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Higher-Order Tree-Level Amplitudes in the Nonlinear Sigma Model

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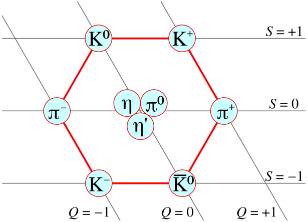
The problem of low-energy QCD

- QCD (Quantum ChromoDynamics)
  - AKA the strong (nuclear) force
  - AKA the colour force

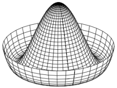
  describes interactions of quarks and gluons.
- High-energy QCD is nice and perturbative — much like QED (Quantum ElectroDynamics).
- Low-energy QCD is not — our ordinary tools are useless.
- Confinement arises, composites are formed $(\pi, K, \eta, \ldots)$ that act as "almost elementary"



- We will model their behaviour at tree-level.

Spontaneous symmetry breaking

- Consider symmetry breaking, e.g.

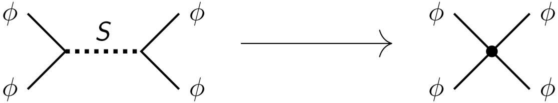
$\mathcal{L} \quad \Rightarrow$ 

- Decompose the fields

$$\Sigma = [v + S] \exp\left(\frac{it^a \phi^a}{v}\right) = [v + S] U(\phi)$$

ϕ^a — rotations in minimum (massless),
v — vacuum expectation value.
S — excitation from minimum (massive),

- At low energies, S becomes hidden



- Resulting theory for ϕ is an effective field theory (EFT)

The general nonlinear sigma model

- Symmetry groups broken $G \to H$.
- Resulting Nambu-Goldstone bosons $\phi = (\phi_1, \phi_2, \ldots)$ tranform under $g \in G$ like

$$\phi \xrightarrow{g} \mathcal{G}_g(\phi), \qquad \mathcal{G}_{g_1}(\mathcal{G}_{g_2}(\phi)) = \mathcal{G}_{g_1 g_2}(\phi), \qquad \mathcal{G}_e(\phi) = \phi$$

- H preserves the vacuum ϕ_0.
  Any other ϕ is found from ϕ_0 via G:

$$\phi = \mathcal{G}_g(\phi_0) = \mathcal{G}_{gh}(\phi_0) \qquad \forall h \in H$$

- This gives a one-to-one correspondence

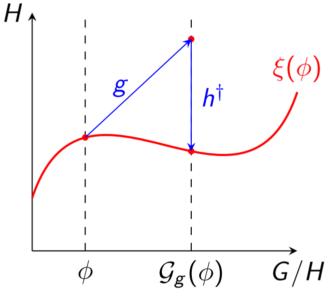
$$\phi \leftrightarrow gH, \qquad gH = \{gh \mid h \in H\} \in G/H$$

- From each such *coset*, pick a representative

$$\phi \to gH \to \xi(\phi) \in gH$$

  to use instead of ϕ.

Coset representative transformations



- $\xi(\phi) \xrightarrow{g} g\,\xi(\phi)$
- But $g\,\xi(\phi) \neq \xi\big(\mathcal{G}_g(\phi)\big)$ in general.
- Compensating transformation: $h\big(\xi(\phi), g\big) \in H$

- Therefore, representatives transform like

$$\xi(\phi) \xrightarrow{g} g\,\xi(\phi) h^\dagger\big(\xi(\phi), g\big).$$

The chiral nonlinear sigma model

- Massless QCD Lagrangian:

$$\mathcal{L}_{\text{QCD}} = i\bar{q}_L\gamma^\mu D_\mu q_L + i\bar{q}_R\gamma^\mu D_\mu q_R - \frac{1}{4}G^a_{\mu\nu}G_a^{\mu\nu}$$

- Invariant under $G = SU(N_f)_L \times SU(N_f)_R$ acting on the N_f quark flavours.
- In low-energy limit, G breaks down to just $H = SU(N_f)_V$.
- $N_f^2 - 1$ Nambu-Goldstone bosons — the light scalar mesons:
  - $N_f = 2$: $\pi^\pm, \pi^0$
  - $N_f = 3$: $\pi^\pm, \pi^0, K^\pm, K^0, \bar{K}^0, \eta$
- Governed by the nonlinear sigma model (NLSM) under G/H.

Building the Lagrangian

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- Choose chiral coset representative $\xi(\phi) = \big(\underbrace{u(\phi)}_{R}, \underbrace{u(\phi)^\dagger}_{L}\big)$.
- Parametrise
$$u(\phi) = \exp\left(\frac{it^a\phi^a}{2F}\right)$$
with t^a generating $SU(N_f)$ and F a constant.
- NLSM Lagrangian consists of all terms constucted from u (respecting symmetries, Lorentz invariance, etc.)
- Instead of u, it is more convenient to use
$$u^2 \equiv U \xrightarrow{g} g_R U g_L^\dagger, \qquad g = (g_R, g_L)$$
- ...and even better to use
$$i\left(u^\dagger \partial_\mu u - u\partial_\mu u^\dagger\right) \equiv u_\mu \xrightarrow{g} h\, u_\mu\, h^\dagger$$
- ...which supports nice covariant derivatives
$$\nabla_\mu u_\nu \xrightarrow{g} h\, \nabla_\mu u_\nu\, h^\dagger, \qquad \nabla_\mu u_\nu = \nabla_\nu u_\mu$$

The NLSM Lagrangian

- Organise the Lagrangian by power-ordering. Each derivative brings down $\mathcal{O}(p)$:

$$\mathcal{L} = \sum_{n=1}^{\infty} \mathcal{L}_{2n}, \qquad \mathcal{L}_{2n} \sim \mathcal{O}(p^{2n})$$

- The simplest term (2 derivatives, $\mathcal{O}(p^2)$) is

$$\mathcal{L}_2 = \frac{F^2}{4}\langle \partial_\mu U^\dagger \partial^\mu U \rangle = \frac{F^2}{4}\langle u_\mu u^\mu \rangle$$

- With 4 derivatives, $\mathcal{O}(p^4)$, the possible terms are

$$\mathcal{L}_4 = L_0 \langle u_\mu u_\nu u^\mu u^\nu \rangle + L_1 \langle u_\mu u^\mu \rangle \langle u_\nu u^\nu \rangle + L_2 \langle u_\mu u_\nu \rangle \langle u^\mu u^\nu \rangle + L_3 \langle u_\mu u^\mu u_\nu u^\nu \rangle$$

- L_i are unknown "low-energy constants" (LECs).
- In QCD, LECs can not be calculated — need lattice or experiments.

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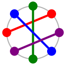
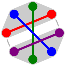
Chord diagrams for more derivatives

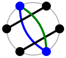
- Terms like "$\langle u^\mu u^\nu u^\rho u_\mu u^\sigma u_\nu u_\rho u_\sigma \rangle$" are not very readable.
- Trace is cyclic — draw something round:

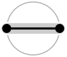
$$\langle u^\mu u^\nu u^\rho u_\mu u^\sigma u_\nu u_\rho u_\sigma \rangle \to$$ 

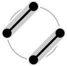
- More than one trace — split the circle:

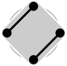
$$\langle u^\mu u^\nu u^\rho u_\mu \rangle \langle u^\sigma u_\nu u_\rho u_\sigma \rangle \to$$ 

- Covariant derivatives — add more lines:

$$\langle u^\mu \nabla^\nu u^\rho u_\mu u^\sigma \nabla_\nu u_\rho u_\sigma \rangle \to$$ 

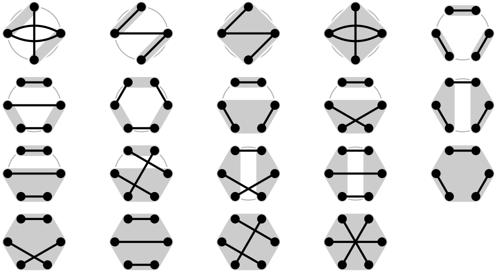
- The Lagrangian is now $\langle u_\mu u^\mu \rangle \to$ ,

$L_0 \langle u_\mu u_\nu u^\mu u^\nu \rangle \to L_0$ , $\quad L_1 \langle u_\mu u^\mu \rangle \langle u_\nu u^\nu \rangle \to L_1$ ,

$L_2 \langle u_\mu u_\nu \rangle \langle u^\mu u^\nu \rangle \to L_2$ , $\quad L_3 \langle u_\mu u^\mu u_\nu u^\nu \rangle \to L_3$ ,

The $\mathcal{O}(p^6)$ Lagrangian

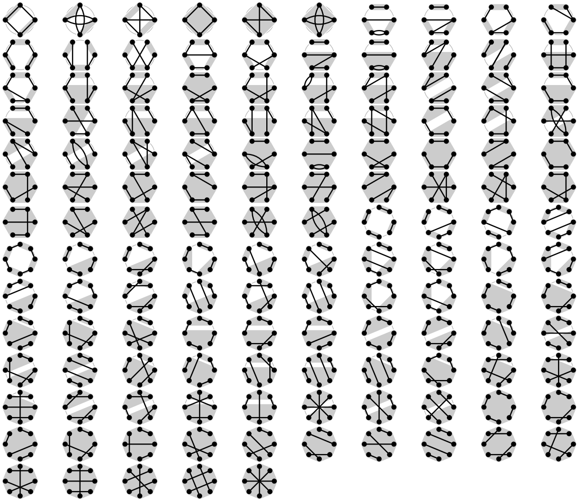
- First version by Bijnens, Colangelo & Ecker (1999)
- This version by Bijnens, Hermansson-Truedsson & Wang (2018) (unpublished)



- Each term has an independent LEC.

The $\mathcal{O}(p^8)$ Lagrangian

- By Bijnens, Hermansson-Truedsson & Wang (2019)



Restrictions on the Lagrangian

- For small N_f, some terms are removed by Cayley-Hamilton:

$$N_f = 2 \quad \Rightarrow \quad AB + BA = \langle AB \rangle, \qquad N_f = 3 \quad \Rightarrow \quad \ldots$$

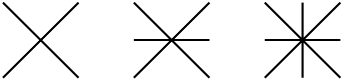
- For small spacetime dimension D, some terms are removed by Schouten:

$$D = 1 \quad \Rightarrow \quad \langle u_\mu u^\mu u_\nu u^\nu \rangle = \langle u_\mu u_\nu u^\mu u^\nu \rangle = \langle u_0 u_0 u_0 u_0 \rangle$$

- Schouten enters above $\mathcal{O}(p^{2D})$
  $\Rightarrow \quad D = 4$ does not affect at $\mathcal{O}(p^8)$.
- Other identities (EOM etc.) affect the general case; see Bijnens, Hermansson-Truedsson & Wang (2019).

Flavour-ordering

- NLSM is perturbative, but difficult to compute in.
- Infinite hierarchy of vertices (only even-legged)



…

- Lots of combinatorics — shortcuts are needed!
- Our main shortcut is "flavour-ordering".
- Based on Kampf, Novotný & Trnka (2013)
- We generalise it to more derivatives and legs.

Power-counting

- Weinberg's power-counting formula:

$$P = 2 + 2L + \sum_d (d-2) N_d$$

  - P: diagram is $\mathcal{O}(p^P)$ overall
  - L: number of loops
  - N_d: number of $\mathcal{O}(p^d)$ vertices
- $\mathcal{O}(p^2)$ vertices are "free".
- $\mathcal{O}(p^4)$ diagrams have a single $\mathcal{O}(p^4)$ vertex.
- $\mathcal{O}(p^6)$ diagrams have one $\mathcal{O}(p^6)$ or two $\mathcal{O}(p^4)$ vertices.
- ...and so on.
- An $\mathcal{O}(p^4)$ vertex can be traded for a loop.

Stripped amplitudes

- We organise amplitudes by group-theoretic factors — flavour structures.
- An $\mathcal{O}(p^m)$ n-point amplitude is decomposed as

$$\mathcal{M}_m^{a_1\dots a_n}(p_1,\dots,p_n) = \sum_{\sigma,R} \mathcal{F}_\sigma(R)\mathcal{M}_{m,R}^{\sigma}(p_1,\dots,p_n)$$

  a_i — flavour indices,
  σ — permutation of $\{1,\dots,n\}$.

- $\mathcal{F}_\sigma(R), R = \{r_1, r_2, \dots\}$ is a flavour structure:

$$\mathcal{F}_\sigma(4) = \langle t^{a_{\sigma(1)}} t^{a_{\sigma(2)}} t^{a_{\sigma(3)}} t^{a_{\sigma(4)}} \rangle,$$
$$\mathcal{F}_\sigma(2,2) = \langle t^{a_{\sigma(1)}} t^{a_{\sigma(2)}} \rangle \langle t^{a_{\sigma(3)}} t^{a_{\sigma(4)}} \rangle, \quad \text{etc.}$$

- We call R a "flavour splitting".

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Stripped amplitudes

- $\mathcal{F}_\sigma(R)$ is symmetric under some group $\mathbb{Z}_R$ acting on σ:

$$\mathbb{Z}_{\{n\}}: \quad \langle\overbrace{1\,2\cdots n}^{\text{cycle}}\rangle, \qquad \mathbb{Z}_{\{2,2,4\}}: \quad \underbrace{\langle\overbrace{1\,2}^{\text{cycle}}\rangle\langle\overbrace{3\,4}^{\text{cycle}}\rangle}_{\text{swap } 12 \leftrightarrow 34}\langle\overbrace{5\,6\,7\,8}^{\text{cycle}}\rangle$$

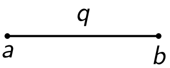
- $\Rightarrow \quad \mathcal{M}^\sigma_{m,R}$ also symmetric under $\mathbb{Z}_R$.
- By Bose symmetry,

$$\mathcal{M}^\sigma_{m,R}(p_1,\ldots,p_n) = \mathcal{M}_{m,R}(p_{\sigma(1)},\ldots,p_{\sigma(n)})$$

where $\mathcal{M}_{m,R} = \mathcal{M}^{\mathrm{id}}_{m,R}$ is *flavour-ordered*.
- $\mathcal{M}_{m,R}$ is the "stripped amplitude" — much easier to calculate, contains all information of the full amplitude.
- Vertex factors can be stripped the same way.

Flavour-ordering

- The NLSM propagator has the form



$$= \frac{i\delta^{ab}}{q^2}$$

- When t^a generate $SU(N_f)$,

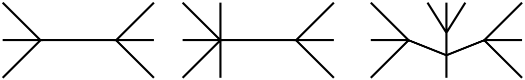
$$\langle X t^a \rangle \delta^{ab} \langle t^b Y \rangle = \langle X t^a \rangle \langle t^a Y \rangle = \langle XY \rangle - \frac{1}{N_f} \langle X \rangle \langle Y \rangle$$

  Propagators *concatenate* flavour structures!
  (up to $\mathcal{O}(N_f^{-1})$ corrections — ignore for now)

- Few contractions contribute to stripped amplitude:

$$\langle 123a \rangle \langle a456 \rangle \to \langle 123456 \rangle \text{ but not } \langle 123a \rangle \langle a645 \rangle \to \langle 123645 \rangle$$

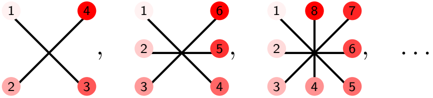
- The reduction is severe:



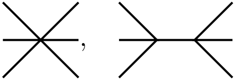
$576 \to 3$ $17\,280 \to 8$ $331\,776 \to 10$

Flavour-ordering in practice, part I

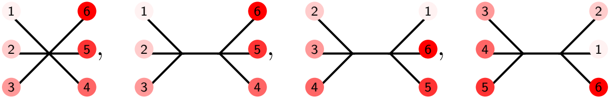
1. Write down vertex factors with legs in cyclic order:



2. Combine into diagrams:



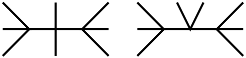
3. Apply $\mathbb{Z}_R$, minus symmetries:



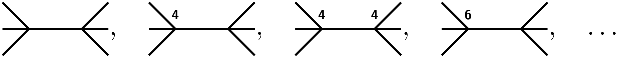
4. Compute the kinematic factor of each diagram. Done!

Generalised flavour-ordering

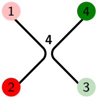
- Diagrams like these are distinct:



- Higher orders introduce new vertices:



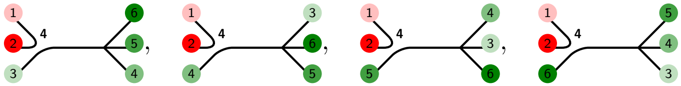
- Also, split flavour structures:



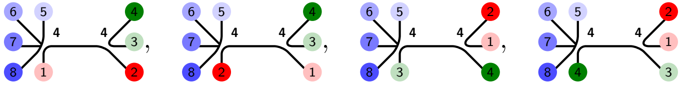
$\rightarrow \langle 12 \rangle \langle 34 \rangle$

Flavour-ordering in practice, part II

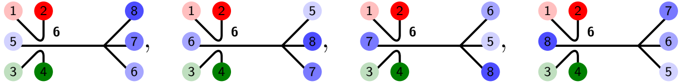
- With flavour splits, separate traces are cycled independently: $\langle 12\rangle\langle 3456\rangle$



- Equal-size traces are swapped: $\langle 12\rangle\langle 34\rangle\langle 5678\rangle$



- . . . or not, if covered by symmetry: $\langle 12\rangle\langle 34\rangle\langle 5678\rangle$



Singlet propagators

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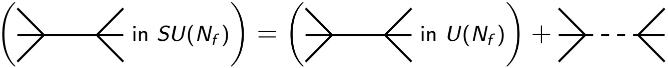
- What about the N_f^{-1} term? It breaks flavour ordering!
- But it doesn't show up in $U(N_f)$:

$$\underbrace{\overbrace{\langle X t^a \rangle \langle t^a Y \rangle = \langle XY \rangle}^{U(N_f)} - \frac{1}{N_f} \langle X \rangle \langle Y \rangle}_{SU(N_f)}$$

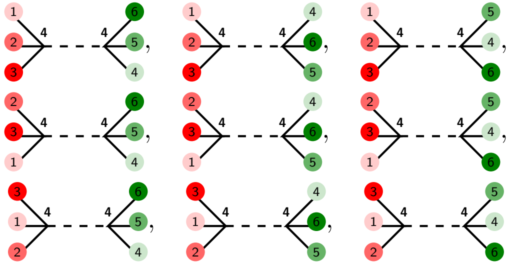
- $U(N_f)$ differs from $SU(N_f)$ by a $U(1)$ singlet $\propto t^0 = \mathbb{1}/\sqrt{N_f}$
- The N_f^{-1} term just subtracts this contribution:



- The extra diagram can be calculated separately in $U(N_f)$.
- Singlet decouples from $SU(N_f)$ at $\mathcal{O}(p^2)$
  $\Rightarrow$ this contributes first at $\mathcal{O}(p^6)$!

Flavour-ordering in practice, part III

- Singlet propagators break symmetry on vertices, so all cyclings are needed: $\langle 123\rangle\langle 456\rangle$



Uniqueness of stripped amplitudes

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- Flavour structures are orthogonal (up to $\mathcal{O}(N^{-2})$ corrections — can be ignored):

$$\mathcal{F}_\sigma(Q)\cdot\left[\mathcal{F}_\rho(R)\right]^* = \frac{N_f^{n-2}(N_f^2-1)}{N_f^{|R|-1}} \times \begin{cases} 1+\mathcal{O}\left(\frac{1}{N_f^2}\right) & \text{if } Q=R \text{ and } \sigma\equiv\rho \pmod{\mathbb{Z}_R}, \\ \mathcal{O}\left(\frac{1}{N_f^2}\right) & \text{if } Q\neq R \text{ or } \sigma\not\equiv\rho \pmod{\mathbb{Z}_R}. \end{cases}$$

- Dot is contraction over all flavour indices.
- Proven by Mangano & Parke (1991), Applied to NLSM by Kampf, Novotný & Trnka (2013), Generalised to multi-trace case by us.
- Stripped amplitudes can be projected out — therefore, unique!

Some more amplitude technology

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Soft behaviour

- For general EFTs, "Adler zeroes" exist

$$\mathcal{M}_n^{a_1 a_2 \cdots a_n}(p_1, \ldots, \varepsilon p_i, \ldots, p_n) \sim \varepsilon^\sigma \quad \text{as } \varepsilon \to 0$$

- $\sigma > 0$ is the "soft degree" — NLSM has $\sigma = 1$
- Uniqueness $\Rightarrow$ Carries over to stripped amplitudes.
- Requires intricate cancellations — useful for error checking.
- Can be used for recursion relations — see Cheung, Kampf, Novotný, Shen & Trnka (2016)
- (Only works at $\mathcal{O}(p^m)$ if $m < n\sigma$)

Soft behaviour

- There is also the "double soft limit"

$$\lim_{\varepsilon\to 0}\mathcal{M}_{m,R}(p_1,\dots,\varepsilon p_i,\varepsilon p_j,\dots,p_{(n+2)})$$
$$=\frac{1}{4F^2}\left(\frac{p_{(j+1)}\cdot(p_i-p_j)}{p_{(j+1)}\cdot(p_i+p_j)}-\frac{p_{(i-1)}\cdot(p_i-p_j)}{p_{(i-1)}\cdot(p_i+p_j)}\right)$$
$$\times\,\mathcal{M}_{m,R'}(p_1,\dots,p_{(i-1)},p_{(j+1)},\dots,p_{(n+2)}),$$

- Remove soft particles, multiply by a kinematic factor.
- One trace must contain $(i-1), i, j, (j+1)$ consecutively — otherwise limit is zero.
- Conjectured by Arkani-Hamed, Cachazo & Kaplan (2010), Proven by Kampf, Novotný & Trnka (2013), Generalised to multi-trace case by us.

Mandelstam variables

- Using momenta directly fails to capture:
  - On-shellness: $p_i^2 = 0$
  - Conservation: $\sum_i p_i = 0$
- With 4 legs, Mandelstam variables solve this:

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2, \qquad u = (p_2 + p_3)^2 = (p_4 + p_1)^2,$$
$$t = (p_1 + p_3)^2 = (p_2 + p_4)^2 = -(s + u)$$

- Can be generalised to n legs. For $n = 6$:

$$s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{61}, \quad s_{123}, s_{234}, s_{345},$$
$$s_{ijk\cdots} = (p_i + p_j + p_k + \ldots)^2$$

- In general, $n(n-3)/2$ variables.
- Fewer if $n \gg D$ (ignore)

n	Invariants
4	2
6	9
8	20
10	35

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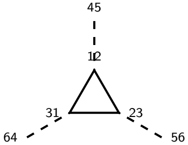
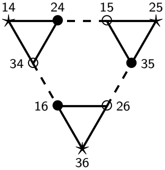
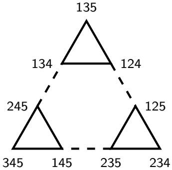
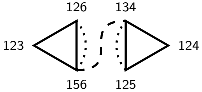
Closed Mandelbases

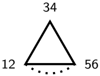
- "Canonical" Mandelstam invariants are not closed under all $\mathbb{Z}_R$
  e.g. $s_{123} \to s_{56} - s_{123} - s_{34} + s_{12}$ under $\mathbb{Z}_{\{2,4\}}$
- Unsolvable under $n > 6$
- Simple under $\mathbb{Z}_{\{2,4\}}$ (Abelian)
- Tricky but solvable under $\mathbb{Z}_{\{3,3\}}$ and $\mathbb{Z}_{\{2,2,2\}}$ (non-Abelian)

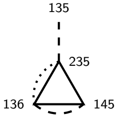


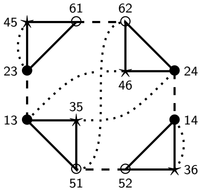










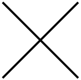


The simplest amplitude: $\mathcal{O}(p^2)$ 4-point

- Vertex factor comes from

$$\mathcal{L}_2 = \frac{1}{F^2}\langle t^a t^b t^c t^d\rangle \left(\frac{1}{6}\phi^a\partial_\mu\phi^b\phi^c\partial^\mu\phi^d - \frac{1}{12}\phi^a\phi^b\partial_\mu\phi^c\partial^\mu\phi^d\right) + \ldots$$

- Stripping leaves single-vertex diagram



$$-iF^2\mathcal{M}_{2,\{4\}} = \frac{t}{2}$$

- Dressing gives the well-known full amplitude

$$\mathcal{M}^{abcd}_{2,4}(s,t,u) = \frac{-4i}{F^2}\left[s\,\delta^{ab}\delta^{cd} + t\,\delta^{ac}\delta^{bd} + u\,\delta^{ad}\delta^{bc}\right]$$

(assuming $N_f = 2$ for simpler group algebra)

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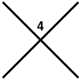
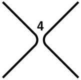
More derivatives: $\mathcal{O}(p^4)$ 4-point

- Two diagrams, singe-trace and two-trace:



$$-iF^4\mathcal{M}_{4,\{4\}} = 2L_3(u^2+s^2) + 4L_0 t^2,$$



$$-iF^4\mathcal{M}_{4,\{2,2\}} = 8L_1 s^2 + 4L_2(t^2+u^2),$$

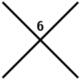
- Dressed amplitude is analogous to $\mathcal{O}(p^2)$:

$$\begin{aligned}
&\mathcal{M}^{abcd}_{4,4}(s,t,u)\\
&= \frac{8i}{F^4}\Big\{ \big[4L_0(u^2+us) + 2L_1 s^2 + L_2(t^2+u^2) + 2L_3 s^2\big]\delta^{ab}\delta^{cd}\\
&\quad + \big[4L_0(s^2+st) + 2L_1 t^2 + L_2(u^2+s^2) + 2L_3 t^2\big]\delta^{ac}\delta^{bd}\\
&\quad + \big[4L_0(t^2+tu) + 2L_1 u^2 + L_2(s^2+t^2) + 2L_3 u^2\big]\delta^{ad}\delta^{bc}\Big\}.
\end{aligned}$$

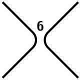
(again with $N_f = 2$)

Higher 4-point amplitudes

- $\mathcal{O}(p^6)$:

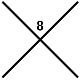


$$-iF^4\mathcal{M}_{6,4} = -L_{6,3}t(s^2+u^2) - 2L_{6,4}t^3$$

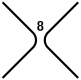


$$\begin{aligned}-iF^4\mathcal{M}_{6,\{2,2\}} = &-2L_{6,1}(t^3+u^3)\\ &+\frac{2}{3}L_{6,2}(s^3+t^3+u^3).\end{aligned}$$

- $\mathcal{O}(p^8)$:



$$-iF^4\mathcal{M}_{8,4} = L_{8,4}s^2u^2 + \frac{1}{2}L_{8,5}t^2(s^2+u^2) + L_{8,6}t^4$$



$$\begin{aligned}-iF^4\mathcal{M}_{8,\{2,2\}} = &L_{8,1}s^2(t^2+u^2) + L_{8,2}(t^4+u^4)\\ &+2L_{8,3}t^2u^2\end{aligned}$$

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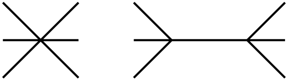
More legs: $\mathcal{O}(p^2)$ 6-point amplitude

- Two familiar diagrams:



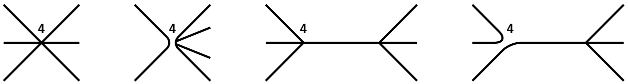
- Stripped amplitude:

$$-4iF^4\mathcal{M}_{2,6} = \left\{ s_{12} - \frac{1}{2}\frac{(s_{12}+s_{23})(s_{45}+s_{56})}{s_{123}} \right\} + [\mathbb{Z}_6]$$

- "$+[\mathbb{Z}_6]$" means sum over cyclic permutations of indices.

The $\mathcal{O}(p^4)$ 6-point amplitude

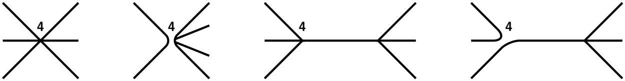
- Four diagrams:



- Single-trace amplitude:

$$
\begin{aligned}
-iF^6\mathcal{M}_{4,6} = L_3\Bigg\{ & s_{12}\left(s_{12}+s_{34}+s_{45}\right) \\
& - \frac{\left(s_{12}+s_{23}\right)\left(s_{45}^2+s_{56}^2\right)}{s_{123}}\Bigg\} + [\mathbb{Z}_6] \\
+ 2L_0\Bigg\{ & s_{12}\left(s_{12}+s_{34}+2s_{45}\right)+s_{123}\left(s_{612}-s_{61}\right) \\
& - \frac{\left(s_{12}+s_{23}\right)\left(s_{45}+s_{56}\right)^2}{s_{123}}\Bigg\} + [\mathbb{Z}_6]
\end{aligned}
$$

The $\mathcal{O}(p^4)$ 6-point amplitude

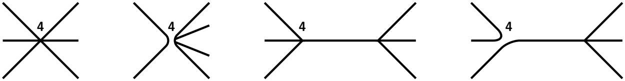


- Two-trace amplitude in a $\mathbb{Z}_{\{2,4\}}$-closed basis $\{t_1, \ldots, t_9\}$:

$$
\begin{aligned}
-iF^6\mathcal{M}_{4,\{2,4\}} = \frac{L_1}{2}\Bigg\{ & t_1\big[t_1 + 2t_2 + t_3 - 3(t_5 + t_6)\big] \\
& + \frac{T_{234}\big[t_3 - 2(t_5 + t_6)\big]}{2t_1}\Bigg\} + [\mathbb{Z}_{\{2,4\}}] \\
+\frac{L_2}{8}\Bigg\{ & t_1\left[t_1 + 2t_2 + \frac{t_3}{2} - 3(t_5 + t_6)\right] + 4t_7^2 - 2t_9^2 \\
& + \frac{[T_{234} + 4T_{789}]\big[t_3 - 2(t_5 + t_6)\big]}{2t_1}\Bigg\} + [\mathbb{Z}_{\{2,4\}}]
\end{aligned}
$$

where $T_{ijk\ldots} = (t_i + t_j + t_k + \ldots)^2$

The $\mathcal{O}(p^4)$ 6-point amplitude

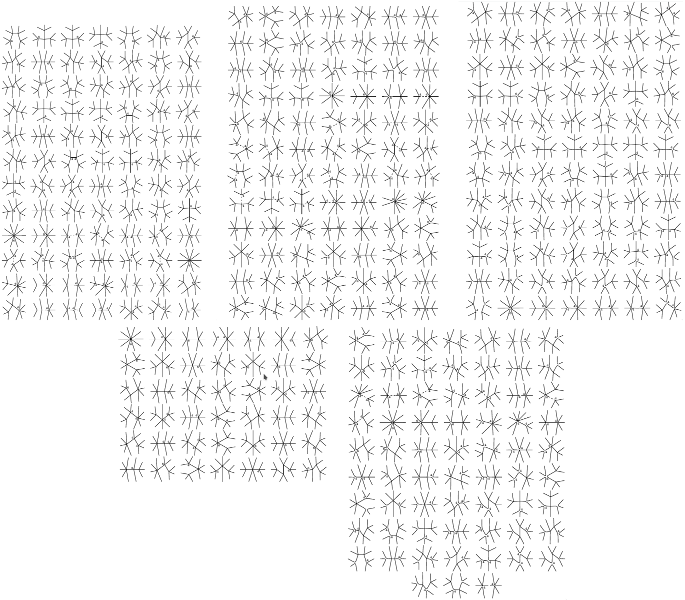


- The first true test of our methods.
- Also our first novel amplitude.
- ...until reproduced by Low & Yin (2019) using soft recursion.
- But soft recursion fails at $\mathcal{O}(p^6)$ 6-point $(m \geq n\sigma)$
- ...and we have computed that amplitude!
- All further amplitudes are too lengthy to display, though.

All my amplitudes

Legs		$\mathcal{O}(p^2)$	$\mathcal{O}(p^4)$	$\mathcal{O}(p^6)$	$\mathcal{O}(p^8)$
4	# diagrams	1	2	2	2
	Done now?	Yes!	Yes!	Yes!	Yes!
6	# diagrams	2	4	7+1	10+1
	Done now?	Yes!	Yes!	Yes!	Yes!
8	# diagrams	4	18	45+5	85+20
	Done now?	Yes!	Yes!	Yes!	Yes!
10	# diagrams	16	90	318+42	???
	Done now?	Yes!	Yes!	No.	No.
12	# diagrams	73	577	???	???
	Done now?	Yes!	No.	No.	No.
14	# diagrams	414	???	???	???
	Done now?	No.	No.	No.	No.

Automatic diagram generation



Generalisation: χPT

- Chiral perturbation theory (χPT) adds masses, electroweak interactions, and more.
- Gauge couplings through fields ℓ_μ and r_μ, recast in χPT as

$$f_\pm^{\mu\nu} \xrightarrow{g} h\, f_\pm^{\mu\nu}\, h^\dagger$$

- Masses and Yukawa couplings through fields s and p, recast in χPT as

$$\chi_\pm \xrightarrow{g} h\, \chi_\pm\, h^\dagger$$

- $\mathcal{O}(p^2)$ Lagrangian extends to

$$\mathcal{L}_2^{\chi\text{PT}} = \frac{F^4}{4}\langle u_\mu u^\mu + \chi_+\rangle$$

- 6 new terms at $\mathcal{O}(p^4)$
- 93 at $\mathcal{O}(p^6)$
- 1705 at $\mathcal{O}(p^8)$...

Generalisation: Adding masses

- Introducing $\chi_\pm$ gives physical mass eigenstates

$$N_f = 3: \quad \phi^a t^a \approx \begin{bmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\eta}{\sqrt{6}} \end{bmatrix}$$

- Giving equal masses to all particles is no problem:

$$\frac{i\delta^{ab}}{q^2} \quad \to \quad \frac{i\delta^{ab}}{q^2 - m^2}$$

but breaks Adler zeroes.

- Inequal masses break flavour-ordering, but can be salvaged:

$$\langle X t^a \rangle \frac{\delta^{ab}}{q^2} \langle t^b Y \rangle \quad \to \quad -\sum_k \frac{x_k(q)}{2} \left\langle \left[X, \tau^{k\dagger}\right] \left[\tau^k, Y\right] \right\rangle$$

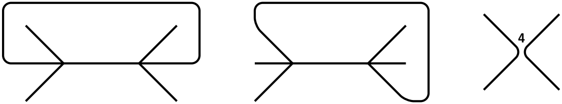
τ^k — generators in basis of mass eigenstates,
$x_k(q)$ — mass-dependent coefficients

Generalisation: Adding loops

- Loops inside a trace are flavour-ordered with

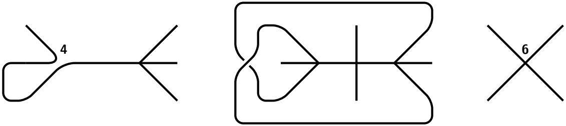
$$\langle X t^a Y t^a \rangle = \langle X \rangle \langle Y \rangle - \frac{1}{N_f} \langle XY \rangle$$

- Second term is a singlet loop,
  first term separates "inside" and "outside": All of



  have same flavour structure.

- Loops between traces connect just like tree-level propagators: All of



  have same flavour structure.

Conclusions

- We have succeeded in our goal: calculate tree-level amplitudes in the NLSM with any number of derivatives and legs.
- Also, some hopes for loops and χPT— explore in the future!

- Much faster than brute-force Feynman diagrams
- ...but limited to NLSM *et al.* with particular groups.

- Optionally: soft recursion — very powerful and general
- ...but broken by masses, loops, $m \geq n\sigma$, etc.

- Also: interesting parallels to gluon scattering?.