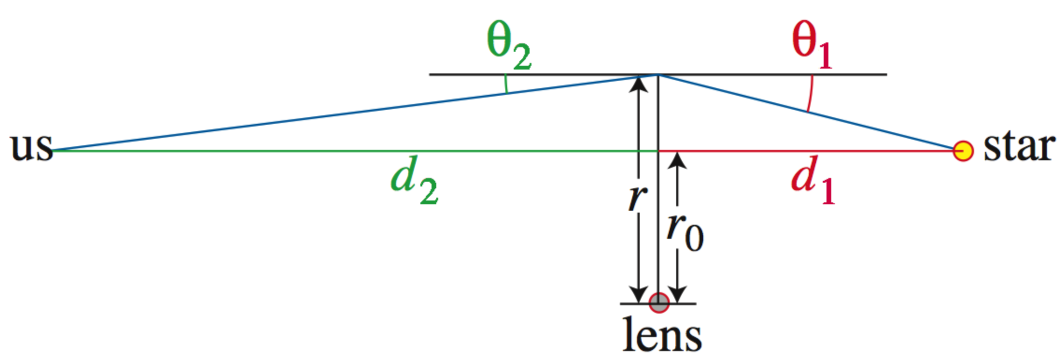
Microlensing and MACHO searches

Suppose a massive particle with velocity v is incident, with impact parameter b, on a fixed deflector of mass M. The deflection angle ϕ due to scattering of this particle via gravitational interaction with the deflector can be computed via classical scattering theory, finding $\phi = 2GM/bc^2$ in the limit as $v \to c$. The proper relativistic treatment for the deflection of light yields twice this value, $\phi = 4GM/bc^2$.

(a) Using this result, **show that light received at your detector from a background source (with an intermediate lens of mass M) is focused and amplified due to this gravitational lensing effect, by a factor**:

$$A = \frac{2+u^2}{u\sqrt{4+u^2}}, \quad u = \frac{b}{r_E}.$$

Here $r_E = \sqrt{GMd/c^2}$ is the Einstein radius, M is the mass of the "lens", and $d = 4d_1d_2/(d_1+d_2)$, where d_1 is the distance from the lens to the source plane, and d_2 is the distance from the lens to the observer. You may assume that all the relevant angles are small, and treat the deflection of the light as occurring abruptly at an infinitely thin "deflection plane". This deflection plane is defined by drawing a line between the star and the source, and then considering the plane perpendicular to this line that intersects the lens.



Hint: Consider first the relationship between the impact parameter and the distance r between the lens and the point of deflection. Once you have obtained this relation, consider how the shape and size of a telescope aperture (as seen from the star) would be distorted by the lensing. You may find Paczynski 1986 (`http://adsabs.harvard.edu/cgi-bin/nph-bibquery?bibcode=1986ApJ...304....1P`) useful – however, you may not use the relations in that paper without proof.

Solution: In terms of the figure, we can write the deflection angle as $\Delta\theta = \theta_1 + \theta_2 = \frac{r-r_0}{d_1} + \frac{r-r_0}{d_2}$, where we have made a small-angle approximation. Setting $r_0 = 0$ (so the lens is along the line of sight), we have $4GM/r = \Delta\theta = \frac{r}{d_1} + \frac{r}{d_2}$, i.e. $r = \sqrt{4GMd_1d_2/(d_1+d_2)} = r_E$. This is the perceived position of a point object directly behind the lens.

Note: in the literature, you will sometimes see the Einstein "radius" defined as $\sqrt{\frac{4GMd_1}{d_2(d_1+d_2)}}$. *Notice that this is actually a dimensionless angular distance* θ_E, *which is simply related to* r_E *by* $r_E = d_2\theta_E$, *the usual relation between angular and linear distances.*

For general r_0, we have $4GM = r(r-r_0)(d_1+d_2)/(d_1d_2)$, i.e. $r_E^2 = r(r-r_0)$. This yields,

$$r_\pm = \frac{1}{2}\left(r_0 \pm \sqrt{r_0^2 + 4r_E^2}\right).$$

These solutions correspond to the light ray passing "above" or "below" the lens in the figure.

To compute the amplification of light from the source, the relevant question is what fraction of light from the source enters the aperture of your detector, relative to the case with no gravitational lensing. One way to work this out is to turn the problem around and ask how large the detector looks from the point of view of the source.

Consider an infinitesimal circular aperture. We define "vertical" to mean the perpendicular direction from the lens to the line joining the star and the observer (as shown in the figure), and "horizontal" to mean the direction perpendicular to both this line and the star-observer line. Suppose the vertical size of the aperture means that the impact parameter r_0 now has a range $r_0 \pm \delta$. The ratio of δ to the radius of the aperture must be the same as $d_2/(d_1+d_2)$, so we can write the radius of the aperture as $\delta(d_1+d_2)/d_2$.

For the solution where the light curves "above" the lens, the top of the aperture, at the deflection plane, appears to be at $r_+ = r_+(r_0+\delta) = r_+(r_0) + \delta\frac{dr_+}{dr_0}$. Likewise the bottom of the aperture is at $r_+ = r_+(r_0-\delta) = r_+(r_0) - \delta\frac{dr_+}{dr_0}$. The vertical height of the aperture is thus:

$$\left|2\delta\frac{dr_+}{dr_0}\right| = \delta\left|1 + \frac{r_0}{\sqrt{r_0^2+4r_E^2}}\right| < 2\delta.$$

The result is exactly the same for the other image, replacing $\left|1 + \frac{r_0}{\sqrt{r_0^2+4r_E^2}}\right|$ with

$\left|1-\frac{r_0}{\sqrt{r_0^2+4r_E^2}}\right|$.

The aperture is "squashed" in the vertical direction.

Now consider rays defining the horizontal edges of the aperture. If we write their horizontal separation at the deflection plane as $2x$, then $2x/r = 2\delta/r_0$ (since δ is defined in terms of the spacing between the lines traced from the edges of the aperture to the star, and the distance from the lens to that point is r_0, whereas the distance from the lens to the deflection point is r). Thus the aperture is "stretched" in the horizontal direction by a factor $|r/r_0|$.

The magnification factor is given by the product of the horizontal and vertical factors. Since there are two images (corresponding to r_+ and r_-), we need to add their contributions together.

$$\begin{aligned}
A &= \frac{|r_+|}{r_0}\frac{1}{2}\left|1+\frac{r_0}{\sqrt{r_0^2+4r_E^2}}\right| + \frac{|r_-|}{r_0}\frac{1}{2}\left|1-\frac{r_0}{\sqrt{r_0^2+4r_E^2}}\right| \\
&= \frac{1}{4r_0}\left[\left(r_0+\sqrt{r_0^2+4r_E^2}\right)\left|1+\frac{r_0}{\sqrt{r_0^2+4r_E^2}}\right| + \left(-r_0+\sqrt{r_0^2+4r_E^2}\right)\left|1-\frac{r_0}{\sqrt{r_0^2+4r_E^2}}\right|\right] \\
&= \frac{1}{4r_0}\sqrt{r_0^2+4r_E^2}\left[\left(1+\frac{r_0}{\sqrt{r_0^2+4r_E^2}}\right)^2 + \left(1-\frac{r_0}{\sqrt{r_0^2+4r_E^2}}\right)^2\right] \\
&= \frac{1}{2r_0}\sqrt{r_0^2+4r_E^2}\left[1+\frac{r_0^2}{r_0^2+4r_E^2}\right] \qquad (1)
\end{aligned}$$

Writing $u = b/r_E = r_0/r_E$, we have:

$$A = \frac{1}{2r_0}\sqrt{r_0^2+4r_E^2}\left[1+\frac{r_0^2}{r_0^2+4r_E^2}\right] = \frac{1}{2r_0}\left[\frac{2r_0^2+4r_E^2}{\sqrt{r_0^2+4r_E^2}}\right] = \frac{r_E}{r_0}\frac{r_0^2/r_E^2+2}{\sqrt{r_0^2/r_E^2+4}} = \frac{1}{u}\frac{u^2+2}{\sqrt{u^2+4}}.$$

This is the required result.

(b) If the dark matter is composed (partly or fully) of massive compact objects, like black holes, then when these objects pass between Earth and background stars, we expect to see transient magnification of the background stars due to gravitational lensing. This fact has been used to place limits on such Massive Compact Halo Objects, or MACHOs, and the possibility that they could constitute the dark matter.

Assume that the dark matter halo density follows $\rho \propto 1/r^2$ (for the scales relevant to this problem, this is a reasonable approximation). The Milky Way's circular velocity in the flat part of the rotation curve is $v \sim 220$ km/s. The Large Magellanic Cloud is roughly 50kpc away, whereas we are 8.5kpc away from the Galactic Center; the angle between the LMC and our line of sight to the Galactic Center is $\alpha = 82°$.

From (a), we see that the magnification is large when $u \ll 1$, i.e. the impact parameter is small compared to r_E. For simplicity, assume that any MACHO whose Einstein radius completely crosses the LMC causes a microlensing event. **Estimate the frequency of gravitational microlensing events of a single star in the Large Magellanic cloud, due to MACHOs in the Galactic halo, as a function of the MACHO mass** (assuming the MACHOs constitute all the dark matter, and all have the same mass). Rather than integrate over the Boltzmann distribution, you can just take the tangential velocity of the MACHOs to be equal to the circular velocity; you can similarly ignore other $\mathcal{O}(1)$ factors in the calculation.

If we can monitor a million stars, roughly how many events caused by solar-mass MACHOs would we expect to see in a year? (Note that this will involve evaluating an integral numerically - to understand the scaling, it is helpful to take all dimensionful factors out the front, so the integral becomes simply a dimensionless number.)

Solution: The Einstein radius $r_E = \sqrt{GMd}$, taking units where $c = 1$; we will write $d = 4r(R_{\rm LMC} - r)/R_{\rm LMC}$, where r is the distance from Earth to the MACHO (we will need to integrate over this variable), and $R_{\rm LMC}$ is the fixed distance from Earth to the LMC.

Consider those MACHOs crossing the line-of-sight to the LMC at a distance r from the Earth; if they are moving with tangential velocity v, the number of MACHOs crossing within an Einstein radius r_E of the line-of-sight in a time Δt will be given approximately by $n(r)(2r_E)dr(v\Delta t)$. Integrating over the Earth-MACHO distance r yields a total crossing rate (dividing by Δt):

$$\int_0^{R_{\rm LMC}} 2vn(r)\sqrt{GM \times 4r(R_{\rm LMC} - r)/R_{\rm LMC}}dr$$

Now $n(r) = n(\sqrt{r^2 + r_\odot^2 - 2rr_\odot \cos\alpha})$, where $r_\odot$ is the Galactocentric radius of the Sun (8.5 kpc) – the argument here is the distance between the MACHO and the Galactic center. Taking $\rho \propto 1/r_{\rm gal}^2$ (where $r_{\rm gal}$ is the distance to the Galactic center), writing $\rho(r_{\rm gal}) = A/r_{\rm gal}^2$, we find:

$$\frac{v^2}{r_{\rm gal}} = \frac{4\pi G}{r_{\rm gal}^2}\int_0^{r_{\rm gal}} r^2\rho(r)dr = \frac{4\pi GA}{r_{\rm gal}},$$

so $A = v^2/4\pi G$. Thus the crossing rate is:

$$\int_0^{R_{\rm LMC}} 2v\frac{v^2}{4\pi GM}\frac{1}{r^2+r_\odot^2-2rr_\odot\cos\alpha}\sqrt{GM\times 4r(R_{\rm LMC}-r)/R_{\rm LMC}}dr$$
$$=\frac{v^3}{\pi\sqrt{GMR_{\rm LMC}}}\int_0^1\frac{\sqrt{\tilde{r}(1-\tilde{r})}}{\tilde{r}^2+\left(\frac{r_\odot}{R_{\rm LMC}}\right)^2-2\tilde{r}\frac{r_\odot}{R_{\rm LMC}}\cos\alpha}d\tilde{r},$$

where we have defined $\tilde{r} = r/R_{\rm LMC}$. Performing the integral numerically we find a crossing rate:

$$0.97\frac{v^3}{\sqrt{GMR_{\rm LMC}}} \sim 6\times 10^{-14}\times\left(\frac{M}{M_\odot}\right)^{-1/2}\text{events/s} \sim 2\times 10^{-6}\times\left(\frac{M}{M_\odot}\right)^{-1/2}\text{events/year.}$$

So if we can monitor a million stars, we expect to see a few crossing events per year.