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GGI Lectures on the Theory of Fundamental Interactions

Kerr-Newman black hole stress tensor from QFT

Jung-Wook Kim (Seoul National University)

Work in collaboration with; Ming-Zhi Chung, Yu-Tin Huang (arXiv:1911.12775 [hep-th])

The “punchline” of the talk

All that is gold does not glitter,
Not all those who wander are lost;

J.R.R. Tolkien

Not all loop contributions are quantum!

Motivations

- Non-relativistic general relativity(NRGR) from QFT
  - Uncountable references!(Iwasaki 1971, Gupta & Radford 1979, Feinberg & Sucher 1988, Donoghue 1994, Holstein & Ross 2008, Neill & Rothstein 2013, Vaidya 2014, Bjerrum-Bohr, Damgaard, Festuccia, Plante & Vanhove 2018, Bern, Cheung, Roiban, Shen, Solon, Zeng 2019, etc.)
  - Works because some loop contributions are actually classical!(Holstein & Donoghue 2004)
- "Aren't loops quantum effects?": A counterexample
  - Stress tensor form factor $< p_2|T_{\mu\nu}(0)|p_1 >$ of scalar QED(Donoghue & Holstein 2001, 2004)

$$\begin{aligned}
T_{00}(\vec{r}) &= \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \left(m - \frac{\alpha\pi|\vec{q}|}{8} - \frac{\alpha\vec{q}^2}{3\pi m}\log\vec{q}^2 \right) + \ldots \\
&= m\delta^3(\vec{r}) + \frac{\alpha}{8\pi r^4} - \frac{\alpha\hbar}{\pi^2 m r^5} + \ldots \\
T_{0i}(\vec{r}) &= 0 \\
T_{ij}(\vec{r}) &= \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} (q_i q_j - \delta_{ij}\vec{q}^2) \left(\frac{\alpha\pi}{16|\vec{q}|} + \frac{\alpha}{6\pi m}\log\vec{q}^2 \right) + \ldots \\
&= -\frac{\alpha}{4\pi r^4}\left(\frac{r_i r_j}{r^2} - \frac{1}{2}\delta_{ij} \right) - \frac{\alpha\hbar}{3\pi^2 m r^5}\delta_{ij} + \ldots
\end{aligned}$$

$$q = p_2 - p_1$$

Motivations

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$q = p_2 - p_1$

No powers in Planck's constant: classical effect!

Stress tensor from EM field sourced by the charged particle

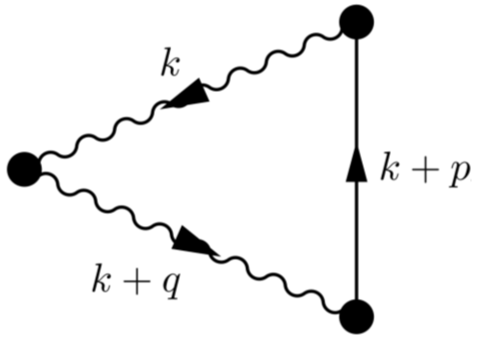
Motivations

- Source of "classicality": mass! (Holstein & Donoghue 2004)
  - Klein-Gordon equation: $\left(\Box + m^2/\hbar^2\right)\phi(x) = 0$
  - The following non-analytic combination carries extra inverse $\hbar$: 1-loop $\hbar$ cancelled!

$$\sqrt{\frac{m^2}{-q^2}} = \frac{m}{\hbar\sqrt{-q^2}}$$

  - The relevant integral is the scalar triangle integral

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2(k+q)^2((k+p)^2-m^2)} = \frac{-i}{32m^2}\frac{m}{\sqrt{-q^2}} + (\text{irrelevant})$$



(Holstein & Ross 2008)

Motivations

- The scalar QED 1-loop form factor generalises to spinning cases!
  - Up to linear order in spin: Spinor QED (Donoghue & Holstein 2001) and Spin-1 QED (Holstein 2006)
  - Extension to higher orders stymied by ineffective tools
    - Dirac spinors, polarisation tensors, Feynman diagrams, etc.
    - Spin-s particle only has up to 2^{2s}-multipoles($= (S^{\mu})^{2s}$ spin order)
- This is a simple computation: Can we go to ***all orders*** in spin?
  - We have better tools!
    - On-shell variables, generalised unitarity, etc.
  - A “testing ground” for computing NRGR higher order spin effects from amplitudes

Problem set-up

- Main strategy
  - Compute 1-loop stress tensor form factor $< p_2|T_{\mu\nu}(0)|p_1 >$
  - Compare with matter stress tensor of the classical theory(GR)
  - Target the simplest spinning charged object: Kerr-Newman BH
- The devil is in the details
  - Only interested in classical contributions: triangle cut, HCL
  - We want ***all orders in spin*** results: access to $s \to \infty$ limit($s\hbar$ fixed $\Leftrightarrow$ $|S^\mu|$ fixed)
  - Form factor in Breit frame matched to classical stress tensor in momentum space

$$p_1 = (E, -\vec{q}/2)$$

$$p_2 = (E, +\vec{q}/2)$$

$$T_{\mu\nu}(\vec{q}) = \int T_{\mu\nu}(\vec{r}) e^{-i\vec{q}\cdot\vec{r}} d^3 r$$

Classical computation

- Kerr-Newman solution in Kerr-Schild coordinates
  - Double copy of QFT generalised to classical solutions (Monteiro, O'Connell, White 2014)
  - Natural representation for comparing QFT computations to classical computations

$$g_{\mu\nu} = \eta_{\mu\nu} - f k_\mu k_\nu$$

$$f = \frac{Gr^2}{r^4 + a^2 z^2}[2Mr - Q^2]$$

$$k_\mu = \left(1, \frac{rx + ay}{r^2 + a^2}, \frac{ry - ax}{r^2 + a^2}, \frac{z}{r}\right)$$

$$A_\mu = \frac{Qr^3}{r^4 + a^2 z^2} k_\mu$$

$$1 = \frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2}$$

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Fall-off $\sim R^{-1}$: tree-level effect

Fall-off $\sim R^{-2}$: 1-loop effect

$$1 = \frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2}$$

Classical computation

- Extracting electromagnetic source current information
  - Maxwell field not in Lorenz gauge; gauge transform!
  - Covariantise by introducing velocity 4-vector $u^\mu \partial_\mu = \partial_t$

$$A'^\mu = \left(u^\mu \cos(a \cdot \partial) + \epsilon^{\mu\nu\alpha\beta} u_\nu a_\alpha \partial_\beta \frac{\sin(a \cdot \partial)}{a \cdot \partial} \right) \frac{Q}{R}$$

  - Obtain expression for the source current

$$j^\mu = Q \int ds \left[u^\mu \cos(a \cdot \partial) + \epsilon^{\mu\nu\alpha\beta} u_\nu a_\alpha \partial_\beta \frac{\sin(a \cdot \partial)}{a \cdot \partial} \right] \delta^4 \left[x - x_{\mathrm{wl}}(s) \right]$$

  - Source term as interaction Hamiltonian → 3pt amplitude: $S_{int} = -4\pi \int d^4x A_\mu j^\mu$

$$M_s^\eta = \epsilon^*(\mathbf{2}) \left[\frac{4\pi Q x^\eta}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\eta \frac{q \cdot S}{m} \right)^n \right] \epsilon(\mathbf{1})$$

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  - Source term as interaction Hamiltonian → 3pt amplitude: $S_{int} = -4\pi \int d^4x A_\mu j^\mu$
    - Minimal coupling argued based on double copy (Arkani-Hamed, Huang, O'Connell 2019)

$$M_s^\eta = \epsilon^*(\mathbf{2}) \left[\frac{4\pi Q x^\eta}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{\boxed{1}}{n!} \left(-\eta \frac{q\cdot S}{m} \right)^n \right] \epsilon(\mathbf{1})$$

Unity Wilson coefficients: minimal coupling!

(Chung, Huang, JWK, Lee 2018, Chung, Huang, JWK 2019)

γ helicity $\eta = \pm 1$

$$M_s^{+1} = \frac{4\pi Q x}{\sqrt{2}} \left(\frac{\langle \mathbf{21} \rangle}{m} \right)^{2s}$$

$$M_s^{-1} = \frac{4\pi Q}{\sqrt{2} x} \left(\frac{[\mathbf{21}]}{m} \right)^{2s}$$

Classical computation

- Stress tensor from electromagnetic fields
  - Target: $T_{\mu\nu} = -\frac{1}{4\pi} F_{\mu\lambda} F_{\nu}{}^{\lambda} + \frac{1}{16\pi} \eta_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}$
  - Computation simplified by holomorphic electromagnetic fields

$$\vec{H} = \vec{E} - i\vec{B} = -\vec{\nabla} e^{i\vec{a}\cdot\vec{\nabla}} \frac{Q}{R} = -\vec{\nabla} \frac{Q}{\sqrt{x^2 + y^2 + (z + ia)^2}} = -\vec{\nabla} Q f(R)$$

  - Fourier transform to momentum space: $T_{\mu\nu}(\vec{q}) = \int T_{\mu\nu}(\vec{r}) e^{-i\vec{q}\cdot\vec{r}} d^3 r$

$$\boxed{\begin{aligned} \frac{8\,T_{\mu\nu}}{Q^2 \pi \sqrt{-q^2}} &= -u_\mu u_\nu J_0(\vec{a}\times\vec{q}) + \left(-u_\mu u_\nu + \frac{q_\mu q_\nu - q^2 \eta_{\mu\nu}}{-q^2} + 2i u_{(\mu} E_{\nu)} \right) \left[\frac{J_1(\vec{a}\times\vec{q})}{\vec{a}\times\vec{q}} \right] \\ &\quad + E_\mu E_\nu \left[\frac{J_2(\vec{a}\times\vec{q})}{(\vec{a}\times\vec{q})^2} \right] \end{aligned}}$$

$$u^\mu = \frac{P^\mu}{m} = (1, \vec{0}) \,, \quad E^\mu = -\frac{1}{m^2} \epsilon^{\mu\nu\lambda\sigma} P_\nu S_\lambda q_\sigma = (0, \vec{a}\times\vec{q}) \qquad q^2 = \eta_{\mu\nu} q^\mu q^\nu = -(\vec{q})^2$$

Quantum field theory computation

- Construct a framework with accessible classical spin limit
  - Stress tensor form factor basis for $s \to \infty$ limit($s\hbar$ fixed $\Leftrightarrow$ $|S^{\mu}|$ fixed)
  - 3pt amplitude in the residue integral representation
- Perform standard QFT computations using modern tools
  - On-shell variables
  - Generalised unitarity
- Compare with classical computation results
  - We have a match!(up to normalisation due to different units)

Approaching the $s \to \infty$ limit

- The form factor basis is usually constructed on a case by case basis
  - Scalar (Donoghue & Holstein 2001)

$$< p_2|T_{\mu\nu}(x)|p_1 >= \frac{e^{i(p_2-p_1)\cdot x}}{\sqrt{4E_2E_1}}\left[2P_\mu P_\nu F_1(q^2) + (q_\mu q_\nu - g_{\mu\nu}q^2)F_2(q^2)\right]$$

  - Spinor (Donoghue & Holstein 2001)

$$< p_2|T_{\mu\nu}|p_1 > \;=\; \bar{u}(p_2)\left[F_1(q^2)P_\mu P_\nu \frac{1}{m} - F_2(q^2)(\frac{i}{4m}\sigma_{\mu\lambda}q^\lambda P_\nu + \frac{i}{4m}\sigma_{\nu\lambda}q^\lambda P_\mu) + F_3(q^2)(q_\mu q_\nu - g_{\mu\nu}q^2)\frac{1}{m}\right]u(p_1)$$

  - Vector (Holstein 2006)

$$\begin{aligned}
< p_2, \epsilon_B|T_{\mu\nu}(x)|p_1, \epsilon_A >_{S=1} = -\frac{e^{i(p_2-p_1)\cdot x}}{\sqrt{4E_1E_2}}[2P_\mu P_\nu \epsilon_B^* \cdot \epsilon_A F_1(^{(S=1)}q^2) \quad &+ \quad (q_\mu q_\nu - \eta_{\mu\nu}q^2)\epsilon_B^* \cdot \epsilon_A F_2^{(S=1)}(q^2) \\
&+ \quad [P_\mu(\epsilon_{B\nu}^*\epsilon_A \cdot q - \epsilon_{A\nu}\epsilon_B^* \cdot q) + P_\nu(\epsilon_{B\mu}^*\epsilon_A \cdot q - \epsilon_{A\mu}\epsilon_B^* \cdot q)]F_3^{(S=1)}(q^2) \\
&+ \quad \left[(\epsilon_{A\mu}\epsilon_{B\nu}^* + \epsilon_{B\mu}^*\epsilon_{A\nu})q^2 - (\epsilon_{B\mu}^* q_\nu + \epsilon_{B\nu}^* q_\mu)\epsilon_A \cdot q\right. \\
&- \quad \left.(\epsilon_{A\mu}q_\nu + \epsilon_{A\nu}q_\mu)\epsilon_B^* \cdot q + 2\eta_{\mu\nu}\epsilon_A \cdot q\epsilon_B^* \cdot q\right] F_4^{(S=1)}(q^2) \\
&+ \quad \frac{2}{m^2}P_\mu P_\nu \epsilon_A \cdot q\epsilon_B^* \cdot qF_5^{(S=1)}(q^2) \\
&+ \quad \frac{1}{m^2}(q_\mu q_\nu - \eta_{\mu\nu}q^2)\epsilon_A \cdot q\epsilon_B \cdot qF_6^{(S=1)}(q^2)]
\end{aligned}$$

Approaching the $s \to \infty$ limit

- The form factor bases look different from spin to spin
  - We cannot go on forever!
  - We need to find a better form factor basis for the $s \to \infty$ limit

- We already have the classical solution!
  - We can construct a basis for the form factor based on classical stress tensor
  - Our suggestion for the form factor basis:

$$\langle p_2 | T_{\mu\nu}(q) | p_1 \rangle = \frac{1}{\sqrt{4E_1E_2}} \left[F_1 P_\mu P_\nu + 2F_2 P_{(\mu} E_{\nu)} + F_3 (q_\mu q_\nu - \eta_{\mu\nu} q^2) + F_4 E_\mu E_\nu \right]$$

$$P^\mu = \frac{p_1^\mu + p_2^\mu}{2} \qquad E^\mu = -\frac{1}{m^2} \epsilon^{\mu\nu\lambda\sigma} P_\nu S_\lambda q_\sigma$$

Approaching the $s \to \infty$ limit

- Usual representations for 3pt amplitudes are not useful for $s \to \infty$ limit
  - Traditional polarisation tensor expressions
    - Expression not simple, extracting information of classical spin vector requires work
  - EFT representation of 3pt amplitudes

$$M_s^{2\eta} = \varepsilon_2^{*\{I_s\}} \left[\sum_{n=0}^{2s} \frac{\kappa m x^{2\eta}}{2} \frac{C_{S^n}}{n!} \left(-\eta \frac{q \cdot S}{m} \right)^n \right] \varepsilon_{1\{J_s\}} \qquad \sum_I \varepsilon_I^{*,\{\mu\}}(p_2) \varepsilon_I^{\{\nu\}}(p_1) = ?$$

    - Spin operators → classical spin vectors, sum over states subtle
  - Massive spinor helicity variables
    - Simple expression, easier sum over states
    - Extracting information of classical spin vector requires work

$$M_s^{+1} = \frac{4\pi Q x}{\sqrt{2}} \left(\frac{\langle \mathbf{21} \rangle}{m} \right)^{2s} \qquad M_s^{-1} = \frac{4\pi Q}{\sqrt{2} x} \left(\frac{[\mathbf{21}]}{m} \right)^{2s}$$

Approaching the $s \to \infty$ limit

- The residue integral representation of 3pt amplitudes
  - Introduced by us in [arXiv:1908.08463]

$$M_s^\eta = M_{s=0}^\eta \oint \frac{dz}{2\pi i z}\left(\sum_{n=0}^{\infty} C_{S^n} z^n\right)\left(\bar{u}(P_2)u(P_1) - \eta \frac{q \cdot S_{1/2}}{mz}\right)^{2s}$$

$$= M_{s=0}^\eta \oint \frac{dz}{2\pi i z}\left(\sum_{n=0}^{\infty} C_{S^n} z^n\right)\left(\frac{[\mathbf{21}] - \langle \mathbf{21}\rangle}{2m} + \frac{\eta}{z}\frac{[\mathbf{2}|q|\mathbf{1}\rangle + \langle \mathbf{2}|q|\mathbf{1}]}{4m^2}\right)^{2s}$$

$\eta = \pm 1$ helicity sign

  - Expresses spin-s 3pt as $2s$ copies of spin-$\frac{1}{2}$ factors attached to scalar amplitude
    - General compact body version of "spin exponentiation" (Arkani-Hamed, Huang, O'Connell 2019)
  - Directly expresses the amplitude in terms of the spin vector $S^\mu_{1/2} = \frac{S^\mu}{2s}$
  - Cut computation easier: $(\sum \text{spin}-s \text{ internal states}) = (\sum \text{spin}-\frac{1}{2} \text{ internal states})^{2s}$

Quantum field theory computation

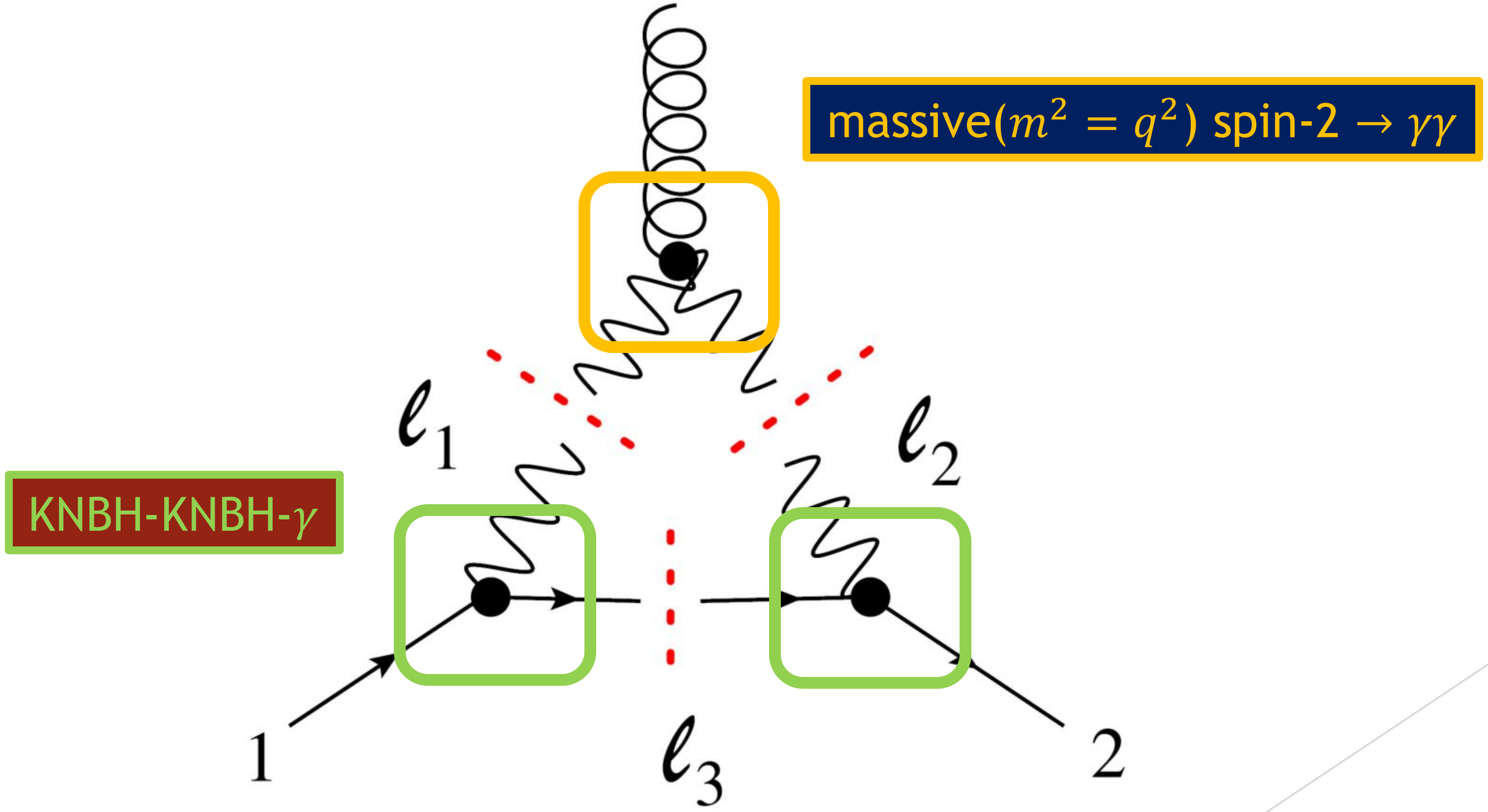
- Modern method of doing 1-loop computations
  - Standard approach: reduction to scalar integrals (Passarino & Veltman 1979, 't Hooft & Veltman 1979, van Neerven & Vermaseren 1984, Ossola, Papadopoulos & Pittau 2006)

$$M_n(p_n) = \sum_{I,i} c_{I,i}(p_n) I_{I,i}(p_n) \qquad I_I = \frac{\mu^{4-D}}{i\pi^{D/2} r_\Gamma} \int d^D l \prod_{j=1}^{I} \frac{1}{(l+q_{j-1})^2 - m_j^2 + i\varepsilon}$$

  - Computing the scalar integral coefficients
    - Branch cut discontinuities related to lower perturbation order amplitudes (Cutkosky 1960)
    - A QFT version of the optical theorem
    - We can compute amplitudes by matching discontinuities: ***Generalised unitarity***! (Bern, Dixon, Dunbar & Kosower 1994, Britto, Cachazo & Feng 2004, Forde 2007, Kilgore 2007, Badger 2008, Mastrolia 2009)
- We only need the *triangle scalar integral* coefficient
  - We compute the triangle unitarity cut!

Quantum field theory computation

- Coefficient for scalar triangle integral: unitarity cuts



Quantum field theory computation

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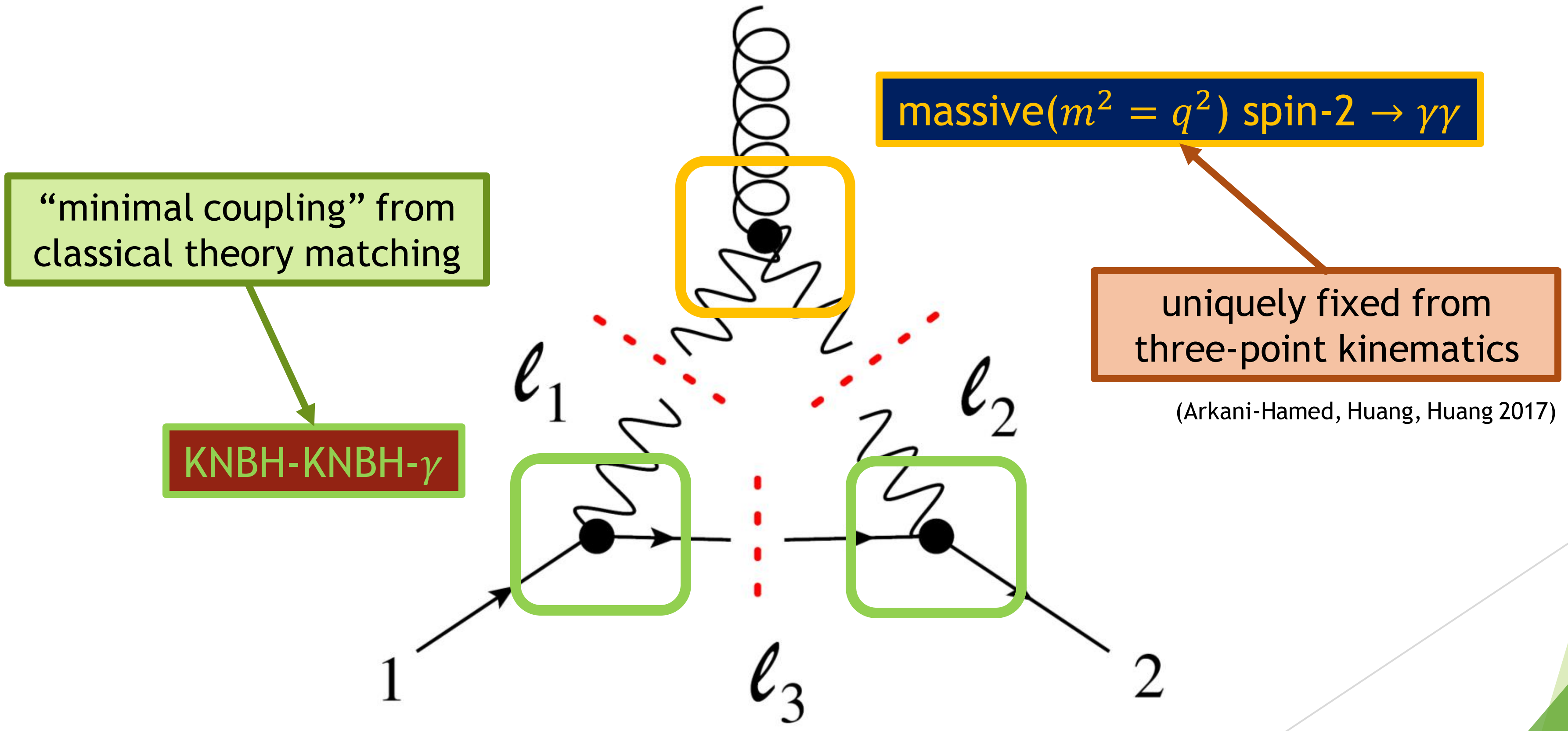

(Arkani-Hamed, Huang, Huang 2017)

Quantum field theory computation

- Write down the cut integrand

$$\frac{1}{4}\sum_{h_1,h_2=\pm 1}\int_{\Gamma_{LS}} d^4L\delta(L^2-m^2)\delta(k_3^2)\delta(k_4^2)$$
$$\times M_3(q,-k_3^{h_1},-k_4^{h_2})\times M_3(1^s,k_3^{-h_1},-L)\times M_3(-2^s,k_4^{-h_2},L)$$

- The amplitude product (KNBH–KNBH–γ)^2 requires following residue integration

$$\oint\frac{dz_1}{2\pi i z_1}\frac{dz_2}{2\pi i z_2}\left(\sum_{n=0}^{\infty}z_1^n\right)\left(\sum_{n=0}^{\infty}z_2^n\right)F(-\mathbf{2},k_4,k_3,\mathbf{1})^{2s}$$

  - Looks simple, but the expression F has too many terms(naïve counting: $4\times 4=16$)
  - Need another tool to reduce computation cost!

Quantum field theory computation

- Holomorphic classical limit (Guevara 2017)
  - We are only interested in long-range effects: study $q^2 = 0$ but $q \neq 0$
  - Computation of cut integrand drastically simplified

$$F(-\mathbf{2}, k_4, k_3, \mathbf{1}) = \left(\frac{[\mathbf{2L}] - \langle \mathbf{2L} \rangle}{2m} + \frac{\eta_1}{z_1} \frac{[\mathbf{2}|k_4|\mathbf{L}\rangle + \langle \mathbf{2}|k_4|\mathbf{L}]}{4m^2} \right) \times \left(\frac{[\mathbf{L1}] - \langle \mathbf{L1} \rangle}{2m} + \frac{\eta_2}{z_2} \frac{[\mathbf{L}|k_3|\mathbf{1}\rangle + \langle \mathbf{L}|k_3|\mathbf{1}]}{4m^2} \right)$$

$$= \bar{u}(2)u(1) - \left(\frac{\eta_1}{z_1} k_4^\mu + \frac{\eta_2}{z_2} k_3^\mu \right) S^\mu_{1/2} + \frac{k_3^\mu}{2m} \left(\frac{p_1^\mu + p_2^\mu}{2m} \bar{u}(2)u(1) - \frac{q^\nu}{2m} \bar{u}(2)\gamma_{\mu\nu}u(1) \right)$$

$$- \left(\frac{\eta_1}{z_1} + \frac{\eta_2}{z_2} \right) \left(\frac{q^2}{8m^2} \bar{u}(2)\gamma^5 u(1) + \frac{k_4^\mu k_3^\nu}{4m^2} \bar{u}(2)\gamma_{\mu\nu}\gamma^5 u(1) \right)$$

$$- \frac{\eta_1 \eta_2}{z_1 z_2} \left(\frac{q^2}{8m^2} \bar{u}(2)u(1) + \frac{k_4^\mu k_3^\nu}{4m^2} \bar{u}(2)\gamma_{\mu\nu}u(1) \right)$$

  - Some information will be lost: $(\vec{a} \times \vec{q})^2 = |\vec{a}|^2 |\vec{q}|^2 - (\vec{a} \cdot \vec{q})^2 \overset{\text{HCL}}{\rightarrow} -(\vec{a} \cdot \vec{q})^2$

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$$
\begin{aligned}
F(-\mathbf{2},k_4,k_3,\mathbf{1}) &= \left(\frac{[\mathbf{2L}]-\langle\mathbf{2L}\rangle}{2m}+\frac{\eta_1}{z_1}\frac{[\mathbf{2}|k_4|\mathbf{L}\rangle+\langle\mathbf{2}|k_4|\mathbf{L}]}{4m^2}\right)\times\left(\frac{[\mathbf{L1}]-\langle\mathbf{L1}\rangle}{2m}+\frac{\eta_2}{z_2}\frac{[\mathbf{L}|k_3|\mathbf{1}\rangle+\langle\mathbf{L}|k_3|\mathbf{1}]}{4m^2}\right)\\
&= \boxed{\bar{u}(2)u(1)-\left(\frac{\eta_1}{z_1}k_4^\mu+\frac{\eta_2}{z_2}k_3^\mu\right)S^\mu_{1/2}}+\frac{k_3^\mu}{2m}\left(\frac{p_1^\mu+p_2^\mu}{2m}\bar{u}(2)u(1)-\frac{q^\nu}{2m}\bar{u}(2)\gamma_{\mu\nu}u(1)\right)\\
&\quad-\left(\frac{\eta_1}{z_1}+\frac{\eta_2}{z_2}\right)\left(\frac{q^2}{8m^2}\bar{u}(2)\gamma^5u(1)+\frac{k_4^\mu k_3^\nu}{4m^2}\bar{u}(2)\gamma_{\mu\nu}\gamma^5u(1)\right)\\
&\quad-\frac{\eta_1\eta_2}{z_1z_2}\left(\frac{q^2}{8m^2}\bar{u}(2)u(1)+\frac{k_4^\mu k_3^\nu}{4m^2}\bar{u}(2)\gamma_{\mu\nu}u(1)\right)
\end{aligned}
$$

Only these terms survive in the HCL

  - Some information will be lost: $(\vec{a}\times\vec{q})^2 = |\vec{a}|^2|\vec{q}|^2-(\vec{a}\cdot\vec{q})^2 \overset{\text{HCL}}{\rightarrow} -(\vec{a}\cdot\vec{q})^2$

Quantum field theory computation

- Take $s \to \infty$ limit at integrand level and perform the integral!
- The result

$$\boxed{\begin{aligned}\langle p_2|T_{\mu\nu}|p_1\rangle = \frac{|\vec{q}|}{32}\frac{\alpha_q^2}{M_{pl}}\Bigg\{ & -\frac{P_\mu P_\nu}{m^2}\left[\frac{I_1(a\cdot q)}{(a\cdot q)} + I_0(a\cdot q)\right] + 2i\frac{P_{(\mu}E_{\nu)}}{m}\frac{I_1(a\cdot q)}{(a\cdot q)} \\ & -\frac{q_\mu q_\nu - \eta_{\mu\nu}q^2}{q^2}\frac{I_1(a\cdot q)}{(a\cdot q)} + E_\mu E_\nu \frac{I_2(a\cdot q)}{(a\cdot q)^2}\Bigg\}\end{aligned}}$$

$$\frac{J_n(\vec{a}\times\vec{q})}{(\vec{a}\times\vec{q})^n} \overset{\text{HCL}}{\to} \frac{I_n(a\cdot q)}{(a\cdot q)^n}$$

  - Compare with classical computations!

$$\boxed{\begin{aligned}\frac{8T_{\mu\nu}}{Q^2\pi\sqrt{-q^2}} = & -u_\mu u_\nu J_0(\vec{a}\times\vec{q}) + \left(-u_\mu u_\nu + \frac{q_\mu q_\nu - q^2\eta_{\mu\nu}}{-q^2} + 2iu_{(\mu}E_{\nu)}\right)\left[\frac{J_1(\vec{a}\times\vec{q})}{\vec{a}\times\vec{q}}\right] \\ & + E_\mu E_\nu\left[\frac{J_2(\vec{a}\times\vec{q})}{(\vec{a}\times\vec{q})^2}\right]\end{aligned}}$$

  - Normalisation: different unit systems

Conclusions

- "Not all loop contributions are quantum!"
- An explicit example to classical loop computation: stress tensor form factor
  - Already shown to reproduce classical linear in spin effects
  - New result: classical spin effects can be reproduced to ***all orders*** in spin
- A "testing ground" for spin effects in NRGR from QFT
  - Existing examples of NRGR from QFT were limited to spinless or low spin orders
  - This is an explicit example of classical physics from QFT to all orders in spin
  - We can have confidence that NRGR from QFT to all orders in spin is possible!

Further remarks

- "Can we reproduce effects of gravitons running in the loop?"
  - Essentially same computations
  - Demonstrated up to linear order in spin (Bjerrum-Bohr, Donoghue, Holstein 2002, Holstein 2006)
  - Obstruction for on-shell techniques: Weinberg-Witten theorem
  - Stress tensor is no longer conserved → covariantly conserved: $\partial_\mu T^{\mu\nu} \neq 0\,,\ \nabla_\mu T^{\mu\nu} = 0$
  - Stress tensor not a gauge-invariant observable
- A possible resolution: probe particle scattering problem
  - Study the $2 \to 2$ scattering problem of probe particle on KBH instead
  - Equivalent to computation of the classical potential in NRGR
  - Obstructed by non-local Compton amplitudes (Chung, Huang, JWK, Lee 2018)

Thank you for listening!