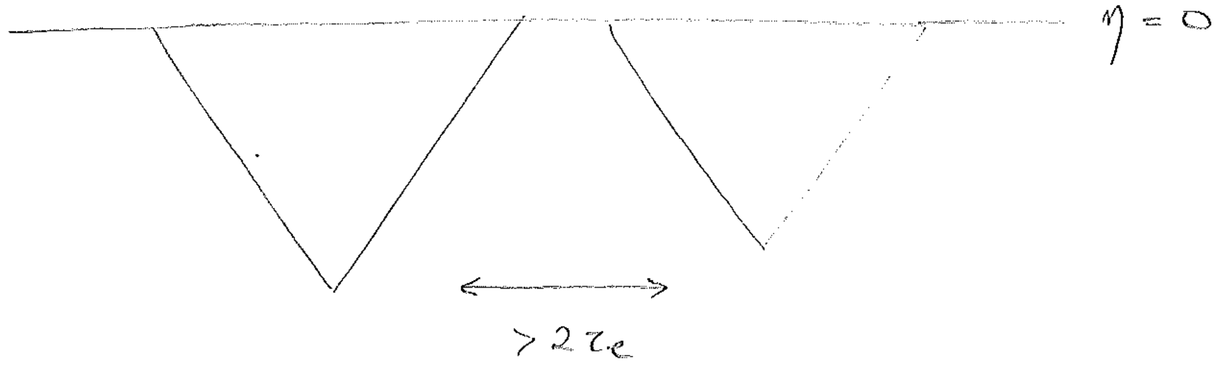
Γ is the probability per unit time and volume

Two different regimes :

$\Gamma \gg H^4$ Bubble percolates and you can reach the new phase in a time $\ll H^{-1}$. No inflation

$\Gamma \ll H^4$ You start inflating but the bubbles of the new phase cannot find each other . No new phase

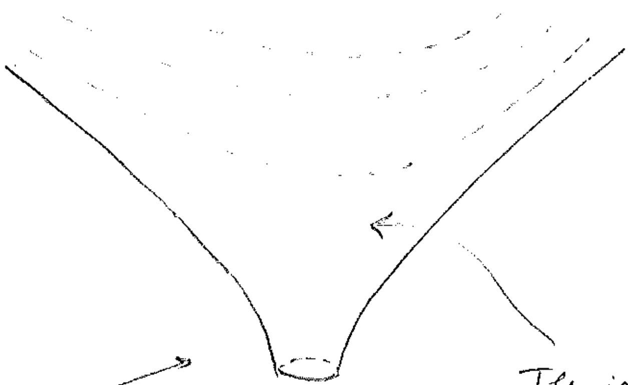
Not enough to wait as the Universe expands . Bubbles separated by > event horizon will never coalesce



But maybe I can live inside the bubble!

It is homogeneous, energy is given by scalar oscillations ...

(Notice the thin wall approximation where I end up in the minimum is misleading: usually I have residual energy in the bubble. Mukhanov's book?!)



The radius of the bubble is always $\lesssim H^{-1}_{\text{inflation}}$

The inside of the bubble can be obtained by analytic continuation of the instanton

It is an open Universe with spatial curvature equal to the one of the bubble: you are immediately curvature dominated!

<u>But</u> nothing prevents to have an episode like this in our past, followed by slow-roll inflation

Landscape ...

How to obtain "graceful exit"?

For old inflation dynamics: Guth, Weinberg NPB 212 (1983) 321
Coleman, de Luccia PRD 21,12 (1980) 3305

SLOW-ROLL INFLATION

Very profound idea:

Take the Lagrangian for a (minimally coupled scalar):

$$S = \int d^4x \sqrt{-g} \left[\frac{M_P^2}{2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right]$$

$$M_P^2 = (8\pi G)^{-1}$$

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = \partial_\mu \phi \partial_\nu \phi + g_{\mu\nu} \left[-\frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - V(\phi) \right]$$

$$\left(\delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g}\, g_{\mu\nu} \delta g^{\mu\nu} \right)$$

Homogeneous solution: $\phi(t)$

$$\rho = T_{00} = \frac{1}{2} \dot{\phi}^2 + V(\phi) \qquad (\rho + p) U_\mu U_\nu + p\, g_{\mu\nu} = T_{\mu\nu}$$

$$p = a^{-2} T_{ii} = \frac{1}{2} \dot{\phi}^2 - V(\phi)$$

We want to be close to dS. Kinetic energy $\ll$ potential energy

<u>Slow-roll</u>: $\frac{\dot{\phi}^2}{2} \ll V(\phi)$ (for many Hubble times)

Nothing changes if I consider

$\longrightarrow$ Non-minimal coupling : $f(\phi) R$

$\longrightarrow$ Coupling with matter : ψ_{SM} coupled to $g_{\mu\nu}\, h(\phi)$

$\longrightarrow$ Take $f(R)$ theory

See exercises

Besides Friedmann equations I have the EoM of scalar :

$$\ddot{\phi} + \underbrace{3H\dot{\phi}}_{\text{Hubble friction}} + V'(\phi) = 0$$

Friction : in the absence of a potential, the kinetic energy of the scalar red-shifts

$$\nabla^{\mu} T_{\mu\nu} = 0 \qquad \dot{\rho} + 3H(\rho + p) = 0$$

Kinetic : $\rho = p \qquad \rho \propto a^{-6}$

Indeed $\partial_t (a^3 \dot{\phi}) = 0 \qquad \frac{\dot{\phi}^2}{2} \propto a^{-6}$

$$\begin{cases} \ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \\ H^2 = \dfrac{1}{3M_p^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \end{cases}$$

I expect, for sufficiently small tilt, to get rid of excessive $\dot{\phi}$ and be dragged to $\sim$ CC solution

Simple example : $V = \frac{1}{2} m^2 \phi^2$ The correct one ?

$$\begin{cases} H^2 = \frac{1}{3M_P^2} \left(\frac{1}{2}\dot\phi^2 + \frac{1}{2} m^2 \phi^2 \right) \\ \ddot\phi + \frac{\sqrt{3/2}}{M_P} \left(\dot\phi^2 + m^2\phi^2 \right)^{1/2} \dot\phi + m^2 \phi = 0 \end{cases}$$

No explicit time dependence $\dot\phi(\phi)$ $\ddot\phi = \frac{d\dot\phi}{d\phi} \dot\phi$

$$\frac{d\dot\phi}{d\phi} = -\frac{1}{\dot\phi} \left[\sqrt{3/2}\, \frac{\dot\phi}{M_P} \left(\dot\phi^2 + m^2\phi^2 \right)^{1/2} + m^2 \phi \right]$$

Focus on $\phi \gg M_P$ (!)

- $\dot\phi^2 \gg m^2\phi^2$: kinetic domination. Opposite to what we are interested in

EOM reduces to $$\frac{d\dot\phi}{d\phi} = -\frac{\sqrt{3/2}}{M_P} |\dot\phi|$$

$$\dot\phi \propto e^{-\frac{\sqrt{3/2}}{M_P}\phi}$$

$\dot\phi$ redshifts exponentially in the direction of motion

Kinetic energy dies off much faster than exponential (exp vs power) and we leave kinetic domin.

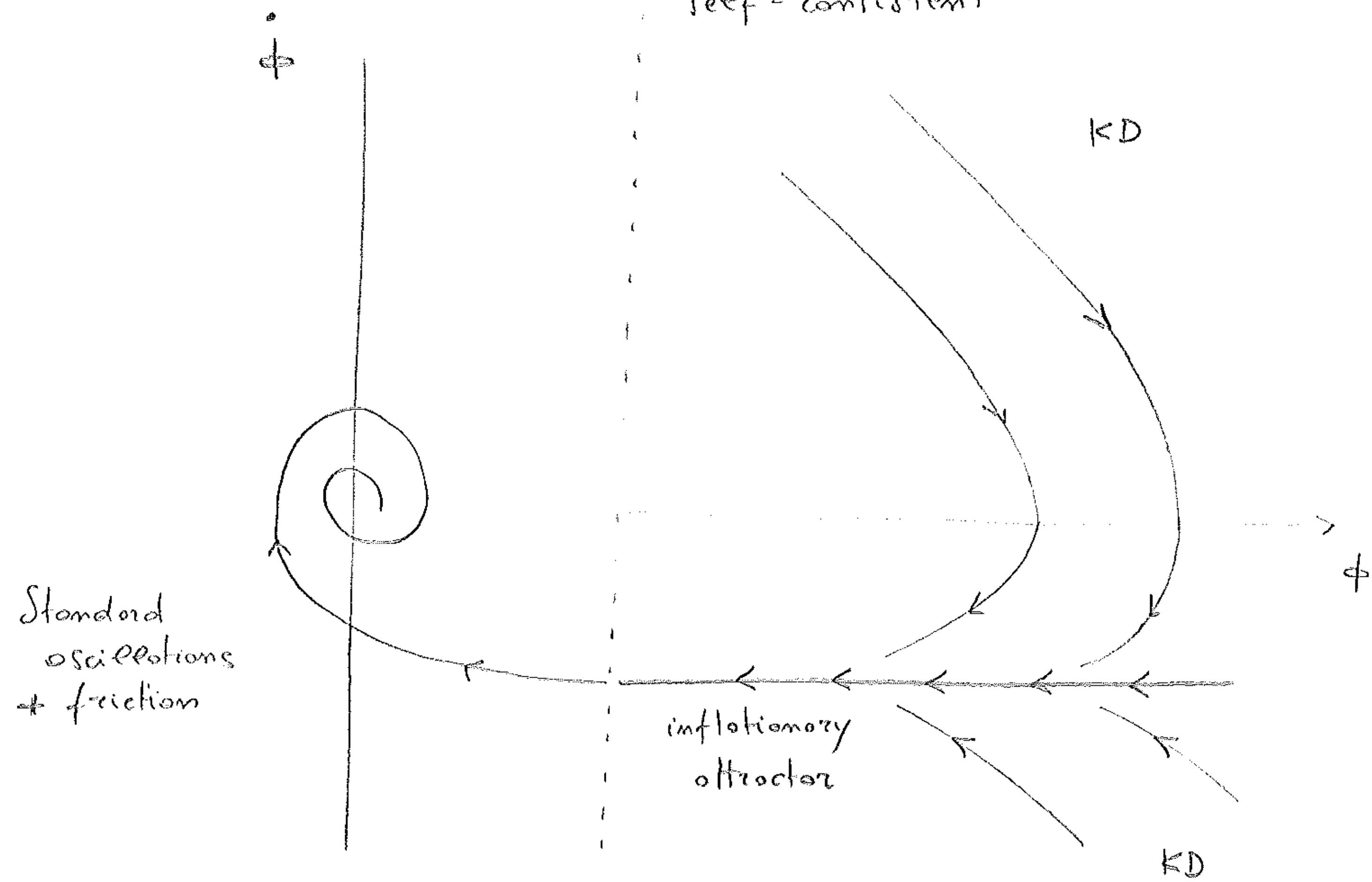
- Inflationary attractor : $\dot\phi^2 \ll m^2\phi^2$ $\phi \gg M_P$

Assume $\frac{d\dot\phi}{d\phi} = 0$

$$\frac{d\dot\phi}{d\phi} \propto \frac{\sqrt{3/2}}{M_P} \dot\phi\, m \phi + m^2 \phi \implies \dot\phi = -\sqrt{\frac{2}{3}}\, m M_P$$

During this phase : $\frac{\dot\phi^2}{2} \simeq m^2 M_P^2 \ll \frac{1}{2} m^2 \phi^2 = V$

self - consistent



$\phi \simeq M_P$
kinetic and potential
energy are similar and
inflation ends

$$\phi = \phi_f + \sqrt{\frac{2}{3}}\, m M_P \,|t - t_f|$$

As $H \simeq \frac{m\phi}{M_P}$ ϕ moves by M_P in an Hubble time. Thus $\phi_f \simeq M_P$ is soon negligible

$$\left(\frac{\dot a}{a}\right)^2 \simeq \frac{1}{3M_P^2}\,\frac{2}{3}\,\frac{1}{2}\, m^4 M_P^2\, |t-t_f|^2 = \frac{1}{9} m^4 |t - t_f|^2$$

$$a(t) = e^{\frac{m^2}{6}|t-t_f|^2}$$

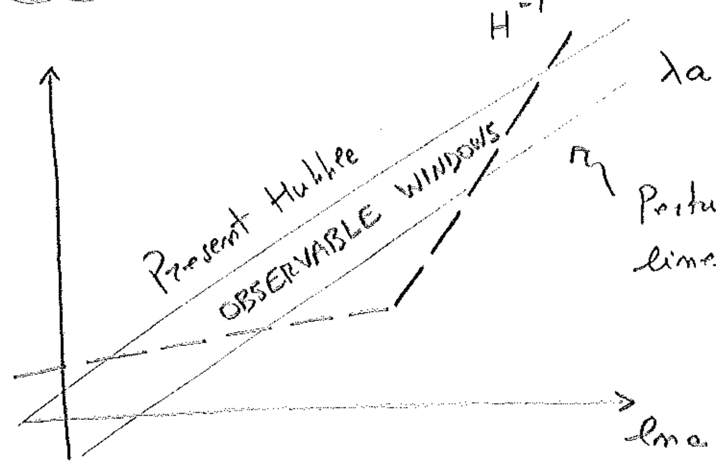
$$N = \frac{m^2}{6}|t-t_f|^2 \qquad \text{Number of e-folds}$$

- Usually we do not an explicit solution for $a(t)$, but I can approximate it as an exponential expansion

$$H(t) = \frac{m^2}{3}|t-t_f| \qquad \frac{\dot{H}}{H^2} \simeq \frac{1}{m^2|t-t_f|^2} \simeq \frac{1}{N}$$

N Hubble times before the end, H is ~ constant up to $\frac{1}{N}$ corrections

Kinematic and the observable window



Perturbations are very non-linear and difficult to study

We can roughly calculate the number of necessary e-folds comparing H_0^{-1} with the Hubble radius at the end of inflation

Rough: RD up to now + reheating at 10^{15} GeV

$$\frac{H_{end}^{-1}\, a_0/a_e}{H_0^{-1}} \sim \frac{\eta_e}{\eta_0} \sim \frac{a_e}{a_0} \sim \frac{T_0}{T_e} \sim \frac{10^{-4}\,eV}{10^{15}\,GeV} \simeq 10^{-28}$$

During inflation $H \simeq \text{const}$, while $a \propto e^{Ht}$

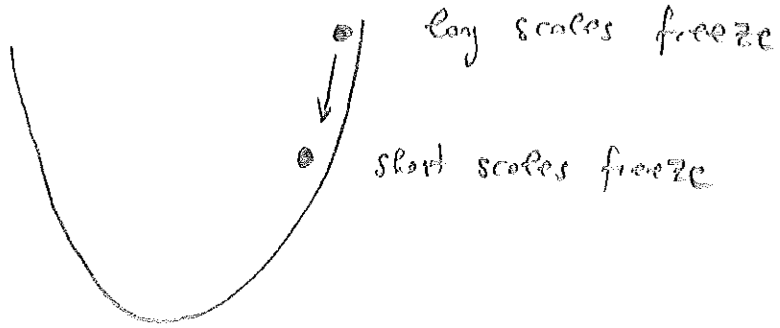
At least $N = \log 10^{28} \simeq 64$

$+$ window $\log\left(\dfrac{10^3\ \text{Mpc}}{0.1\ \text{Mpc}}\right) \simeq 9$

- Required # of e-folds depends logarithmically on the scale of inflation + reheating temperature + how I glue together inflation and RD

  Experimentally we are only confident $T_{RH} \gtrsim 1$ MeV for nucleosynthesis

- In any model:



- No bound on the total number of e-folds

Graceful exit + reheating

Now we clearly have a graceful (i.e. smooth) exit from inflation

In the quadratic case we can follow the solution after it leaves the attractor

$$\dot{\phi}^2 + m^2\phi^2 = 6 M_P^2 H^2$$

$$\dot{\phi} = \sqrt{6}\, M_P H \,\mathrm{sen}\,\theta \qquad m\phi = \sqrt{6}\, M_P H \cos\theta$$

$$\theta = \arctg \frac{\dot{\phi}}{m\phi} \qquad \dot{\theta} = \frac{1}{1 + \frac{\dot{\phi}^2}{(m\phi)^2}} \left[\frac{\ddot{\phi}}{m\phi} - \frac{\dot{\phi}^2}{m\phi^2} \right] =$$

$$= \frac{1}{1 + \left(\frac{\dot{\phi}}{m\phi}\right)^2} \left[\frac{-3H\dot{\phi} - m^2\phi}{m\phi} - \frac{\dot{\phi}^2}{m\phi^2} \right] = -m - 3H \frac{\dot{\phi}/m\phi}{1 + \left(\frac{\dot{\phi}}{m\phi}\right)^2}$$

$$= -m - \frac{3H}{2} \,\mathrm{sen}\, 2\theta$$

while $\quad H^2 = \frac{1}{3M_P^2} \frac{1}{2} \left(\dot{\phi}^2 + m^2\phi^2 \right)$

$$2H\dot{H} = \frac{1}{6M_P^2} \left(2\dot{\phi}\ddot{\phi} + 2m^2\phi\dot{\phi} \right) = \frac{\dot{\phi}}{3M_P^2} \left(-3H\dot{\phi} \right)$$

$$\begin{cases} \dot{H} = -3H^2 \,\mathrm{sen}^2\theta \\ \dot{\theta} = -m - \frac{3}{2} H \,\mathrm{sen}\, 2\theta \end{cases}$$

$H \simeq m$ at the end of inflation, but eventually $H \ll m$

$$\theta \simeq -mt$$

$$\dot{H} = -3H^2 \,\mathrm{sen}^2 mt$$

$$H(t) = \frac{2}{3t} \left(1 - \frac{\mathrm{sen}\, 2mt}{2mt} \right)^{-1} \qquad a(t) \propto t^{2/3}$$

for $mt \gg 1$
i.e. $H \ll m$

Indeed :

✓ Inflaton oscillations are just a coherent state of massive particles with zero velocity (homogeneous) : MD

✓ $\rho = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}m^2\phi^2$

$p = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}m^2\phi^2$ $\qquad \langle p \rangle = 0$ over many periods

$$\left(\text{Indeed the amplitude of oscillations } \propto \frac{1}{t} \propto a^{-3/2} \implies \rho \propto a^{-3} \right)$$

Reheating

- Oscillations around the bottom of the potential are a coherent state with density

$$n_\phi = \frac{\rho_\phi}{m} = \frac{1}{2m}\left(\dot{\phi}^2 + m^2\phi^2\right) \simeq \frac{1}{2} m \Phi^2$$

where Φ is the amplitude

(For $m = 10^{13}$ GeV, $\Phi \simeq M_P$ I get $10^{92}\,\text{cm}^{-3}$!)

- Perturbative decay of the inflaton with decay rate Γ (into what? what are the couplings?)

The decay completes when $H \simeq \Gamma$

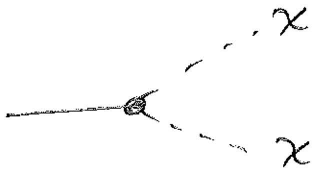
$$g_* T^4 \simeq H^2 M_P^2 \simeq \Gamma^2 M_P^2 \implies T_{RH} \simeq \sqrt{M_P \Gamma}\, g_*^{-1/4}$$

If the decay is very fast RD starts with $H_{RD}^{max} \simeq H_{inflation}$, otherwise I may have a long period of oscillation

Preheating

Coherent effects are usually important and the treatment above is not correct

$$\mathcal{L} \supset \left(\frac{m_\chi^2}{2} - g\phi \right) \chi^2$$



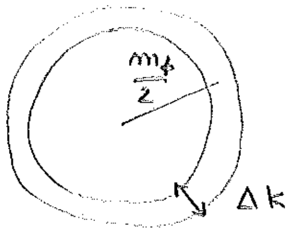
Example to show that the perturbative calculation can be wrong

$$\Gamma = \frac{g^2}{8\pi m_\phi}$$

$$\left(\frac{m_\phi}{2}\right)^2 = k^2 + \underbrace{m_\chi^2 - 2g\,\Phi \cos mt}_{\text{Assume } \ll m_\phi}$$

χ particles are produced in a narrow range of k and Bose-enhancement becomes relevant

$$\Delta k \simeq \frac{4g\Phi}{m}$$



Phase space density

$$\hat{n}_{k=\frac{m}{2}} \simeq \frac{n_\chi}{(4\pi k_0^2 \Delta k)/(2\pi)^3} = \frac{2\pi^2 n_\chi}{m g \Phi} = \frac{\pi^2 \Phi}{g} \frac{n_\chi}{m_\phi}$$

Bose condensation is relevant when

$$n_\chi > n_\phi \cdot \frac{g}{\pi^2 \Phi}$$

To have $m_\chi < m_\phi$

$$\frac{g\Phi}{\pi^2 \Phi^2} < \frac{m_\phi^2}{\pi^2 \Phi^2} \sim 10^{-12}$$

Coherent, non-perturbative production

(Preheating starts with: Linde, Kofman, Starobinsky
hep-th/9405187)

- Very <u>complicated</u> and model dependent

- In most cases <u>unobservable</u>

Predictions of inflation will be (non-linearly) insensitive to what happens when modes are out of H^{-1}

Insensitive to many interesting things: which particles?
Phase transitions?

- Everything originates from a very <u>out-of-equilibrium</u> process. Quite $\neq$ from standard hot FRW cosmology

Good place for <u>baryogenesis</u> for example

General potential: slow-roll approximation

For a general potential the analysis is more complicated, but everything simplifies in the slow-roll regime

$$\begin{cases} \ddot\phi + 3H\dot\phi + V'(\phi) = 0 \\ H^2 = \dfrac{1}{3M_p^2}\left(\dfrac{1}{2}\dot\phi^2 + V(\phi)\right) \end{cases} \xrightarrow[\text{SR}]{} \begin{cases} \dot\phi \simeq -\dfrac{V'}{3H} \\ H^2 \simeq \dfrac{V}{3M_p^2} \end{cases}$$

Slow-roll parameters

- $\frac{1}{2}\dot\phi^2 \ll V \qquad \dfrac{V'^2}{H^2} \ll V$

$$\epsilon = \frac{1}{2} M_p^2 \left(\frac{V'}{V}\right)^2 \ll 1 \qquad \text{Also} \quad \frac{\dot H}{H^2} = -\epsilon$$

Parametrizes the departure from de Sitter

$\dot H$: $\quad 2H\dot H = \dfrac{1}{3M_p^2}\left(\dfrac{1}{2}\dot\phi^2 + V\right)^{\cdot} = \dfrac{1}{3M_p^2}\left(\dot\phi\ddot\phi + V'\dot\phi\right) = -\dfrac{H\dot\phi^2}{M_p^2}$

$$\frac{\dot H}{H^2} \simeq -\frac{1}{2}\frac{\dot\phi^2 M_p^{-2}}{\frac{V}{3}M_p^{-2}} = -\frac{3}{2}\frac{\dot\phi^2}{V} = -\frac{\cancel{3}}{2}\frac{V'^2}{\cancel{3}3H^2}\frac{1}{V} = -\frac{1}{2}\frac{V'^2}{V^2}M_p^2 = -\epsilon$$

- $\ddot\phi \simeq \left(-\dfrac{V'}{3H}\right)^{\cdot} \underset{\substack{\text{up to}\\ \epsilon \text{ corrections}}}{\simeq} \dfrac{V''}{H}\dot\phi \simeq \dfrac{V''V'}{H^2} \ll V'$ in EOM for ϕ

$$\Rightarrow V'' \ll H^2$$

$$\eta \equiv M_p^2 \frac{V''}{V} \qquad (\text{notice} \gtrless 0)$$

$$\frac{\ddot{H}}{H\dot{H}} \underset{\uparrow}{=} \frac{1}{H}\frac{2\dot{\phi}\ddot{\phi}}{\dot{\phi}^2} = 2\epsilon - \frac{2}{3}\frac{V''}{H^2} = 2\epsilon - 2\eta$$

$$\dot{H} = -\frac{1}{2}\frac{\dot{\phi}^2}{M_P^2} \qquad \ddot{\phi} \simeq \left(-\frac{V'}{3H}\right)^{\cdot} = \frac{V''}{3H}\frac{V'}{3H} + \underbrace{\frac{V'}{3H^2}\dot{H}}_{\epsilon H \dot{\phi}}$$

So we are talking about the relative variation of H and $\dot{H}$ (which will be the only things appearing in EFTI)

- Slow-roll conditions are necessary, but not sufficient for inflation. I can always start with a large velocity... Anyway I expect there is an attractor as in the $m^2\phi^2$ case

- One can define higher-order slow-roll parameters, requiring $\epsilon, \eta \ll 1$ for many Hubble times

Not enough to have a flex point

E.g. $\delta\eta = M_P^2 \dfrac{\delta V''}{V}$ + terms $O(\epsilon, \eta)$

$$M_P^2 \frac{V''' \dot{\phi}/H}{V} = M_P^2 \frac{V''' V'}{V} H^{-2} = M_P^4 \frac{V' V'''}{V^2} \equiv \xi^2$$

and so on

Quartic: $\quad M_P^4 \dfrac{V' V^{(4)} \cdot V'/H^2}{V^2} \ll 1$

$$V^{(4)} \ll \frac{H^2}{M_P^2}\frac{1}{\epsilon} \sim 10^{-10}$$

Precisely this combination is fixed by the power spectrum

<u>Tremendously weakly coupled!</u>

Very rough "classification" of slow-roll models

- Nothing happens at the end of inflation

$$V \propto \phi^n \qquad \epsilon ; \eta \sim \left(\frac{M_P}{\phi}\right)^2$$

$$\epsilon \sim \eta \sim \frac{1}{N} \qquad \Delta\phi \gg M_P$$

$\left(\frac{1}{\sqrt{\epsilon}} \frac{H}{M_P} \right.$ is fixed by experiments to $\left. 10^{-5} \right)$

H is large, $\Delta\phi$ is super Planckian : high energy models

As ϵ is not very small GWs are observable

$\left(\frac{H}{M_P} \right.$ gives GWs contribution $\left. \right)$

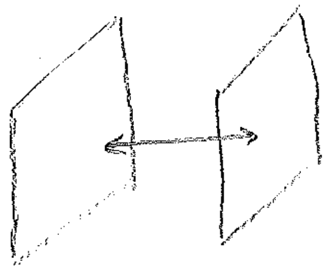
- Some kind of phase transition at the end of inflation

Hybrid models:

$$V = \frac{1}{2} m^2\phi^2 + \frac{1}{4}\lambda(\psi^2 - M^2)^2 + \frac{1}{2}\lambda'\psi^2\phi^2$$

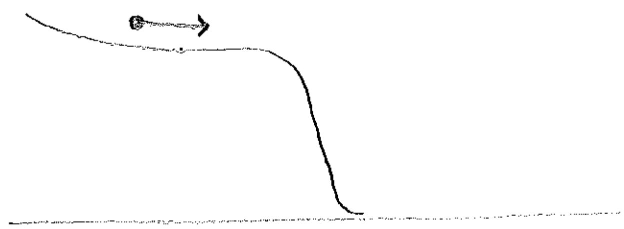
ϕ rolls and triggers some kind of phase transition

Brane inflation:



Inflaton is the interbrane separation
At the end they annihilate + create matter

Some kind of plateau in energy



$V_0 \qquad \epsilon \lll 1$

$$\Delta\phi \simeq \dot{\phi}/H \simeq V'/H^2 \simeq \frac{V'}{V} M_P^2 \simeq M_P\sqrt{\epsilon}$$

$$\epsilon \lll 1 \quad \Longrightarrow \quad \Delta\phi \ll M_P \quad \Longleftrightarrow \quad \text{No GWs}$$

$$\Longrightarrow \quad H \lll 10^{13}\ \text{GeV}$$

<u>Low-energy inflation</u>

Notice that it does not mean $|\eta| \lll 1$

$$\eta = \frac{V''}{V} M_P^2 \qquad \sqrt{\epsilon} = \frac{V'}{V} M_P$$

$$\frac{\sqrt{\epsilon}}{\eta} \simeq \frac{1}{M_P}\frac{V'}{V''} \ll 1$$ for sub Planckian models

Both these cases have been argued to be generic / extremely tuned

- $\Delta\phi \gg M_P$ is silly !?
- $\epsilon \lll 1$ is silly !?

The point is <u>vital for GW detection</u>!

Why is it so difficult to find an inflaton

1) Can we have $\Delta\phi > M_P$?

2) Can we have a sufficiently flat potential ?

- About (1) notice that when we say $\phi > M_P$ we still have $\rho \ll M_P^4$

E.g. $V = \frac{1}{2} m^2 \varphi^2$ I still have a large number of e-folds before losing control of GR

- About (2). Even for subPlanckian model you have to face UV physics

$$V = V_0(\varphi) + V_0(\varphi)\frac{\varphi^2}{M_P^2} + \dots$$

For superPlanckian models, this is not even an expansion ...

$\uparrow$

This generates $\eta \sim 1$. So I have to control M_P-suppressed operators.

UV-sensitivity of inflation

Many years in BSM physics taught us that there are two ways to keep a potential flat (like the mass of the Higgs)

- shift-symmetry : $\phi \longrightarrow \phi + c$
- Supersymmetry

Approximate shift symmetry (PNGB) Natural inflation

$$V = \Lambda^4 \cos(\phi/f)$$

Slow-roll requires $f \gg M_P$ to inflate

- Perturbatively there is no problem
- Non-perturbatively I do not expect gravity to respect any global symmetry (wormholes...)
  But you can use a gauge symmetry
- Not easy to realize in string theory (essentially I cannot make a coupling small as this also increases M_P).
  Recent axion monodromy McAllister, Silverstein, Westphal 0808.0706

Supersymmetry

which must be promoted to SUGRA

$$V_F = e^{K/M_P^2}\left[K^{\varphi\bar\varphi} D_\varphi W \overline{D_\varphi W} - \frac{3}{M_P^2}|W|^2\right]$$

$$D_\varphi W = \partial_\varphi W + M_P^{-2}(\partial_\varphi K) W$$

$$K^{\varphi\bar\varphi}\partial_\mu\varphi\partial_\mu\bar\varphi \qquad e^{K^{\varphi\bar\varphi}|\varphi|^2}\left[\quad V \quad\right]$$

$\rightsquigarrow \eta \sim 1$ $\qquad$ η-problem in SUGRA

$\hbar \neq 0$ — Harmonic oscillator + squeezing

(see e.g. Allen gr-qc/9604033)

For $\hbar = 0$ the Universe would be driven (if inflation lasts long enough) to a completely homogeneous + flat Universe

We want to claim: all the structure in our Universe is created by quantum fluctuations! WOW!

We do not need initial conditions, which in a classical theory will always be there. QM does the job: we are making predictions about initial conditions

It seems the first to think about this possibility was Dirac

A first example: as always an harmonic oscillator

$$\omega_i \longrightarrow \omega_f$$
$$\sqrt{\frac{g}{l_i}} \longrightarrow \sqrt{\frac{g}{l_f}}$$

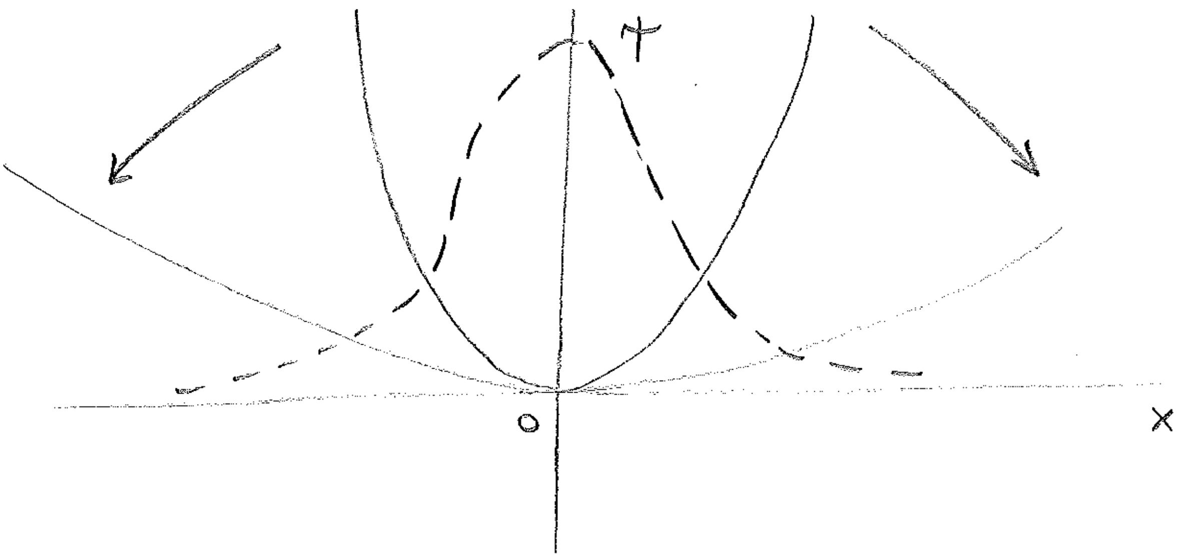
- Classically nothing happens (if I start from the vacuum)
- QM. Two possible behaviours

$\longrightarrow$ $T \gg \omega_i^{-1}; \omega_f^{-1}$ adiabatic limit

The system follows the vacuum at every step and you end up in the final vacuum at $t = t_f$

$\longrightarrow$ $T \ll \omega_i^{-1}; \omega_f^{-1}$ $\left(\frac{\dot{l}}{l} \gg \omega \text{ at every moment} \right)$

Diabotic limit



The potential varies but there is no time to adjust so ψ remains in the old vacuum, which is an excited state of the new harmonic oscillator

$$-\frac{\hbar^2}{2m}\nabla^2\psi + \frac{1}{2}m\omega^2 x^2\psi = E_n\psi$$

$$-\frac{\hbar}{2m\omega}\nabla^2\psi + \frac{1}{2}\frac{m\omega}{\hbar}x^2\psi = \frac{E_n}{\hbar\omega}\psi$$

The equation is just a function of $\sqrt{\alpha}x$

$$\psi_n(x) = N_n H_n(\sqrt{\alpha}\,x)\, e^{-\frac{\alpha}{2}x^2} \qquad \alpha = \frac{m\omega}{\hbar}$$

Old vacuum: $\psi(x) = \left(\frac{\alpha_i}{\pi}\right)^{1/4} e^{-\frac{\alpha_i}{2}x^2}$

Decomposed wrt new eigenstates: $\psi(x) = \sum c_n \psi_n(x)$

$c_n = 0 \qquad n$ odd

$$c_{2n} = (\alpha_i\alpha_f)^{1/4}\sqrt{\frac{2(2n)!}{\alpha_i+\alpha_f}}\,\frac{1}{n!}\left[\frac{\omega_f-\omega_i}{2(\omega_f+\omega_i)}\right]^n$$

$$E_f = \sum_{n=0}^{\infty}\left(2n+\frac{1}{2}\right)\hbar\omega_f\,|c_{2n}|^2 = \hbar\omega_f\left(\frac{1}{2}+\frac{(\omega_f-\omega_i)^2}{4\omega_f\omega_i}\right)$$

For $\omega_f \ll \omega_i \qquad E_f = \frac{1}{4}\hbar\omega_i \qquad N_f = \frac{1}{4}\frac{\omega_i}{\omega_f} \gg 1$

$\left(\hat{a}_{NEW} \text{ and } \hat{a}^{\dagger}_{NEW} \text{ satisfy } [\hat{a}; \hat{a}^{\dagger}] = 1 \quad \text{check!} \right)$

$$|\xi\rangle = S(\xi)|0\rangle$$ is a <u>squeezed state</u>

(Not a coherent state!)

In particular the old vacuum is a squeezed state.
$|0\rangle_{old}$ must be annihilated by $\hat{a}_{old}$, a linear combination of $\hat{a}_{new}$ and $\hat{a}^{\dagger}_{new}$: $S\hat{a}_{NEW}S^{\dagger}$

$$S\hat{a}_{NEW}S^{\dagger}|0_{old}\rangle = 0 \qquad S\hat{a}_{NEW}S^{\dagger}\underbrace{S|0_{NEW}\rangle}_{|0_{old}\rangle}$$

What are the features of this new state?

The <u>uncertainty is squeezed in one direction</u>

$$Y_1 + iY_2 \equiv (X_1 + iX_2)e^{-i\theta/2} \qquad [Y_1; Y_2] = i$$

$$S^{\dagger}(Y_1 + iY_2)S = Y_1 e^{-\tau} + iY_2 e^{\tau}$$

$$\langle(\Delta Y_1)^2\rangle = \frac{1}{2}e^{-2\tau} \qquad \langle(\Delta Y_2)^2\rangle = \frac{1}{2}e^{2\tau}$$

$$\langle \Delta Y_1 \Delta Y_2 \rangle = \frac{1}{2}$$

Expectation values in the squeezed state can be calculated as

$$\langle 0|S^{\dagger} \text{ Operator } S|0\rangle$$

The uncertainty is squeezed in one direction and enlarged in the other

You end up in a very excited state

$E_f = \frac{1}{4}\hbar\omega_i$ — same ψ with potential energy → 0

Energy is equally split potential / kinetic so I have $\frac{1}{2}$ of the initial energy

Usual definition of $\hat{a}$ and $\hat{a}^+$

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\,\hat{x} + i\,\frac{1}{\sqrt{m\omega\hbar}}\,\hat{p}\right) \equiv (\hat{x}_1 + i\hat{x}_2)/\sqrt{2}$$

$$\hat{a}^+ = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\,\hat{x} - i\,\frac{1}{\sqrt{m\omega\hbar}}\,\hat{p}\right) \equiv (\hat{x}_1 - i\hat{x}_2)/\sqrt{2}$$

If I change ω, I change the linear combination that defines $\hat{a}$ and $\hat{a}^+$. In particular the old vacuum is annihilated by $\hat{a}_{old}$, a linear combination of $\hat{a}_{NEW}$ and $\hat{a}^+_{NEW}$

I am interested in a transformation mixing $\hat{a}$ and $\hat{a}^+$, such that the commutator $[\hat{a};\hat{a}^+] = 1$ remains the same

Squeeze operator:
$$S(\xi) = e^{\frac{1}{2}\xi^*\hat{a}^2 - \frac{1}{2}\xi\hat{a}^{+2}}$$
$$\xi = \tau e^{i\theta}$$

$$S^+(\xi) = S^{-1}(\xi) = S(-\xi)$$

Using
$$e^A B e^{-A} = B + [A;B] + \frac{1}{2}[A[AB]] + \dots$$

$$S^+(\xi)\,\hat{a}\,S(\xi) = \hat{a}\,\mathrm{ch}\,\tau - \hat{a}^+ e^{i\theta}\,\mathrm{sh}\,\tau$$
$$S^+(\xi)\,\hat{a}^+ S(\xi) = \hat{a}^+\mathrm{ch}\,\tau - \hat{a}\,e^{-i\theta}\,\mathrm{sh}\,\tau$$

Bogolubov transformation

check that Y_1 and Y_2 are squeezed

$$X_1 = \frac{1}{\sqrt{2}}\left(\hat{a} + \hat{a}^{+}\right)$$

$$X_2 = \frac{1}{\sqrt{2}}\,\frac{1}{i}\left(\hat{a} - \hat{a}^{+}\right)$$

$$Y_1 = \sqrt{2}\,\hat{a}\,e^{-i\theta/2} + \text{h.c.}$$

$$\xrightarrow{S} \sqrt{2}\left[\hat{a}\,\mathrm{ch}\,r\,e^{-i\theta/2} - \hat{a}^{+}\,\mathrm{sh}\,r\,e^{i\theta/2} + \hat{a}^{+}\,\mathrm{ch}\,r\,e^{i\theta/2} - \hat{a}\,\mathrm{sh}\,r\,e^{-i\theta/2}\right]$$

$$= e^{-r}\sqrt{2}\left[\hat{a}\,e^{-i\theta/2} + \text{h.c.}\right]$$

Time evolution of the squeezing parameter

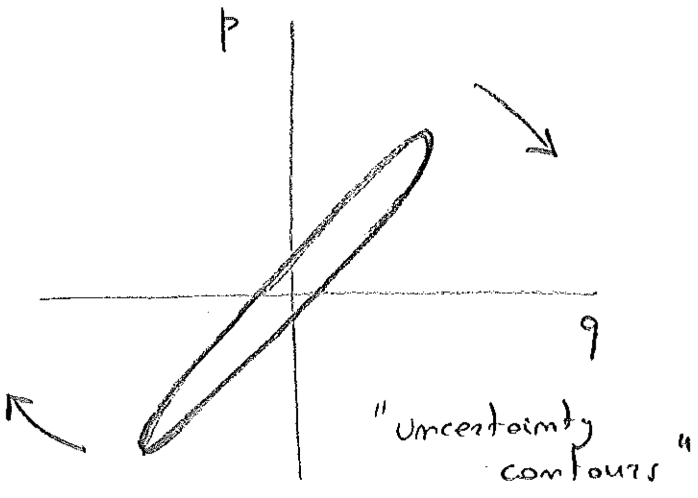
$$e^{-iHt}\,S|0\rangle \qquad\qquad e^{-iHt}\,S\,e^{+iHt}$$

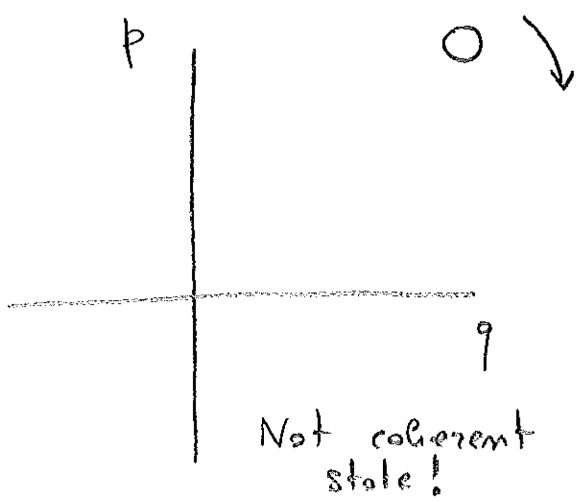
Writing S as an exponential series, I simply get the evolution of each of the $\hat{a}$ and $\hat{a}^{+}$

$$S = e^{\frac{1}{2}\xi^{*}\hat{a}^{2} - \frac{1}{2}\xi\,\hat{a}^{+2}} \qquad\qquad \xi = \xi_0\,e^{-2i\omega t}$$

The state remains squeezed, but the uncertainty rotates clockwise as in the classical evolution

(Notice the time evolution would have e^{-iHt} on the right, but we are interested in the evolution of $S|0\rangle$ ...)





Decoherence without decoherence :

in the limit in which I can forget about the decaying mode I have a classical stochastic variable, I can assign a probability to its amplitude

It is not a mathematic property of squeezed states (I can always define observables such that the state is fully quantum) but a reasonable assumption about decoherence :

any measurement will be sensitive only to a coarse grained description of the modes in which the decaying mode can be neglected

Massless scalar in an expanding Universe

We want to see that each Fourier mode behaves quite similarly to the example of a single HO

$$S = \int d^4x \sqrt{-g}\left[-\frac{1}{2} g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi\right] =$$

$$= \int d^4x\, a^2\left[\frac{1}{2}(\partial_\eta\phi)^2 - \frac{1}{2}(\partial_i\phi)^2\right]$$

To make the Lagrangian more similar to the case of an HO where only the frequency is changing I define

$$y = a\phi$$

(Polarski, Starobinsky gr-qc/9504030)

$$S = \int d^4x \;\frac{1}{2}\left(\partial_\eta y - \frac{a'}{a}y\right)^2 - \frac{1}{2}(\partial_i y)^2 =$$

$$= \int d\eta \int \frac{d^3k}{(2\pi)^3}\;\frac{1}{2}\left[y'(k) - \frac{a'}{a}y(k)\right]\left[y'(-k) - \frac{a'}{a}y(-k)\right] - \frac{k^2}{2}y(k)y(-k)$$

$$p(k) = \frac{\partial\mathcal{L}}{\partial y'(k)} = y'(k)^\dagger - \frac{a'}{a}y(k)^\dagger \qquad \text{The field is real...}$$

$$\mathcal{H} = \int_{\text{Half space}} \frac{d^3k}{(2\pi)^3}\;\frac{1}{2}p(k)p(k)^\dagger + \frac{1}{2}k^2 y(k)y^\dagger(k) + \frac{1}{2}\frac{a'}{a}\left[y(k)p^\dagger(k) + p(k)y^\dagger(k)\right]$$

$$\hat{b}(k) = \frac{1}{\sqrt{2}}\left(\sqrt{k}\,y(k) + \frac{i}{\sqrt{k}}p(k)\right)$$

$$y(\vec{k}) = \frac{\hat{b}(k) + \hat{b}^\dagger(-k)}{\sqrt{2k}} \qquad p(\vec{k}) = -i\sqrt{\frac{k}{2}}\left(\hat{b}(k) - \hat{b}^\dagger(-k)\right)$$

(Notice I want $\hat{b}_k e^{i\vec{k}\vec{x} - i\omega t}$ and $\hat{b}_k^\dagger e^{-i\vec{k}\vec{x} + i\omega t}$, so actually I have some $-\vec{k}$ in $\hat{b}^\dagger$)

$$\mathcal{H} = \int \frac{d^3k}{(2\pi)^3} \frac{k}{2}\left[\hat{b}(k)\,\hat{b}^+(k) + \hat{b}^+(k)\,\hat{b}(k)\right]$$
$$+ i\,\frac{a'}{a}\left[\hat{b}^+(k)\,\hat{b}^+(-k) - \hat{b}(k)\,\hat{b}(-k)\right]$$

$$\begin{pmatrix} \hat{b}(k) \\ \hat{b}^+(-k) \end{pmatrix}' = \begin{pmatrix} -ik & \frac{a'}{a} \\ \frac{a'}{a} & ik \end{pmatrix} \begin{pmatrix} \hat{b}(k) \\ \hat{b}^+(-k) \end{pmatrix}$$

- The evolution mixes $\hat{b}$ and $\hat{b}^+$, you'll end up in a squeezed state

- Adiabatic / diabatic = inside / outside H^{-1}

$$k \gtrless \frac{a'}{a} = aH$$

- In the limit of small k we have

$$\begin{pmatrix} \ \end{pmatrix}' = \frac{a'}{a}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \ \end{pmatrix}$$

$$y(\vec{k}) \propto a \qquad\qquad \phi = \text{const}$$
$$p(\vec{k}) \propto a^{-1}$$

Staying constant for ϕ describes the squeezing

- The squeezing ratio is the ratio of the scale factors between when the mode goes out and comes back in H^{-1}; more than 60 e-folds

- Each mode behaves as a classical stochastic variable at large squeezing

2-point function in de Sitter

$$a = a_0 e^{Ht} \qquad \eta = \int \frac{dt}{a_0 e^{Ht}} = -\frac{1}{a_0 H} e^{-Ht} = -\frac{1}{aH}$$

$$\eta \in (-\infty; 0)$$

$$S = \int d^4x \, \frac{1}{2H^2\eta^2} \left[(\partial_\eta \phi)^2 - (\partial_i \phi)^2 \right]$$

$$\phi(\eta; \vec{x}) = \int \frac{d^3k}{(2\pi)^3} \phi_k(\eta) e^{i\vec{k}\cdot\vec{x}}$$

EOM: $\partial_\eta \left[\eta^{-2} \partial_\eta \phi \right] + \eta^{-2} k^2 \phi = 0$

$$\boxed{\phi_k^{cl} = \frac{H}{\sqrt{2k^3}} (1 - ik\eta) e^{ik\eta}}$$

$$\phi_k(\eta) = \phi_k^{cl}(\eta)\, \hat{a}_k^+ + \phi_k^{cl}(\eta)^* \, \hat{a}_{-k}$$

- ϕ_k^{cl} is a solution of the EOM

$$\partial_\eta \left[\eta^{-2} \left((\cancel{1} - ik\eta) ik - \cancel{ik} \right) e^{ik\eta} \right] + k^2 \eta^{-2} (1 - ik\eta) e^{ik\eta} = 0$$

- Minkowski normalization

$$\phi_k^{cl} \xrightarrow[k \to +\infty]{} \frac{H}{\sqrt{2k^3}} (-ik\eta) e^{ik\eta_0} e^{ik/a \, \Delta t}$$

$$= \frac{1}{\sqrt{2k/a}} a^{-3/2} e^{ik/a \cdot \Delta t}$$

The factor of $a^{-3/2}$ comes from the different normalization of $\hat{a}$ and $\hat{a}^+$ in comoving coordinates, and the integral over k comoving

$$\left[\hat{a}_k ; \hat{a}_{k'}^+ \right] = (2\pi)^3 \delta(\vec{k} + \vec{k}')$$

$$\left[\hat{a}_p ; \hat{a}_{p'}^+ \right] = (2\pi)^3 \delta(\vec{p} + \vec{p}') \qquad \hat{a}_p = a^{3/2} \hat{a}_k$$

$$\int d^3p \ \hat{a}_p = \int d^3k \ a^{-3} \ \hat{a}_k \ a^{3/2} \quad \checkmark$$

$$\langle \phi_{\vec{k}}(\eta) \ \phi_{\vec{k}'}(\eta) \rangle = (2\pi)^3 \delta(\vec{k}+\vec{k}') \ |\phi_k^{cl}|^2 =$$

$$= (2\pi)^3 \delta(\vec{k}+\vec{k}') \frac{H^2}{2k^3} (1 + k^2\eta^2) \simeq (2\pi)^3 \delta(\vec{k}+\vec{k}') \frac{H^2}{2k^3}$$

large scales

In Minkowski I would get

$$\langle \phi_{\vec{p}}(t) \ \phi_{\vec{p}'}(t) \rangle = (2\pi)^3 \delta(\vec{p}+\vec{p}') \frac{1}{2p}$$

$$\boxed{\langle \phi_k(\eta) \ \phi_{\vec{k}'}(\eta) \rangle \cong (2\pi)^3 \delta(\vec{k}+\vec{k}') \frac{H^2}{2k^3}}$$

Celebrated scale invariant result

In real space

$$\langle \phi(\vec{x};t) \ \phi(\vec{y};t) \rangle \simeq - \frac{H^2}{(2\pi)^2} \log \frac{|x-x'|}{L}$$

$$ds^2 = \frac{1}{H^2\eta^2} \left(-d\eta^2 + d\vec{x}^2\right) \qquad \eta \longrightarrow \lambda\eta \quad \vec{x} \longrightarrow \lambda\vec{x}$$

$$\varphi_{\vec{k}} \longrightarrow \lambda^3 \varphi_{\vec{k}/\lambda} \qquad \langle \varphi_{\vec{k}_1}, \varphi_{\vec{k}_2} \rangle = (2\pi)^3 \delta(\vec{k}_1+\vec{k}_2) \frac{1}{k_1^3} F(k\eta)$$

As fields become time-independent I get the k^{-3} behaviour

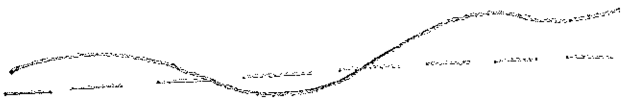
Independent of the Lagrangian!

Getting the results, we will justify them later

We are going through the full construction later as it connects with NG. Now we are sloppy

You have to buy two things

Spatially flat gauge: $g_{ij} = a^2(t)\,\delta_{ij}$ (assume no GWs)

φ  $t = \text{const}$

$$\phi(t;\vec{x}) = \phi_0(t) + \varphi(t;\vec{x})$$

Buy 1 The quadratic action for φ is the one of a massless scalar in dS up to slow-roll corrections

(Indeed the mass is slow-roll suppressed, dS up to ϵ and when I solve the constraints I will get $\epsilon; \eta$ suppression)

I am interested in the metric on constant inflaton surfaces as these will become constant T surfaces in RD, while φ won't have any meaning

$$\delta t = -\frac{\varphi(t;\vec{x})}{\dot{\phi}}$$

Now this is the $t = \text{const}$ surface

The spatial metric in the new coordinates will be

$$a^2(t+\delta t)\,\delta_{ij} = a^2(t)\underbrace{\left[1 - 2\frac{H\varphi}{\dot{\phi}}\right]}_{1+2\zeta(\vec{x};t)}\delta_{ij}$$

Constant inflaton surfaces (and later on T = const surfaces) are not flat anymore. Non-flat induced metric

$$^{(3)}R = -\frac{4}{a^2}\nabla^2 \zeta$$

Buy 2 ζ is constant outside H^{-1} (up to corrections $\frac{k^2}{a^2H^2}$)

This result will be true non-linearly and it is valid without slow-roll approximation

I am interested in the spectrum of ζ when it is comfortably out of H^{-1}

$$\langle \zeta_{\vec{k}}(\eta)\, \zeta_{\vec{k}'}(\eta') \rangle = (2\pi)^3 \delta(\vec{k}+\vec{k}') \underbrace{\frac{H^2}{2k^3}}_{\text{free scalar in dS}} \underbrace{\frac{H^2}{\dot{\phi}^2}}_{\text{to go to } \zeta \text{ variable}} \Bigg|_{\text{horizon crossing}}$$

This quantity is quite close to observations. E.g. in the large scale limit (Sachs-Wolfe)

$$\frac{\delta T}{T} = -\frac{1}{5}\zeta$$

Normalization of the spectrum

Experimental results are given in terms of $\frac{k^3 P_\zeta}{2\pi^2}$

where $\langle \zeta \zeta \rangle = (2\pi)^3 \delta(\ \ ) P_\zeta$

(This describes the contribution to the variance per logarithmic k-interval

$$\int \frac{d^3k}{(2\pi)^3} P_\zeta = \int \frac{dk}{k} \frac{P_\zeta(k)\, k^3}{2\pi^2}$$
)

$$\frac{k^3 P_\zeta}{2\pi^2} = (2.4 \pm 0.1)\ 10^{-9}$$

WMAP 7 + others

(the old COBE normaliz)

Typical amplitude of $\left(\frac{\delta T}{T}\right)^2$ in the CMB

Roughly: $\langle \zeta \zeta \rangle \simeq \frac{H^4}{\dot\phi^2}$

Amplitude $\sim \frac{H^2}{\dot\phi}$ Ratio between quantum perturbations (H) and classical motion in one Hubble time $\dot\phi / H$

It can also be written as: $\sim \frac{H}{M_P} \frac{1}{\sqrt{\epsilon}}$

(What ~~not to do~~ to estimate the spectrum

$$\frac{\delta\rho}{\rho} \simeq \frac{V'}{V} \delta\phi \simeq \frac{V'}{V} H \simeq \sqrt{\epsilon} \frac{H}{M_P}$$ WRONG!

Think of the inflaton as a clock, not something that carries energy
)

E.g. $\quad V = \frac{1}{2} m^2 \phi^2$

$$\epsilon = \frac{1}{2} M_p^2 \left(\frac{V'}{V}\right)^2 \qquad \eta = M_p^2 \frac{V''}{V}$$

$$\phi_{end} \simeq M_p$$

$$d\phi \simeq -\frac{V'}{3H} dt = -\frac{V'}{3H^2} dN = -\frac{V'}{V} M_p^2 dN$$

$$N(\phi) = M_p^{-2} \int_{\phi_{end}}^{\phi} \frac{V}{V'} d\phi \underset{V = \frac{1}{2} m^2 \phi^2}{=} M_p^{-2} \int \frac{\phi}{2} d\phi = M_p^{-2} \left[\frac{\phi^2}{4} - \frac{\phi_{end}^2}{4}\right]$$

$\phi \simeq 2 M_p \sqrt{N} \qquad \phi_{end}$ is irrelevant

$$\frac{H^4}{(2\pi)^2 \dot{\phi}^2} : \quad \frac{H^2}{2\pi \dot{\phi}} \simeq \frac{\frac{1}{3M_p^2} \frac{1}{2} m^2 \phi^2}{2\pi \sqrt{\frac{2}{3}} m M_p} = \frac{1}{4\pi\sqrt{6}} \frac{m M_p^2 4N}{M_p^3}$$

$$= \frac{1}{\pi\sqrt{6}} \frac{m}{M_p} N \implies m \simeq 1.3 \; 10^{13} \text{ GeV}$$

Scale dependence: As H and $\dot\phi$ are not exactly constant I get a small deviation from scale invariance

$$k^{-3+(n_s-1)}$$

$$n_s - 1 = \frac{d}{d\log k} \log \frac{H^4}{\dot\phi^2}\Big|_{\text{crossing}} = H^{-1}\frac{d}{dt}\log\frac{H^6}{V'^2}$$

$k \sim aH \qquad a \sim e^{Ht}$

$d\log k = H dt$

$$= \frac{6\dot H}{H^2} - 2\frac{V''}{V'}\underbrace{\frac{\dot\phi}{H}}_{-\frac{V'}{3H^2}} = -6\epsilon + 2\eta$$

Red spectrum $n_s - 1 < 0$ More power at low k

Blue spectrum $n_s - 1 > 0$ " at high k

E.g. $V = \frac{1}{2}m^2\phi^2 \qquad \epsilon = \eta = \frac{1}{2N}$

$$n_s - 1 = -6\epsilon + 2\eta = -8\frac{M_P^2}{\phi^2} = -\frac{2}{N}$$

Indeed we saw $P_\zeta \propto N^2$

$$n_s - 1 = -\frac{d}{dN}\log N^2 = -\frac{1}{N}\frac{d}{d\log N}\log N^2 = -\frac{2}{N}$$

$\simeq 0.97$ with $N = 60$

WMAP 7: $n_s = 0.963 \pm 0.012$

Reheating enters in the predictions

A <u>non-trivial prediction of inflation</u>: we are in dS, but only approximately!

Typically (?): $|n_s - 1| \sim 1/N$

We may object: we did the calculation at leading order in slow-roll and we got the tilt that is slow-roll suppressed. Did I lose some piece?

$$\langle \zeta \zeta \rangle \sim \frac{H^4}{\dot{\phi}^2}\left(1 + O(\epsilon, \eta) + O\left(\frac{k^2}{a^2H^2}\right)\right)$$

slow-roll corrections in the action that I neglected

ζ becomes constant out of H^{-1}

These corrections will also be slowly varying with k as $H^4/\dot{\phi}^2$ does. So they will give small corrections to the spectrum and to the tilt

(Alternatively I could switch to ζ for all the modes at the end of inflation. But in this case φ does evolve out of H^{-1}:

slow-roll evolution $\times$ N e-folds can be large

I should take into account its mass, the massless de Sitter approx. is not enough)

Gravitational waves

Any light field γ ($m \ll H$) gets excited during inflation.
Tensor modes behave quite similarly to a scalar field

$$S_{(2)} = \frac{M_P^2}{8} \int d\eta \, d^3x \; a^2 \left[\gamma_{ij}'^2 - (\partial_e \gamma_{ij})^2 \right]$$

$$\gamma_{ij} = \int \frac{d^3k}{(2\pi)^3} \sum_{s=\pm} \epsilon_{ij}^s(k) \, \gamma_{\vec{k}}^s(\eta) \, e^{i\vec{k}\vec{x}}$$

Polarization tensors are transverse, traceless $\epsilon_{ii} = k^i \epsilon_{ij} = 0$
and normalized $\epsilon_{ij}^s(k) \, \epsilon_{ij}^{s'} = 2\delta_{ss'}$

$$\langle \gamma_{\vec{k}}^s \, \gamma_{\vec{k}'}^{s'} \rangle = (2\pi)^3 \delta(\vec{k} + \vec{k}') \frac{1}{2k^3} \frac{2H^2}{M_P^2} \delta_{ss'}$$

Notice a factor 2 comes from the normaliz. of ϵ_{ij}

$$\frac{k^3 P_T}{2\pi^2} = 4 \cdot \frac{1}{2\pi^2} \frac{H^2}{M_P^2} = \frac{2}{\pi^2} \frac{H^2}{M_P^2}$$

2 polarizations, each normalized to 2

Notice we have no $\frac{1}{\epsilon}$ enhancement

$$r \equiv \frac{P_T}{P_\zeta} = \frac{8H^2/M_P^2}{H^4/\dot{\phi}^2} = -16 \frac{\dot{H}}{H^2} = 16\epsilon$$

E.g. $\quad V = \frac{1}{2} m^2 \phi^2 \quad \Rightarrow \quad r = 16\epsilon = \frac{8}{N} \simeq 0.13$

$$V^{1/4} \simeq \left(\frac{r}{0.01} \right)^{1/4} 10^{16} \, \text{GeV}$$

Gravitational waves are very robust: they just probe H during inflation. They do not care about one field, two fields ...

A real probe of inflation!

WMAP 7: $r < 0.24 \quad 2\sigma$

Lyth's bound

$$r = \frac{8}{M_P^2}\frac{\dot{\phi}^2}{H^2} = \frac{8}{M_P^2}\left(\frac{d\phi}{dN}\right)^2$$

$$\frac{\Delta\phi}{M_P} = \int_{N_{end}}^{N_{CMB}} dN \sqrt{\frac{r}{8}} \implies \frac{\Delta\phi}{M_P} = O(1)\left(\frac{r}{0.01}\right)^{1/2}$$

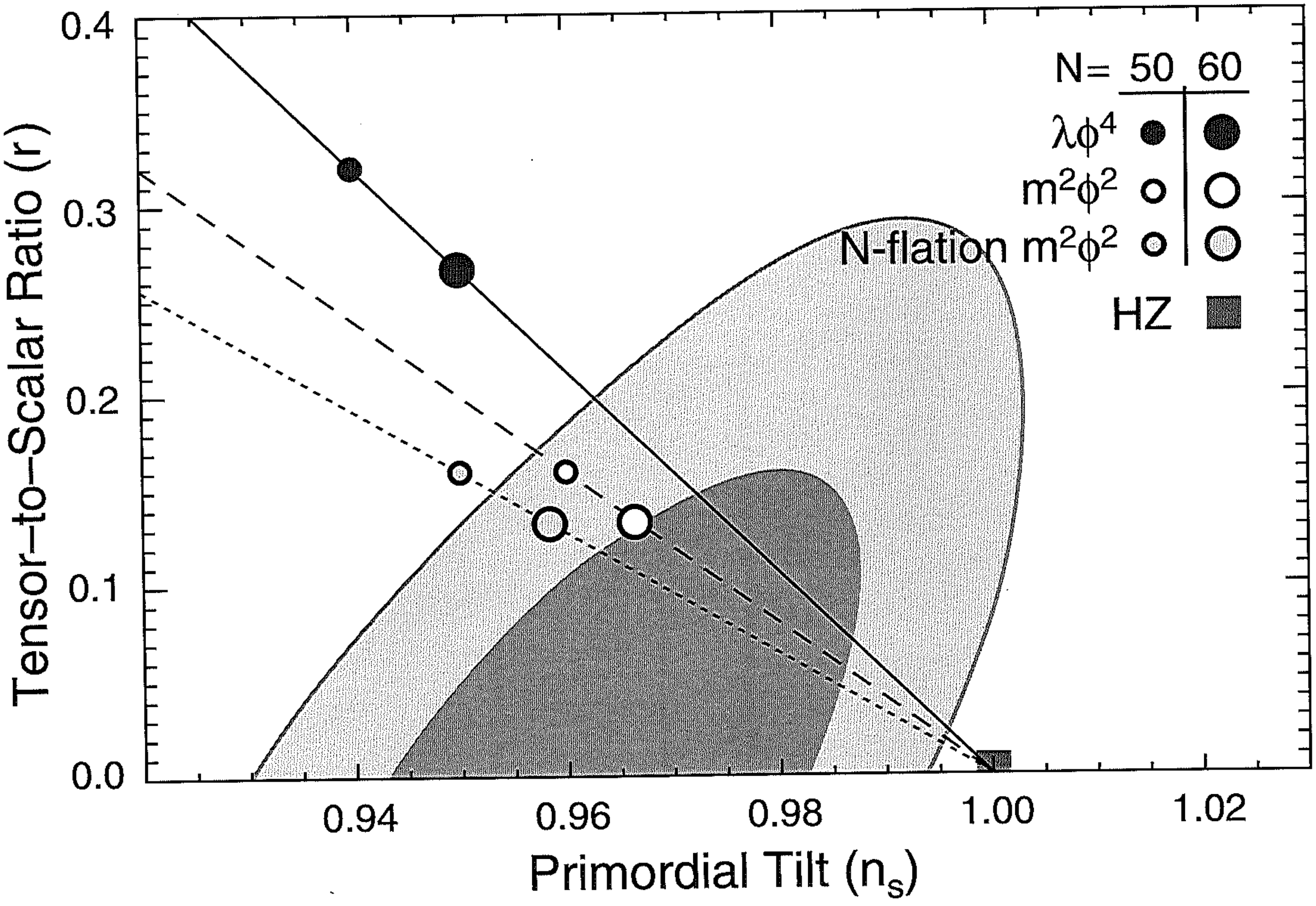
Detection of GWs $\implies$ Super Planckian displacement

Tilt of the GW spectrum + consistency relation

$$n_t = \frac{d\log H^2}{d\log k} = -2\epsilon \qquad \text{Always red as } \dot{H} < 0$$

This is fixed by $r = 16\epsilon$: consistency relation

Quite challenging to probe!



Tensor–to–Scalar Ratio (r)
0.4
0.3
0.2
0.1
0.0
0.94
0.96
0.98
1.00
1.02
Primordial Tilt (n_s)
N= 50 60
$\lambda\phi^4$
$m^2\phi^2$
N-flation $m^2\phi^2$
HZ

Perturbations: helicity decomposition + gauges

- The background is homogeneous $\longrightarrow$ Conservation of 3d momentum

  E.g. No mixing of $\vec{k}$ at linear order

(Notice we do not have Poincaré invariance: I cannot boost, only rotate. We are looking for the irreducible representations of rotations)

- Helicity: I can classify states according to the way they transform under rotation around $\vec{k}$

  Scalar, vector and tensors. At first order there is no mixing

$$ds^2 = a^2(\eta)\Big\{-(1+2\phi)\,d\eta^2 + 2(\partial_i B - S_i)\,d\eta\, dx^i$$
$$+\left[(1-2\psi)\delta_{ij} + 2\partial_i\partial_j E + 2\partial_i F_j + \gamma_{ij}\right]dx^i dx^j\Big\}$$

  All vectors are transverse $\partial^i S_i = \partial^i F_i = 0$

  γ is transverse and traceless: $\gamma^i{}_i = \partial^i \gamma_{ij} = 0$

- In the absence of sources we would have only tensors, but the perturbations of the scalar will mix with the other scalar perturbations

  We can forget about vectors

- Gauge transformations

$$\tilde{\eta} = \eta + \xi^0 \qquad\qquad \tilde{x}^i = x^i + \partial^i \xi + \bar{\xi}^i$$
$$\partial_i \bar{\xi}^i = 0$$

I have two scalar transformations and one vector

E.g. I can choose $\delta\varphi = 0$ $\quad E = 0$ $\quad$ T-gauge

$\Psi = E = 0$ $\quad$ Spatially flat gauge

$B = E = 0$ $\quad$ Newtonian gauge

- I have 5 scalar perturbations (inflaton + 4 metric) with 2 gauge freedoms and (we will see) two constraint equations. Left with 1 final scalar mode

- 2 vectors, 1 gauge and 1 constraint equation. Nothing left

2.1 Spectrum of perturbations in spatially flat gauge

Let us complete the calculation of the spectrum of scalar and tensor perturbations in the spatially flat gauge. We will see that in this gauge the action for scalar perturbations is the one of a massless scalar field at leading order in slow-roll, as the mixing with gravity only induces corrections which are explicitly slow-roll suppressed. We start from the action

$$S = \int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] . \tag{2.1}$$

It is useful to write the metric in the so-called ADM form

$$ds^2 = -N^2dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt) . \tag{2.2}$$

We can now rewrite the action using the Gauss-Codazzi relation, $R = {}^{(3)}R + K_{ij}K^{ij} - K^2$, which relates the 4d Ricci scalar R with the 3d one ${}^{(3)}R$ and the extrinsic curvature of surfaces of constant time K_{ij} defined as

$$K_{ij} \equiv N^{-1}E_{ij} \qquad E_{ij} \equiv \frac{1}{2}(\dot{h}_{ij} - \nabla_i N_j - \nabla_j N_i) , \tag{2.3}$$

where the covariant derivatives are with respect to the 3d induced metric. The action takes the form

$$\begin{aligned} S = \int d^4x\sqrt{h}\left\{\frac{M_P^2}{2}\left[N\,{}^{(3)}R + N^{-1}(E_{ij}E^{ij} - E^2)\right]\right. \\ \left. + \frac{1}{2}N^{-1}(\dot\phi - N^i\partial_i\phi)^2 - \frac{1}{2}Nh^{ij}\partial_i\phi\partial_j\phi - NV\right\} , \end{aligned} \tag{2.4}$$

Inverse of ADM metric

where we have used $\sqrt{-g} = N\sqrt{h}$. It is time to choose the gauge. We choose to set to zero the 2 scalar degrees of freedom in the spatial part of the metric: this is called spatially flat gauge. Actually the spatial metric is not flat, but perturbed only by transverse and traceless perturbations, i.e. gravitational waves. Perturbations are therefore of the form

$$h_{ij} = a^2(t)(\delta_{ij} + \gamma_{ij}) \qquad \partial_i\gamma_{ij} = 0 \quad \gamma_{ii} = 0 \tag{2.5}$$

$$\phi = \phi_0(t) + \varphi(\vec{x}, t) \tag{2.6}$$

This choice of gauge does not affect the ADM variables N and N^i. Notice that these variables appear in the action (2.4) without time derivatives. This means that these degrees of freedom do not propagate and act as Lagrange multipliers, whose equations of motion are constraint equations. These constraint equations will allow us to relate N^i and N to the scalar degree of freedom φ: plugging the solutions back into the action (2.4), we will get an action where only the physical propagating degrees of freedom appear: φ and the tensor modes γ_{ij}. We need to express N and N^i in terms of φ at first order in perturbations: a second order term would go to multiply $\partial S/\partial N$ and $\partial S/\partial N^i$ at zeroth order, i.e. on the unperturbed solution. But the unperturbed solution solves the constraint equations, so these terms vanish. From now on we take $\gamma = 0$ as at linear order there is no mixing between tensor and scalar modes.

With these simplifications the equations obtained by varying with respect to N^i and N are:

$$M_P^2 \partial_i \left[N^{-1}(E^i_j - \delta^i_j E)\right] - \partial_j \varphi \, \dot\phi_0 = 0 \qquad (2.7)$$

$$\frac{M_P^2}{2N^2}(E_{ij}E^{ij} - E^2) + N^{-2}\frac{\dot\phi^2}{2} + V = 0 \qquad (2.8)$$

One can write $N^i = \partial_i \alpha + N^i_T$, with N^i_T transverse $\partial_i N^i_T = 0$. The vector mode N^i_T is not sourced by the scalar φ in eq. (2.7) and can thus be set to zero: vector and scalar perturbations cannot mix at linear order. Using that $E_{ij} = a\dot a \delta_{ij}$ for the unperturbed solution, and $E^{(1)i}_j = -\partial^i \partial_j \alpha$ for the linear perturbation, eq. (2.7) takes the form $\quad \frac{\dot\phi^2}{H^2 M_P^2} \simeq \epsilon$

$$\partial_i (2 H M_P^2 \delta N - \dot\phi_0 \varphi) = 0\,. \qquad (2.9)$$

The other unknown α dropped from this equation so one can solve for the perturbation in the lapse

$$\delta N = \frac{\dot\phi_0}{2 H M_P^2}\varphi\,. \qquad (2.10) \qquad \delta N \simeq \frac{H}{M_P}\sqrt{\epsilon}$$

Equation (2.8) takes the form

$$\frac{M_P^2}{2}\left[2\delta N \cdot 6H^2 + \frac{4}{a^2} H \partial^2 \alpha\right] - \delta N \dot\phi_0^2 + \dot\phi_0 \dot\varphi + V'\varphi = 0 \qquad (2.11)$$

which gives using (2.10)

$$\partial^2 \alpha = -\frac{a^2}{2 H M_P^2}\left[\left(\frac{\dot\phi_0 V}{H M_P^2} + V'\right)\varphi + \dot\phi_0 \dot\varphi\right] \qquad (2.12) \qquad \sim \frac{H}{M_P} \quad \frac{\dot\phi}{H M_P}$$

Now we see what we were looking for: (2.10) and (2.12) imply that δN and α are slow-roll suppressed with respect to φ. This implies that plugging these solutions back into the action will give corrections to the action for φ which are slow-roll suppressed. Let us see this explicitly. The quadratic terms that we get expanding (2.4) are of the form

$$\frac{1}{2}\dot{\varphi}^2 - \frac{1}{2a^2}(\partial\varphi)^2 - \frac{1}{2}V''\varphi^2$$
$$- 2HM_P^2\partial^2\alpha\delta N - 3H^2M_P^2\delta N^2 - \dot{\phi}_0\delta N\dot{\varphi} + \dot{\phi}_0\partial^2\alpha\varphi - \delta NV'\varphi\,, \tag{2.13}$$

where we have integrated by parts to obtain the term before the last in the second line. Let us concentrate on the second line, which comes from the mixing of ϕ with gravity. The contributions with α cancel using the expression for δN and we are left with

$$-3H^2M_P^2\frac{\dot{\phi}_0^2}{4H^2M_P^4}\varphi^2 - \frac{\dot{\phi}_0^2}{2HM_P^2}\varphi\dot{\varphi} - \frac{\dot{\phi}_0}{2HM_P^2}V'\varphi^2\,. \tag{2.14}$$

The second term can be integrated by parts using $\sqrt{h} = a^3$ and its dominant contribution in slow-roll cancels with the first term. Rewriting the last term using the slow-roll parameter ϵ, we are thus led (neglecting terms further suppressed in slow-roll) to the action for φ

$$S = \int d^4x a^3 \left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}V''\varphi^2 + 3\epsilon H^2\varphi^2\right]\,. \tag{2.15}$$

At leading order in slow-roll the action for φ is the one of a massless scalar: the mixing with gravity just induces an ϵ suppressed mass term.[1] For reference, the exact action at all orders in slow-roll is

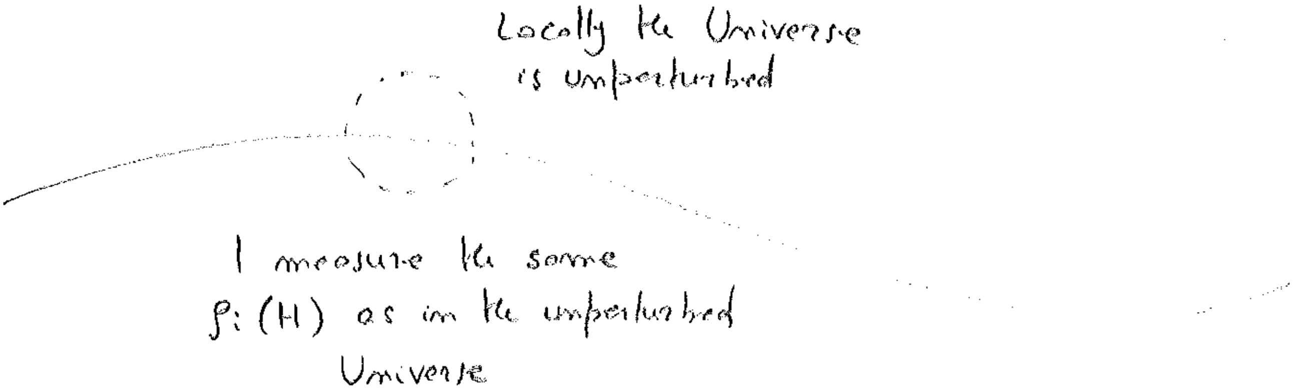
$$S = \int d^4x a^3 \left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}V''\varphi^2 + \frac{1}{a^3}\frac{d}{dt}\left(\frac{a^3\dot{H}}{H}\right)\varphi^2\right]\,. \tag{2.16}$$

Of course this procedure induces cubic and higher-order terms in the action

[1] One can calculate the tilt of the spectrum of ζ also starting from the spectrum of φ and converting to ζ at the end. This requires keeping the mass terms in the action (2.16) and the deviation of the metric from exact de Sitter.

Conservation of ζ

The intuitive idea behind it is the one of separate Universes



No relative perturbations that can be locally measured

$\zeta(x)$ is just an <u>unobservable rescaling of the spatial coordinates</u> which matters only when the modes come back in

(Notice that this picture is useful also when there are more fields, only the local perturbation of the fields matters and not the gradients : in this case we have different homogeneous evolutions we will discuss about it later)

We are in ζ-gauge ($\varphi = 0$, $E = 0$). In general this gauge can be defined as velocity orthogonal, the velocity of the fluid is $\perp$ constant time surfaces

Indeed $U^\mu \propto \partial^\mu \phi$

$$ds^2 = -dt^2 + a^2(t)\, d\vec{x}^{\,2}$$

$\vec{x}\,' = \vec{x}\, e^{-\lambda(t)}$ is an unfixed diff. at $k=0$

I am not touching the slicing, $\delta\varphi = 0$ remains and I am not inducing $g_{ij} \supset \partial_i \partial_j E$

Is the $k \to 0$ limit of a physical mode?

I love to check whether some equation disappears in the $k \to 0$ limit

3d covariant derivative (which is the 3d projection of the 4d covariant derivative)

$$D^a \left(K_{ab} - K^c_c\, h_{ab} \right) = +8\pi G \left(h_b^c\, T_{ca}\, n^a \right)$$

$$\partial_b \left(H\, \delta N - \dot{\zeta} \right) \qquad\qquad 0$$

because U^μ is perpendicular to $t = \text{const}$

But $N = 1$ as we are not redefining the slicing so

$$\zeta = \text{const}$$

- This statement <u>does not depend on the equation of state</u> or any details of the unperturbed evolution

- Notice what happens if I look at the momentum constraint in the diff induced solution

$$\vec{x}' = e^{-\lambda(t)}\, \vec{x}$$

$$t' = t$$

$$N = 1 \qquad N_i = g_{0i} = \frac{\partial x^\mu}{\partial x^0} \frac{\partial x^\nu}{\partial x^i}\, g_{\mu\nu} = \dot{\lambda}\, e^{2\lambda} a^2 x^i$$

Spatial metric: $a^2(t)\, e^{2\lambda(t)} = (a e^\lambda)^2$

$$E_{ij} = \frac{1}{2}\left(\dot{h}_{ij} - \nabla_i N_j - \nabla_i N_j \right) = \frac{1}{2}\left(2 a e^\lambda \left(\dot{a} e^\lambda + \dot{\lambda} a e^\lambda \right) \delta_{ij} - 2\dot{\lambda} e^{2\lambda} a^2 \delta_{ij} \right) = e^{2\lambda} a^2 \delta_{ij} \left(H + \cancel{\dot{\lambda}} \right) - \cancel{\dot{\lambda} e^{2\lambda} a^2 \delta_{ij}}$$

So of course it is satisfied but in a very $\neq$ way that finite K solution, where $\partial_i N_j$ cancel

- Non-linearly

$$\nabla_i \left[N^{-1} \left(E^i_j - \delta^i_j E \right) \right] = 0$$

$N = 1$ For scalar shift $N^i = \partial^i \alpha$

$$\partial_i \left(- \partial_i \partial_j \alpha + \delta^i_j \partial^2 \alpha \right) = 0$$

(This assuming N^i_T vanish at large scales)

As $E_{ij} = e^{2\lambda} a^2 (H + \dot{\lambda})$

we have $\dot{\zeta} = 0$

The gauge solution has $N^i_T = 0$

$\nabla^2 N^i_T$

- In the presence of more fields

$$h^c{}_b T_{ca} n^a \neq 0$$

We are velocity orthogonal to one of them, but others give other surfaces. This solution always exists but there are more.

- Notice also the Hamiltonian constraint is subtle though it does not contain an explicit derivative

$$(3H^2 + \dot{H}) \delta N + H \partial_i N^i = -\frac{\nabla^2}{a^2} \zeta + 3H \dot{\zeta}$$

This cancels $\dot{\zeta}$ in the gauge mode, but in a physical solution is suppressed as $k \to 0$

- What goes wrong if I take $\lambda(t)$ and make it slightly $\vec{x}$ dependent? Helmholtz decomposition fails

- Genesis case : $\begin{cases} H \delta N \approx \dot{\zeta} & H \propto t^{-3} \quad \delta N \propto t \dots \\ \text{it is not a perfect fluid: RHS} \end{cases}$

The effective field theory of inflation (EFTI)

Cheung et al 0709.0293

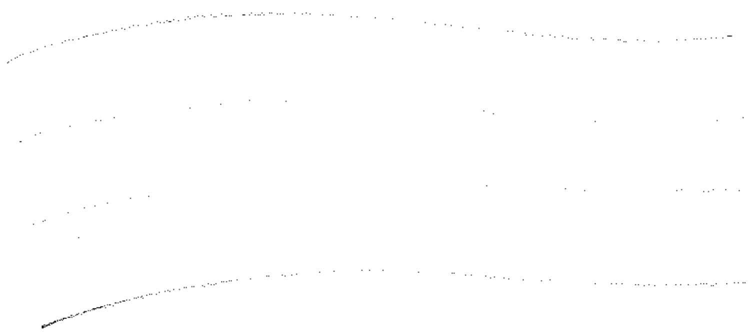
EFT approach : • identify the relevant degrees of freedom (at a given energy scale)

• identify the symmetries at play

• write the lowest dimension operators compatible with the symmetries

Let us do this for (single-field) inflation and see what happens

What is inflation? It is ~ de Sitter with a clock



Privileged slicing

Inflation must end and lead to a decelerated expansion :
I need a clock to know when to terminate

I can use this privileged slicing as t, but now

$$t \longrightarrow t + \xi^0(t;\vec{x})$$ is not a diff anymore

On the other hand (time-dependent) spatial diff are still a good symmetry

$$x^i \longrightarrow x^i + \xi^i(t;\vec{x})$$

Given that I have less symmetries I can write many more operators besides the diff. invariant ones

For example g^{00} is invariant under $x^{\tilde i}(x^i; t)$

[Indeed: $$g^{\tilde\mu\tilde\nu} = \frac{\partial x^{\tilde\mu}}{\partial x^\mu}\frac{\partial x^{\tilde\nu}}{\partial x^\nu}\, g^{\mu\nu}$$

so that g^{00} transforms like a scalar under diff]

$$S = \int d^4x\,\sqrt{-g}\,\Big[\frac{1}{2}M_P^2 R - c(t)\,g^{00} - \Lambda(t) + \frac{1}{2}M_2(t)^4\,(g^{00}+1)^2 + \frac{1}{6}M_3(t)^4\,(g^{00}+1)^3 + \dots$$
$$+ \text{ higher derivative terms}\Big]$$

Notice I can write down generic functions of time

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}}$$

$c(t)$ and $\Lambda(t)$ contributes to the unperturbed Einstein equations

$$\supset M_P^2\dot H\, g^{00} - M_P^2(3H^2+\dot H) \qquad \text{Ex}$$

Indeed: $$\underbrace{-c(t)(g^{00}+1)}_{\text{only contributes to } \rho = +2c}\ \underbrace{-\Lambda(t)+c(t)}_{\text{like a cc}}$$

$$\dot{H} = -\frac{1}{2M_p^2}(\rho + p) = -\frac{c}{M_p^2}$$

$$H^2 = \frac{1}{3M_p^2}(\Lambda + c) = \frac{\Lambda}{3M_p^2} - \frac{\dot{H}}{3}$$

$$\Lambda = M_p^2(3H^2 + \dot{H})$$

Usually we assume "slow-variation", things vary slowly compared to the Hubble time. In particular we will be close to dS: $|\dot{H}| \ll H^2$

Let us open a parenthesis about the symmetries of the problem even before talking about the dynamics

Spatial translations + rotations + dilations

$$ds^2 = \frac{1}{H^2\eta^2}\left(-d\eta^2 + d\vec{x}^2\right)$$

$$(\vec{x};\eta) \longrightarrow \lambda(\vec{x};\eta) \qquad + \text{ slow-variation}$$

Eventually we will be interested in the correlation functions of $\zeta_{\vec{k}}$

$$\langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle, \quad \langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \zeta_{\vec{k}_3} \rangle \ \ldots$$

The metric of constant inflaton, and eventually constant ζ surfaces, is (no tensors)

$$e^{2\zeta(\vec{x};t)} \delta_{ij}\, dx^i dx^j$$

One can think of ζ as $\frac{\delta T}{T}$, so these are the correlation function in the CMB map

An important property of ζ is that it is constant in time on super-H scales, independently of the evolution of the Universe (phase transitions?, degrees of freedom?).
We will come back to it later

Dilation symmetry acts as

$$\zeta_{\vec{k}} \longrightarrow \lambda^3 \zeta_{\lambda \vec{k}}$$

[Indeed: $\zeta_{\vec{k}} = \int d^3x \, e^{i\vec{k}\vec{x}} \zeta(\vec{x})$

$$\longrightarrow \int d^3x \, e^{i\vec{k}\vec{x}} \zeta(\underbrace{\lambda^{-1}\vec{x}}_{\tilde{x}}) = \lambda^3 \int d^3\tilde{x} \, e^{i\vec{k}\cdot\lambda\tilde{x}} \zeta(\tilde{x}) =$$

$$= \lambda^3 \zeta_{\lambda\vec{k}}$$]

Translations + rotations + dilations

$$\Longrightarrow \langle \zeta_{\vec{k}} \zeta_{\vec{k}'} \rangle = (2\pi)^3 \delta(\vec{k}+\vec{k}') \cdot \frac{c}{k^3}$$

[There could be a function of $k\eta$, but ζ becomes η-independent]

c is fixed by experiment.
Usually one puts constraints on

$$\Delta_\zeta^2 = \frac{c}{2\pi^2} = 2.4 \; 10^{-9}$$

Notice scale invariance is just a consequence of symmetries: I don't care about c_s, or change dispersion relations, or massive fields mixed with the inflaton ...

Nothing profound:

$$\int d^4x \sqrt{-g}\left[-\frac{1}{2}(\partial\phi)^2 - V(\phi)\right] \longrightarrow \int d^4x \sqrt{-g}\left[-\frac{\dot\phi_0^2(t)}{2} g^{00} - V(\phi_0(t))\right]$$

if I start with

$$P(X,\phi) \longrightarrow \int d^4x \sqrt{-g}\; P\left(\dot\phi_0(t)^2 g^{00}; \phi(t)\right)$$

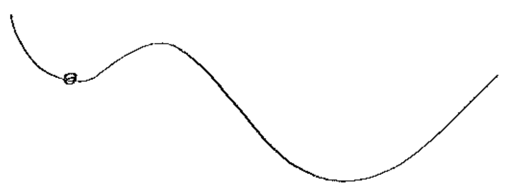
If you feel uncomfortable

$$g^{00} \longrightarrow g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi \qquad t \longrightarrow \phi$$

and you get a "standard" Lagrangian with the solution $\phi = t$ and the correct stress-energy tensor

$\longrightarrow$ I get rid of ambiguity of field redefinitions $\tilde\phi(\phi)$

$\longrightarrow$ Operators are written order by order in perturbation. In the standard case, changing an operator I have to restart the whole process



Like writing operators around a given minimum while we are interested in a different one!

We are describing an extra degree of freedom as the theory is not invariant under t-diff any more. We can make it explicit changing gauge

$$\int d^4x \sqrt{-g}\, A(t) \longrightarrow \int d^4\tilde{x} \sqrt{-\tilde{g}}\, A(\tilde{t} - \xi^0)$$

$$\tilde{t} = t + \xi^0(x)$$

$$\int d^4x \sqrt{-g}\, A(t + \pi(x)) \qquad \text{with} \quad \pi(x) \longrightarrow \tilde{\pi}(\tilde{x}) = \pi(x) - \xi^0(x)$$

This is the analogue of

$$-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu$$

$$A_\mu \longrightarrow A_\mu + \partial_\mu \Lambda \qquad \text{is not a symmetry}$$

But I can write

$$-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 (A_\mu - \partial_\mu \pi)^2$$

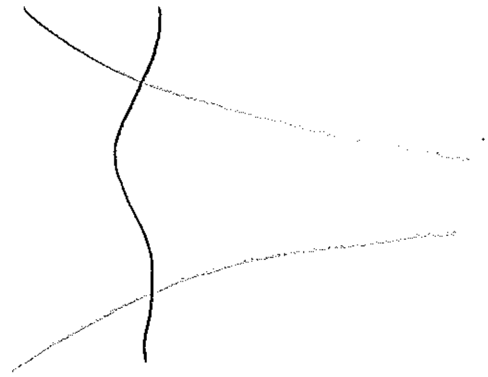
$$\pi \longrightarrow \pi + \Lambda$$

An example : DBI inflation

Alishahiha, Silverstein, Tong 04

Probe D3-brane in AdS_5

$$ds^2 = \frac{r^2}{R^2}\left(-dt^2 + d\vec{x}^{\,2}\right) + \frac{R^2}{r^2}\,dr^2$$



Metric on the brane

$$\frac{r^2}{R^2}\eta_{\mu\nu} + \frac{R^2}{r^2}\partial_\mu r\,\partial_\nu r$$

$$S = -\frac{1}{g_{YM}^2}\int d^4x\,\frac{\phi^4}{\lambda}\left[-\det\left(\eta_{\mu\nu} + \frac{\lambda}{\phi^4}\partial_\mu\phi\,\partial_\nu\phi\right)^{1/2} - 1\right] - V(\phi)$$

$R \sim \sqrt{\alpha'}\,\lambda^{1/4}$
$\phi \sim r/\alpha'$

- For homogeneous solutions: $\sqrt{1 - \lambda\dfrac{\dot\phi^2}{\phi^4}}$

  Speed limit in extra dimension.
  All higher-derivative operators become relevant

- Dual interpretation of motion towards the origin of moduli space. At large λ

  $$\frac{\dot\phi}{\phi^2} \ll 1$$

  I neglect the on-shell production of states becoming light as $\phi \to 0$

- $c_s \ll 1$ because of a geometrical effect: excitations transverse to the brane gets red-shifted

In calculating primordial perturbations the energy is fixed $E \sim H$, but it turns out to be in the decoupling regime (up to slow-roll corrections)

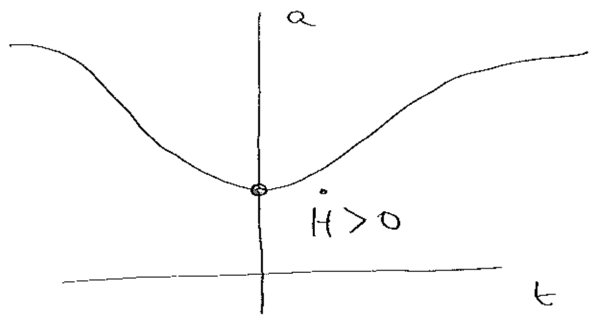
- $+ M_P^2 \dot{H} \frac{(\partial_i \pi)^2}{a^2}$ is fixed by the background and pathological for $\dot{H} > 0$

Not to be confused with $\ddot{a} > 0$, i.e. inflation

$$\dot{H} = \frac{\ddot{a}}{a} - H^2$$

Violation of NEC, $T_{\mu\nu} k^\mu k^\nu \geq 0$ $\forall k$ null

Necessary to have a smooth bounce without curvature



- $c_s^{-2} = 1 - \frac{2 M_2^4}{M_P^2 \dot{H}}$

$$-\frac{M_P^2 \dot{H}}{c_s^2}\left(\dot{\pi}^2 - c_s^2 \frac{(\partial_i \pi)^2}{a^2}\right) + M_P^2 \dot{H}\left(1 - \frac{1}{c_s^2}\right)\left(\dot{\pi}^3 - \dot{\pi}\frac{(\partial_i \pi)^2}{a^2}\right)$$
$$- \frac{4}{3} M_3^4 \dot{\pi}^3 + \dots$$

k-inflation, Garriga - Mukhanov 99

DBI - inflation, Alishahiha - Silverstein - Tong 04

It is easy to realize that

$$B(t)\, g^{00} \longrightarrow B(t+\pi(x))\, \frac{\partial(t+\pi(x))}{\partial x^\mu}\, \frac{\partial(t+\pi(x))}{\partial x^\nu}\, g^{\mu\nu}$$

It is ok when $\pi = 0$ and $t+\pi(x)$ transform like a scalar ...

If we now write down the π action

$$S_\pi = \int d^4x \sqrt{-g} \Big[\frac{1}{2} M_P^2 R - M_P \dot{H} \Big(\dot{\pi}^2 - \frac{(\partial_i \pi)^2}{a^2} \Big)$$

$$+ 2 M_2^4 \Big(\dot{\pi}^2 + \dot{\pi}^3 - \dot{\pi} \frac{(\partial_i \pi)^2}{a^2} \Big) - \frac{4}{3} M_3^4 \dot{\pi}^3 + \dots \Big]$$

Why neglecting metric perturbations?

E.g. $\quad (g^{00} + 1 + 2\dot{\pi} \dots)^2$

The mixing with the metric contains one less derivative so it is expected to be irrelevant at sufficiently high energy

In a gauge theory I have

$$-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu \underbrace{- \frac{1}{2} m^2 (\partial_\mu \pi)^2 + m^2 A_\mu \partial^\mu \pi}_{-\frac{1}{2}(\partial_\mu \pi_c)^2 + m A_\mu \partial^\mu \pi_c}$$

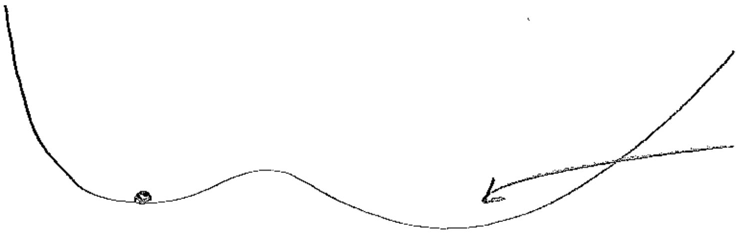
Mixing is irrelevant for $E \gg m$

Advantages of the EFTI

- Written in terms of what experiments directly measure

  NG_3 limits on c_s and M_3

  Similar to writing operators around the correct minimum



It does make sense to expand operators around <u>this</u> minimum

- $t + \pi(t, \vec{x})$

  We probe small π fluctuations around the unperturbed solution. We <u>do not know</u> whether the theory can be extrapolated

  at $\pi \sim t$

  Sometimes it cannot: Ghost Condensate
  Our Galileon example

- Makes explicit the symmetry relation among operators

  E.g. $\quad M_p \dot{H} (\partial\pi)^2 \quad$ and $\quad$ NEC

  $\quad c_s^2 \qquad\qquad \dot{\pi}(\nabla\pi)^2$

- Invariance under field-redefinition

  $\tilde{\phi}(\phi)$

- Decoupling limit

Concentrating on π, say in spatially flat gauge, we are neglecting the effects coming from solving for $\delta N, N^i$ and plugging back in the action

(of course this is suppressed by M_P, but I have to compare with extra powers of $\dot{\pi}$, which are $\frac{1}{\sqrt{\epsilon}} \frac{H}{M_P}$)

As in the minimal case I expect this extra terms to be slow-roll suppressed. Take P_ζ and c_s fixed, I can take the limit $\epsilon \to 0$ and these terms go to zero

(Notice this holds also for the 4pf, with constraints at second order if written in terms of π

$$M_P^2 H^2 \delta N \sim (\partial \varphi)^2 \sim \dot{\phi}^2 (\partial \pi)^2$$

This will give terms like $\sim \epsilon^2 (\partial \pi)^4$ compared to $\epsilon (\partial \pi)^2$

So it is slow-roll suppressed)

I cannot make a theorem: eventually I have to calculate the constraints at n^{th} order ...

(Not really an issue of energy scales: if ϵ becomes large the quadratic action for π in spatially flat gauge becomes $\neq$ from scalar in dS at H^{-1} crossing)

• To take into account fluctuations of the metric I should use ADM formalism:

$$ds^2 = -N^2 dt^2 + h_{ij}\overset{a^2}{\gamma}(dx^i + N^i dt)(dx^j + N^j dt)$$

N and N^i appear in the action without time derivatives so their eom are not dynamical and can be used to solve for N and N^i

$$\left.\begin{array}{l} N(\pi) \\ N^i(\pi) \end{array}\right. \quad \text{back in the action}$$

J. Maldacena astro-ph/0210603. Read it!

• For a standard slow-roll inflation

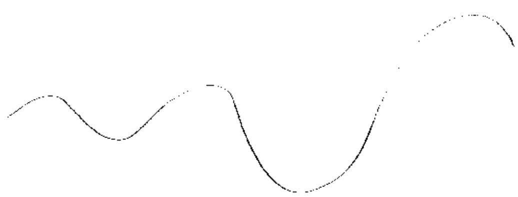
$$\langle \zeta_{\vec{k}}\, \zeta_{\vec{k}'} \rangle = (2\pi)^3\, \delta(\vec{k}+\vec{k}')\, \frac{H^4}{\dot{\phi}^2}\, \frac{1}{2k^3}$$

Comparison between quantum and classical motion

<u>What not to do!</u>

$$\frac{\delta\rho}{\rho} = \frac{V'\delta\phi}{V} \simeq \sqrt{\epsilon}\,\frac{H}{M_P}$$

Perturbations do not change the geometry sensibly: it is more a clock fluctuations



1) Spectrum

$$\langle \pi_{\vec{k}} \, \pi_{\vec{k}'} \rangle = (2\pi)^3 \delta(\vec{k} + \vec{k}') \, \frac{c_s^2}{2 M_P^2 |\dot{H}|} \, \frac{H^2}{2k^3} \, \frac{1}{c_s^3}$$

$$\zeta = -H\pi$$

$$\langle \zeta_{\vec{k}} \, \zeta_{\vec{k}'} \rangle = (2\pi)^3 \delta(\vec{k} + \vec{k}') \, \frac{1}{c_s} \, \frac{H^2}{4 M_P^2} \, \frac{1}{\epsilon} \, \frac{1}{k^3}$$

(Mode freeze at $\partial_t \sim H$, $k \sim \frac{H}{c_s}$)

$\frac{1}{c_s}$ enhancement

next $\longrightarrow$

2) 3-point function

Compare quadratic and cubic action for $c_s^2 \ll 1$ at $\partial \sim H$

$$\frac{\mathcal{L}_3}{\mathcal{L}_2} \simeq \frac{M_P^2 \dot{H}/c_s^2 \; \dot{\pi} (\partial_i \pi)^2}{M_P^2 \dot{H} \; (\partial_i \pi)^2} \simeq \frac{\dot{\pi}}{c_s^2} \simeq \frac{H\pi}{c_s^2} \simeq \zeta / c_s^2$$

$$NG_3 \simeq \frac{\zeta}{c_s^2} \simeq 10^{-5} \cdot \frac{1}{c_s^2}$$

$$f_{NL} \sim \frac{1}{c_s^2}$$

Symmetries relate a small speed of sound with large NGs! Not so obvious to see in the standard language

The same operator changes c_s and induce NG_3. Notice there is also an independent operator $\dot{\pi}^3$

A pause on non-Gaussianities:

To get a feeling of how big can NG_3 be experimentally, think that WMAP map has $2 \cdot 10^6$ pixels

Like measuring the skewness of a distribution $NG_3 \lesssim \frac{1}{\sqrt{N_{pix}}} \sim 10^{-3}$

We already know, given no detection, that deviations from Gaussianity, i.e. interactions are small!

Whatever gave rise to primordial perturbations was pretty weakly coupled

What about non-derivative interactions?

$$M_p^2 \dot{H} (\partial \pi)^2 - H^2 M_p^2 \epsilon^3 \pi^3 H^3$$

I pay a slow-roll parameter every derivative

$$(\partial \pi_c)^2 - \underbrace{\frac{H^5 M_p^2 \epsilon^3}{M_p^3 \epsilon^{3/2} H^3}}\, \pi_c^3$$

$$H \cdot \underbrace{\frac{H}{M_p} \frac{1}{\sqrt{\epsilon}}} \cdot \epsilon^2 \, \pi_c^3$$

Maldacena 02

$$NG_3 \sim \mathcal{J} \cdot \epsilon^2 \sim 10^{-5} \epsilon^2$$

If NG comes from the breaking of dilation invariance (time-shift symmetry), it is very constrained!

Terms through the constraints are larger $NG_3 \sim 10^{-5} \epsilon$

It is even more evident if we look at the 4-point f.

$$M_p^2 \dot{H} (\partial\pi)^2 - H^2 M_p^2 \epsilon^4 \pi^4 H^4$$

$$(\partial\pi_c)^2 - \underbrace{\frac{H^6 M_p^2 \epsilon^4}{M_p^4 \epsilon^2 H^4}}_{\frac{H^2}{M_p^2} \cdot \frac{1}{\epsilon} \epsilon^3 \sim \epsilon^3 10^{-10} \ !} \pi_c^4$$

Dimensionless parameter

Not exactly like the Higgs!

What we have learned in these years is that the absence of interactions is not "intrinsic" to inflation, i.e. forced by the symmetries. Derivative interactions are allowed

The tilt is a measure of deviation from time-translation symmetry

$$\langle \zeta(\vec{k}_1)\,\zeta(\vec{k}_2)\rangle = (2\pi)^3\,\delta(\vec{k}_1+\vec{k}_2)\cdot\frac{1}{4c_s}\,\frac{H^4}{|\dot{H}|\,M_P^2}\,\frac{1}{k_1^3}$$

ζ is constant when confortably outside the Hubble radius so that the tilt comes from the time dependence of the amplitude

$$n_s - 1 = \frac{d}{d\log k}\log\frac{H^4}{|\dot{H}|\,c_s} \underset{k/a\,\sim\,H}{=} 4\,\frac{\dot{H}}{H^2} - \frac{\ddot{H}}{\dot{H}H} - \frac{\dot{c}_s}{c_s H}$$

(This would also change the GW consistency relation)

Let us calculate the 3pf induced by a small c_s

Normally in QFT we are interested

$$\langle \Omega | T\, \phi(x)\, \phi(y) | \Omega \rangle$$

In interaction picture

$$\phi_I(t;\vec{x}) = U_0^{-1}(t;t_0)\, \phi(t_0;\vec{x})\, U_0(t;t_0)$$

Notice we will have a time-dependent H_0 in cosmology

$$\tilde{U}(t;t_0) \equiv U_0^{-1}(t;t_0)\, U(t;t_0)$$

$$= \lim_{T \to +\infty(1-i\epsilon)} \langle 0 | \tilde{U}(T;x_0)\, \phi_I(x)\, \tilde{U}(x_0;y_0)\, \phi_I(y)\, \tilde{U}(y_0;-T) | 0 \rangle$$

$$\tilde{U}: \quad \frac{d}{dt}\tilde{U}(t;t_0) = -i \underbrace{\left[U_0^{-1}(H-H_0)\, U_0 \right]}_{H_I} \tilde{U}$$

Indeed:
$$\frac{d}{dt} U_0(t) = -i H_0(t)\, U_0(t)$$

$$0 = \frac{d}{dt}\left(U_0(t)\, U_0^{-1}(t) \right) \Rightarrow \frac{d}{dt} U_0^{-1} = i\, U_0^{-1} H_0$$

$$\frac{d}{dt}\tilde{U}(t) = \frac{d}{dt}\left(U_0^{-1}(t)\, U(t) \right) = i\, U_0^{-1} H_0\, U + U_0^{-1}(-iH)\, U$$

$$= -i \left[U_0^{-1}(H - H_0)\, U_0 \right] \tilde{U}$$

$$\tilde{U} = T\exp\left(-i \int_{t_0}^{t} H_I(t)\, dt \right)$$

Given Lorentz invariance I can project into the interacting vacuum with a slight tilt in Euclidean time, both in the past and in the future

$$|\Omega\rangle = \lim_{T\to+\infty(1-i\epsilon)} \tilde{U}(t_0;-T)|0\rangle$$

$$\langle\Omega| = \lim_{T\to+\infty(1-i\epsilon)} \langle 0|\tilde{U}(T;t_0) \qquad \text{up to a normalization}$$

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \lim_{T\to+\infty(1-i\epsilon)} \frac{\langle 0|T\phi(x)\phi(y)e^{-i\int_{-T}^{T}H_I(t)dt}|0\rangle}{\langle 0|e^{\overline{\quad}}|0\rangle}$$

Everything is time-ordered

In cosmology I cannot do that, because only in the past all modes behave like in Minkowski and I can define the vacuum state

$$\langle 0|\tilde{U}^{-1}(t;-T)\underbrace{\phi_I(x)\phi_I(y)}_{\text{same time}}\tilde{U}(t;-T)|0\rangle$$

From $\frac{d}{dt}(\tilde{U}\tilde{U}^{-1})=0 \qquad \frac{d}{dt}\tilde{U}^{-1}(t;t_0) = -i\,\tilde{U}^{-1}(t;t_0)H_I(t)$

$$\tilde{U}^{-1}(t;t_0) = \bar{T}e^{i\int_{-\infty-i\epsilon}^{t}H_I(t)}$$

$\neq$ prescription for past and future $\pm i\epsilon$

So we have objects like:

$$\langle 0|\bar{T}e^{i\int_{-\infty-i\epsilon}^{t}H_I(t)}\phi_I(x)\phi_I(y)\,T e^{-i\int_{-\infty+i\epsilon}^{t}H_I(t)}|0\rangle$$

IN-IN calculation

Let us do $-\frac{4}{3}M_3^4\dot{\pi}^3$ as it is simpler

$$-\frac{M_P^2\dot{H}}{c_s^2}\left(\dot{\pi}^2-c_s^2\frac{(\partial_i\pi)^2}{a^2}\right)-\frac{4}{3}M_3^4\dot{\pi}^3$$

$$\pi_c=\frac{\sqrt{2}M_P|\dot{H}|^{1/2}}{c_s}\pi$$

$$\frac{1}{2}\left(\dot{\pi}_c^2-c_s^2\frac{(\partial_i\pi_c)^2}{a^2}\right)-\underbrace{\frac{4}{3}M_3^4\frac{c_s^3}{(\sqrt{2}M_P|\dot{H}|^{1/2})^3}}_{A}\dot{\pi}_c^3$$

$$\mathcal{H}_{int}=A\int d\eta\,\frac{-1}{H\eta}\pi'^3 \qquad \text{Drop canonical}$$

$$\langle\pi_{k_1}(\eta=0)\pi_{k_2}(\eta=0)\pi_{k_3}(\eta=0)\rangle$$

$$=-i\int_{-\infty}^{0}d\eta'\langle[\pi_{k_1}(0)\pi_{k_2}(0)\pi_{k_3}(0);\mathcal{H}_{int}(\eta')]\rangle$$

$$\pi_{\vec{k}}(\eta)=\pi_k^{cl}(\eta)\hat{a}^{+}_{\vec{k}}+\pi_k^{cl\,*}(\eta)\hat{a}_{-\vec{k}}$$

$$\pi_k^{cl}(\eta)=\frac{H}{\sqrt{2c_s^3k^3}}(1-ikc_s\eta)e^{ic_sk\eta}$$

$$\pi_k^{cl\,\prime}(\eta)=\frac{H}{\sqrt{2c_s^3k^3}}c_s^2k^2\eta\,e^{ic_sk\eta}$$

(In general $\mathcal{H}_I\neq-\mathcal{L}_I$. But at cubic order the "new" terms are

$$\underbrace{p\dot{\pi}(p)}_{\text{at 2nd order}}-\Big(\underbrace{\frac{\dot{\pi}^2}{2}(p)\cdots}_{\text{one I have to take it unperturbed}}\Big)=p\dot{\pi}(p)$$
)

$$\langle \pi_{k_1} \pi_{k_2} \pi_{k_3} \rangle = (2\pi)^3 \delta(\Sigma_i \vec{k}_i) \frac{H^6}{\Pi_i 2c_s^3 k_i^3} (-i) A$$

$$\int d\eta \frac{-1}{H\eta} c_s^6 k_1^2 k_2^2 k_3^2 \eta^3 e^{i k_t c_s \eta} + cc \qquad \times 6 \text{ perm.}$$

$$\int_{-\infty - i\epsilon}^{0} d\eta\, \eta^2 e^{i k_t c_s \eta} = \frac{2i}{(k_t c_s)^3}$$

$$\langle \pi_{k_1} \pi_{k_2} \pi_{k_3} \rangle = (2\pi)^3 \delta(\Sigma_i k_i) \frac{H^6}{\Pi_i 2c_s^3 k_i^3} \frac{-k_1^2 k_2^2 k_3^2 \cdot 4 c_s^6}{H(k_t c_s)^3}$$

$$\cdot 6A$$

$$\langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \zeta_{\vec{k}_3} \rangle = (2\pi)^3 \delta(\Sigma_i k_i) \frac{4}{3} M_3^4 \frac{c_s^6 H^3}{(\sqrt{2} M_p |\dot{H}|^{1/2})^6}$$

$$\cdot \frac{H^6}{\Pi_i 2c_s^3 k_i^3} \frac{+24 k_1^2 k_2^2 k_3^2}{H k_t^3} \cdot c_s^3 \qquad \zeta = -H\pi$$

$$P_\zeta = \frac{H^4}{4 M_p^2 |\dot{H}|} \cdot \frac{1}{c_s}$$

$$\langle \quad \rangle = (2\pi)^3 \delta(\Sigma_i \vec{k}_i) P_\zeta^2 \cdot c_s^2 \frac{M_3^4}{M_p^2 |\dot{H}|}$$

$$\cdot 8 \cdot \frac{1}{\Pi_i k_i^3} \frac{k_1^2 k_2^2 k_3^2}{k_t^3} \qquad f_{NL}$$

$$\frac{M_p^2 |\dot{H}|}{c_s^2} \dot{\pi}^2 \gtrsim M_3^4 \dot{\pi}^3$$

$$NG_3 \sim \zeta \cdot \frac{M_3^4}{M_p^2 |\dot{H}| / c_s^2}$$

f_{NL}^{local} and f_{NL}^{equil}

Another popular shape of non-Gaussianity is the so-called "local" shape

$$\zeta(x) = \zeta_g(x) - \frac{3}{5} f_{NL}^{loc} \left(\zeta_g(x)^2 - \langle \zeta_g^2(x) \rangle \right)$$

We will see that it is not only simple, but the justified

$$\langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \zeta_{\vec{k}_3} \rangle = - \frac{3}{5} f_{NL}^{loc} \cdot 2 \cdot P_\zeta^2 \left(\frac{1}{k_1^3 k_2^3} + \frac{1}{k_1^3 k_3^3} + \frac{1}{k_2^3 k_3^3} \right)$$

Compare this with the previous shape in the limit $k_1 \to 0$

local: $\dfrac{1}{k_1^3 k_s^3}$

equilateral: $\dfrac{1}{k_1 k_s^5} \ll$ local

$\longrightarrow$ Derivative interactions induce correlation among modes with similar wavelength. Equilateral

$\longrightarrow$ local is completely non-local in Fourier space

Multifield models : local NG

A (test) scalar is excited during inflation and it becomes important for the cosmological evolution later on

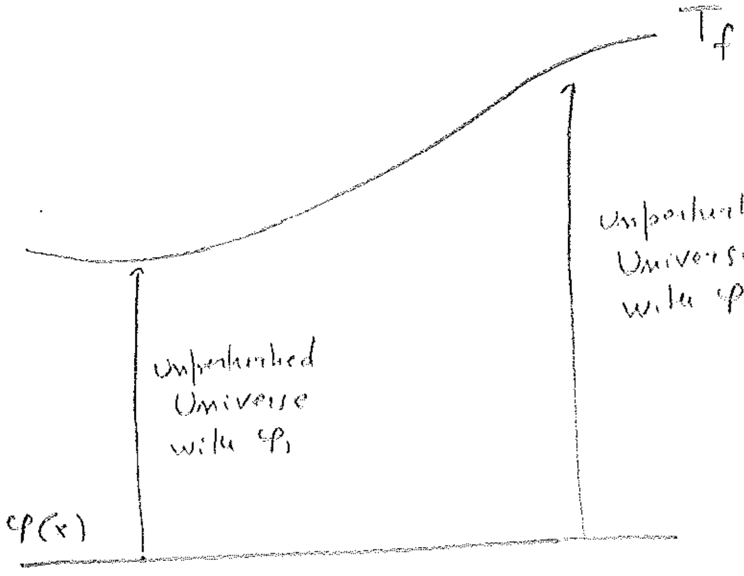
→ Curvaton : its energy-density becomes relevant when it oscillates and then decay

→ Modulated reheating : reheating is affected by the value of φ

it can change some mass, some coupling constant ...

→ Modulated something : e.g. RHN go out of T equilibrium and their decay can be affected by φ

To calculate the effect of φ we just need a homogeneous calculation, as it is relevant only when modes are out of H^{-1}



Assume no inflaton perturbations

$$e^{2\zeta(\vec{x})} = \frac{a(T_f)}{a(T_i)}(\varphi(\vec{x}))$$

The effect of φ must take place during an out of equilibrium otherwise entropy conservation fixes the ratio of a's

E.g. Variable decay. $\Gamma \ll H_{end}$

$$\underbrace{\frac{a(T_f)}{a(t_{decay})}}_{RD} \cdot \underbrace{\frac{a(t_{decay})}{a(t_i)}}_{MD} \propto H_{decay}^{-1/6} \propto \Gamma^{-1/6}$$

$$a_{RD} \propto t^{1/2} \propto H^{-1/2} \qquad a_{MD} \propto t^{2/3} \propto H^{-2/3}$$

Γ changes the time you spend in MD and RD from t_i to T_f

In this regime: $\quad \zeta = -\frac{1}{6}\frac{\delta\Gamma}{\Gamma} = \alpha \frac{\varphi}{\varphi_0}$

Many sources of NGs:

$\zeta(\Gamma(\varphi(x)))$ local, non-linear relation

$$e^{\zeta} \propto \Gamma^{-1/6} \qquad \zeta = -\frac{1}{6}\Bigg(\underbrace{\frac{\delta\Gamma}{\Gamma_0}}_{\equiv \zeta_g} - \frac{1}{2}\left(\frac{\delta\Gamma}{\Gamma_0}\right)^2 + \dots\Bigg)$$

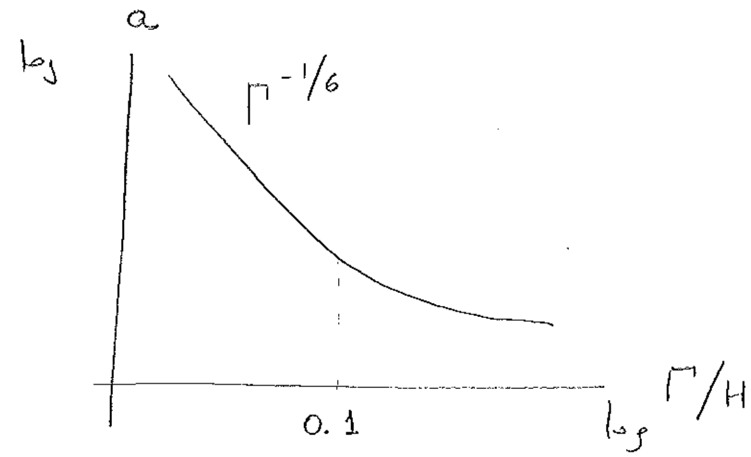
$$\zeta_g \underbrace{+\,3}_{= -\frac{3}{5} f_{NL}^{loc}} \zeta_g^2 \implies f_{NL}^{loc} = -5$$

+ other sources of NG: $\Gamma(\varphi)$ and φ non-G.

- All these sources give a local NG as they take place out of H^{-1}

- $f_{NL}^{loc} \sim$ few is a kind of lower limit

E.g.



If Γ becomes large reheating becomes instantaneous and there is no effect in perturbing it

$$\zeta = \alpha \left(\frac{\delta\Gamma}{\bar{\Gamma}} + \# \left(\frac{\delta\Gamma}{\bar{\Gamma}} \right)^2 \right)$$

$$f_{NL}^{local} \sim \frac{\#}{\alpha}$$

The amplitude of ζ is fixed and we are forced to make $\frac{\delta\Gamma}{\bar{\Gamma}}$ larger and larger, enhancing f_{NL}^{loc}

$\boxed{\text{NG from inefficiency}}$

$f_{NL}^{equil}, f_{NL}^{loc} \sim 1$ is an important threshold

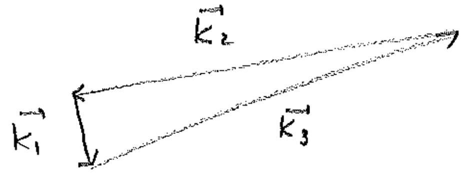
- Approximate lower bound for non-minimal inflation
- Experimentally around the corner with Planck and scale-dependent bias
- Limit of all possible 2[nd] order GR effects

No matter what we are entering in a new regime!

Planck: $f_{NL}^{local} \sim 3-5 \qquad f_{NL}^{equil} \sim 20$

Consistency relation for the 3pf

- Property of the 3pf which is true in <u>any single field model</u>



Squeezed limit
$|\vec k_1| \to 0$

We already saw that there is a $\not\!f$ behaviour, let us make it general

In the limit $k_1 \ll k_{2;3}$ this mode exits H^{-1} much before the others and it acts as a background (classical background) for them

$$\langle \bar\zeta(\vec x_1)\,\zeta(\vec x_2)\,\zeta(\vec x_3)\rangle = \langle \bar\zeta(\vec x_1)\,\langle \zeta(\vec x_2)\,\zeta(\vec x_3)\,|\,\bar\zeta\rangle\rangle$$

This is nothing, but now $\langle \zeta\zeta|\bar\zeta\rangle$ is easy to calculate

$$\langle \zeta(\vec x_2)\,\zeta(\vec x_3)\,|\,\bar\zeta\rangle \simeq \xi(\vec x_3 - \vec x_2) + \bar\zeta(\vec x_+)\left[(\vec x_3 - \vec x_2)\cdot\vec\nabla\,\xi(|\vec x_3 - \vec x_2|)\right]$$

+ higher order in $\bar\zeta$

The 2pf is the same as in the absence of $\bar\zeta$, the only difference is that the coordinates are rescaled

$$\langle \zeta(\vec x_2)\,\zeta(\vec x_3)\,|\,\bar\zeta\rangle = \xi(e^{\bar\zeta}\,|\vec x_2 - \vec x_1|)$$

$$\langle \bar\zeta(\vec x_1)\,\zeta(\vec x_1)\,\zeta(\vec x_2)\rangle = \langle \bar\zeta(\vec x_1)\,\zeta(\vec x_+)\rangle\left[(\vec x_3 - \vec x_2)\,\vec\nabla\,\xi(|\vec x_3 - \vec x_2|)\right]$$

$$= \int \frac{d^3k_L}{(2\pi)^3}\frac{d^3k_s}{(2\pi)^3}\, e^{i\vec k_L\cdot(\vec x_1 - \vec x_+)}\, P(k_L)\,P(k_s)\left[\vec k_s\cdot\frac{\partial}{\partial \vec k_s}\right] e^{i\vec k_s\cdot\vec x_-}$$

$$\vec x_+ = \frac{\vec x_1 + \vec x_2}{2} \qquad \vec x_- = \vec x_2 - \vec x_1$$

Integrate by parts and insert $1 = \int d^3k_1 \, \delta(\vec{k}_1 + \vec{k}_L)$

$$\frac{\partial}{\partial \vec{k}_s}\left(\vec{k}_s \, P(k_s)\right) = 3P + \vec{k}_s \, \frac{\partial P}{\partial k_s} \underbrace{\left(\frac{\partial k_s}{\partial \vec{k}_s}\right)}_{\frac{\vec{k}_s}{k_s}} = 3P + \frac{\partial P}{\partial \log k_s}$$

$$= P \cdot \frac{d}{d \log k_s} \log k_s^3 P(k_s)$$

$$= - \int \frac{d^3k_1 \, d^3k_2 \, d^3k_s}{(2\pi)^3} \, e^{-i\vec{k}_1 \cdot \vec{x}_1 - i \vec{k}_L \cdot \vec{x}_+ + i \vec{k}_s \cdot \vec{x}_-}$$
$$\left[(2\pi)^3 \delta(\vec{k}_1 + \vec{k}_L) \, P(k_1) \, P(k_s) \, \frac{d \log k_s^3 P(k_s)}{d \log k_s} \right]$$

$$\vec{k}_L = \vec{k}_2 + \vec{k}_3 \qquad \vec{k}_s = (\vec{k}_2 - \vec{k}_3)/2$$

$$\langle \zeta_L(\vec{k}_1) \, \zeta(\vec{k}_2) \, \zeta(\vec{k}_3) \rangle = -(2\pi)^3 \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) \, P(k_1) \, P(k_s) \, \frac{d \log k_s^3 P(k_s)}{d \log k_s} + O\left(\left(\frac{k_1}{k_s}\right)^2\right)$$

it corresponds to $f_{NL}^{\text{local}} \sim (n_s - 1)$ very small

<u>Any single field model gives a negligible local NG</u>

- It is quite possible to observe NGs and to see that it has a large local f_{NL}^{loc}. This would rule out all single field models

Probes of NG

Notice that some of them are only sensitive to squeezed limits

- Observations of statistical properties of linear scales in CMB and LSS

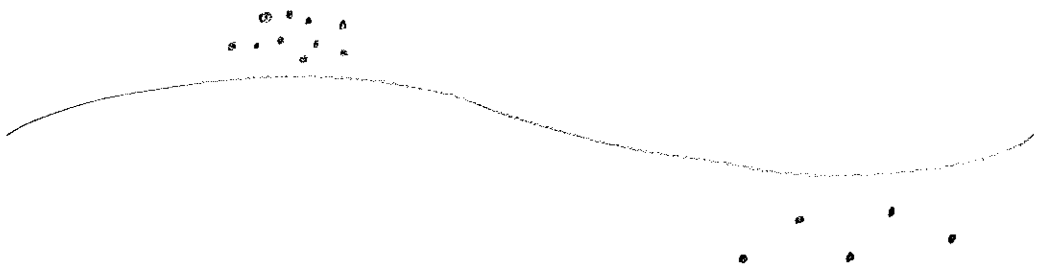
Until 2008

$$\Delta f_{NL}^{loc} = 21 \longrightarrow 3-5$$

$$\Delta f_{NL}^{equil} = 140 \longrightarrow 20$$

Planck

- Scale dependent bias



The bias relates the matter overdensity δ_m with $\delta(M)$, overdensity of objects with a certain mass M

$$\delta(M) = b(M)\,\delta_m$$

In the presence of local NG the short scale power spectrum is enhanced (or reduced) by the presence of the long mode and this affects the bias

Not degenerate with the other as $\langle \Phi \Phi \rangle_{short} \propto \Phi_L \propto \frac{\delta_{m,L}}{k^2}$

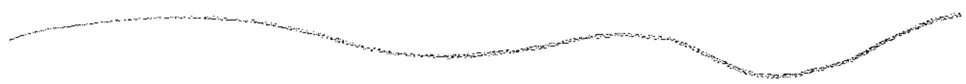
scale dependent

$$\Delta f_{NL}^{local} \sim 20$$

$\longrightarrow 1$ in the future

- Quite similar to observing isocurvature perturbations

Single field:



Multi-field:

Adiabaticity: $\rho_\gamma \propto a^{-4}$ $\quad\quad$ $\rho_{DM} \propto a^{-3}$

$$\frac{\delta\rho_\gamma}{\rho_\gamma} = -4\,\frac{\delta a}{a} \qquad\qquad \frac{\delta\rho_{DM}}{\rho_{DM}} = -3\,\frac{\delta a}{a}$$

$$\frac{1}{4}\,\frac{\delta\rho_\gamma}{\rho_\gamma} = \frac{1}{3}\,\frac{\delta\rho_{DM}}{\rho_{DM}}$$

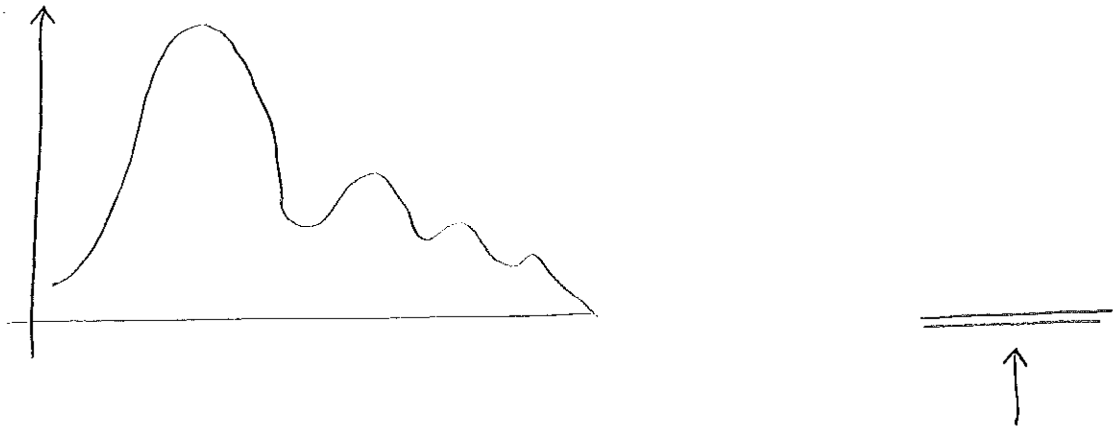
In single field models perturbations are locally unobservable: Just a diff. of the unperturbed solution

The consistency relations say the same thing: the effect of a long mode must be unobservable locally. Its effect on 2pf is just a rescaling of coordinates

New way of probing isocurvature perturbations, which does not need a conserved quantity not to be washed out

- μ - distorsion

Modes entering H^{-1} before recombination in a given window



They enter when kinetic equilibrium (elastic Compton) is restored, but not chemical. These modes induce a departure from thermal spectrum : μ distortion

Again in the presence of f_{NL}^{loc} a long mode will be correlated with short-scale power spectrum, i.e. μ - distortion

$$\Delta f_{NL}^{local} < 10^3$$ but on very short scales