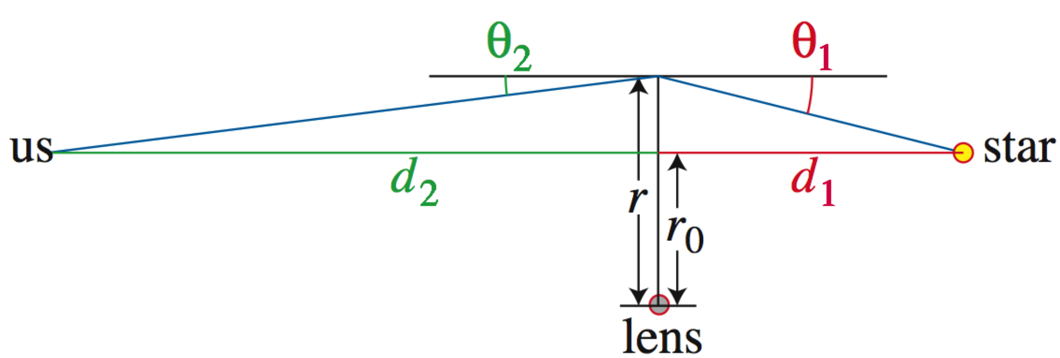
Microlensing and MACHO searches

Suppose a massive particle with velocity v is incident, with impact parameter b, on a fixed deflector of mass M. The deflection angle ϕ due to scattering of this particle via gravitational interaction with the deflector can be computed via classical scattering theory, finding $\phi = 2GM/bc^2$ in the limit as $v \to c$. The proper relativistic treatment for the deflection of light due to the gravitational field of a point source of mass M yields twice this value, $\phi = 4GM/bc^2$.

(a) Using this result, **show that light received at your detector from a background source (with an intermediate lens of mass M) is focused and amplified due to this gravitational lensing effect, by a factor**:

$$A = \frac{2+u^2}{u\sqrt{4+u^2}}, \quad u = \frac{b}{r_E}.$$

Here $r_E = \sqrt{GMd/c^2}$ is the Einstein radius, M is the mass of the "lens", and $d = 4d_1d_2/(d_1+d_2)$, where d_1 is the distance from the lens to the source plane, and d_2 is the distance from the lens to the observer. You may assume that all the relevant angles are small, and treat the deflection of the light as occurring abruptly at an infinitely thin "deflection plane". This deflection plane is defined by drawing a line between the star and the source, and then considering the plane perpendicular to this line that intersects the lens.



Hint: Consider first the relationship between the impact parameter and the distance r between the lens and the point of deflection. Once you have obtained this relation, consider how the shape and size of a telescope aperture (as seen from the star) would be distorted by the lensing. You may find Paczynski 1986 (http://adsabs.harvard.edu/cgi-bin/nph-bibquery?bibcode=1986ApJ...304....1P) useful – however, you may not use the relations in that paper without proof.

(b) If the dark matter is composed (partly or fully) of massive compact objects, like black holes, then when these objects pass between Earth and background stars, we expect to see transient magnification of the background stars due to gravitational lensing. This fact has been used to place limits on such Massive Compact Halo Objects, or MACHOs, and the possibility that they could constitute the dark matter.

Assume that the dark matter halo density follows $\rho \propto 1/r^2$ (for the scales relevant to this problem, this is a reasonable approximation). The Milky Way's circular velocity in the flat part of the rotation curve is $v \sim 220$ km/s. The Large Magellanic Cloud is roughly 50kpc away, whereas we are 8.5kpc away from the Galactic Center; the angle between the LMC and our line of sight to the Galactic Center is $\alpha = 82°$.

From (a), we see that the magnification is large when $u \ll 1$, i.e. the impact parameter is small compared to r_E. For simplicity, assume that any MACHO whose Einstein radius completely crosses the LMC causes a microlensing event. **Estimate the frequency of gravitational microlensing events of a single star in the Large Magellanic cloud, due to MACHOs in the Galactic halo, as a function of the MACHO mass** (assuming the MACHOs constitute all the dark matter, and all have the same mass). Rather than integrate over the Boltzmann distribution, you can just take the tangential velocity of the MACHOs to be equal to the circular velocity; you can similarly ignore other $\mathcal{O}(1)$ factors in the calculation.

If we can monitor a million stars, roughly how many events caused by solar-mass MACHOs would we expect to see in a year? (Note that this will involve evaluating an integral numerically - to understand the scaling, it is helpful to take all dimensionful factors out the front, so the integral becomes simply a dimensionless number.)