Lecture 2: From Diagrams to Cross Sections

Last time, I introduced the master formula for collider physics

$$AB \to 1\,2 \cdots n$$

$$\sigma_{obs} = \frac{1}{2E_{cm}^2} \sum_{n=2}^{\infty} \int d\Phi_n \left|\mathcal{M}_{AB\to 12\cdots n}\right|^2 f_{obs}(\Phi_n)$$

Things to remember:

- In principle, $1 2 \cdots n$ consists only of collider stable (or quasi-stable) particles

  <u>yes</u> $p, e^-, \gamma, \mu^-, K_L, \pi^\pm, K^\pm$, etc.

  <u>no</u> $\pi^0 \to \gamma\gamma$ (though in practice, collinear $\gamma\gamma$ can often be tagged as π^0)

  <u>no</u> $d, u, s, c, b, W, Z, h, t$

  If we want to write, e.g. $pp \to t\bar{t}$, we need to massage master formula.

- You must define a specific measurement function f_{obs}. This is true even if you want a "total cross section", since how do you treat elastic $pp \to pp$?

- You must include all configurations that could contribute to $f_{obs}(\Phi_n)$. This is very important for jets.

With 100s to 1000s of particles in each LHC collision, this seems like a daunting problem.

Luckily, these are simplifications *if* you choose an appropriate measurement function.

If you are sufficiently insensitive to what happens to beam remnant, and if you use a suitable jet algorithm then you can often replace:

$$p_A\, p_B \to \text{hadron}_1 + \text{hadron}_2 + \cdots + \text{hadron}_n \qquad f_{obs}(\text{hadrons})$$

$$\Downarrow$$

$$\text{parton}_A\ \text{parton}_B \to \text{parton}_1 + \text{parton}_2 + \cdots + \text{parton}_m \qquad \tilde{f}_{obs}(\text{partons})$$

with $m << n$.

This replacement depends on choice of f_{obs}, and has only been proved in a small subset of cases. But this is crucial for making predictions at the LHC, which is why I keep emphasizing it.

More formally, we call this *factorization* if:

$$|\mathcal{M}|^2 f_{obs} \simeq \text{``}|\mathcal{M}_A|^2\text{''}\ \text{``}|\mathcal{M}_B|^2\text{''}\ \text{``}|\mathcal{M}_C|^2\text{''} \cdots \tilde{f}_{obs}$$

+ small controlled corrections.

This is a remarkable statement when true. It says that the full quantum mechanical interference in $|\mathcal{M}|^2$ can be neglected and replaced by a series of semi-classical probabilities (up to corrections).

For proton-proton collisions, the most important factorization is for parton distribution functions:

$$pp \to X \quad \Rightarrow \quad \sum_{ij} A_i \, B_j \to X$$

where i, j could be quarks, anti-quarks, or gluons.

This is <u>not</u> true in general, and only holds (conjecturally) for observables that are insensitive to "beam remnants".

Instead of fixed p_{CM} (see later discussion of kinematics), we have a variable $\hat{p}_{CM}$.

$$\hat{p}_{CM} = x_A \underbrace{(E_{beam}, 0, 0, E_{beam})}_{\text{proton 1}} + x_B \underbrace{(E_{beam}, 0, 0, -E_{beam})}_{\text{proton 2}}$$

momentum fraction carried by parton

We then have a probability distribution over x_A and x_B:

$$f_i(x_A, \mu) \qquad f_j(x_B, \mu)$$

"factorization scale"

(Scale where you are resolving proton structure)

Parton distribution functions are non-perturbative objects with perturbative renormalization group flow. I'll only briefly touch on DGLAP evolution in these lectures.

Most convenient to rewrite master formula in terms of parton luminosity function:

$$\sigma_{obs} = \sum_{ij} \int d\hat{s}\; \mathcal{L}_{ij}(\hat{s})\; \underbrace{\frac{1}{2\hat{s}} \sum_{n=2}^{\infty} \int d\Phi_n(\hat{s}) \left| \mathcal{M}_{ij\to 12\cdots n} \right|^2 f_{obs}(\Phi_n)}_{\text{cross section for } ij\to 12\cdots n \text{ at } \hat{E}_{cm}^2 = \hat{s}}$$

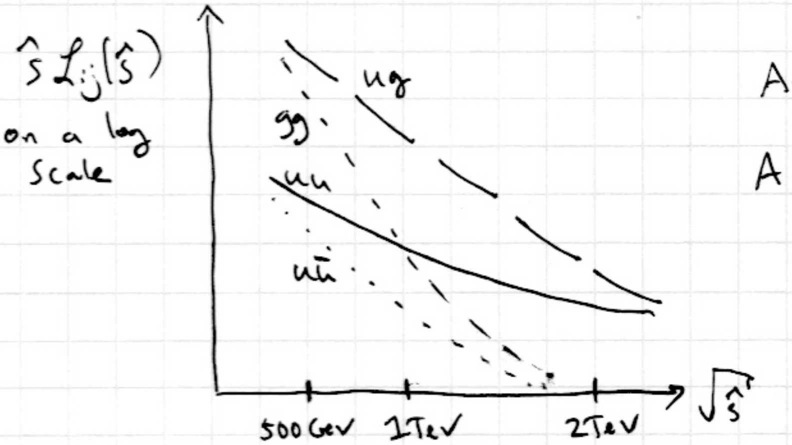
all possible partonic channels ↗ (under $\sum_{ij}$)

↑ all possible collision energies of partons (under $d\hat{s}$)

↑ parton luminosity function (under $\mathcal{L}_{ij}(\hat{s})$)

where $$\mathcal{L}_{ij}(\hat{s}) = \int dx_A \int dx_B\, f_i(x_{A,p})\, f_j(x_{B,p})\, \delta(\hat{s} - x_A x_B \underset{\substack{\uparrow \\ E_{cm}^2 \text{ for protons.}}}{s})$$

So proton-proton collider is really a set of quark/gluon colliders at different center of mass collision energies.



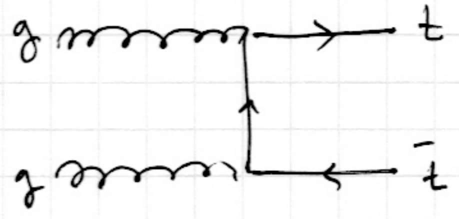
At low energies, gluons dominate

At high energies, valence quarks (up & down) dominate.

Because parton luminosities (and cross sections) fall with $\hat{s}$, scattering dominated by low $\hat{s}$ processes.

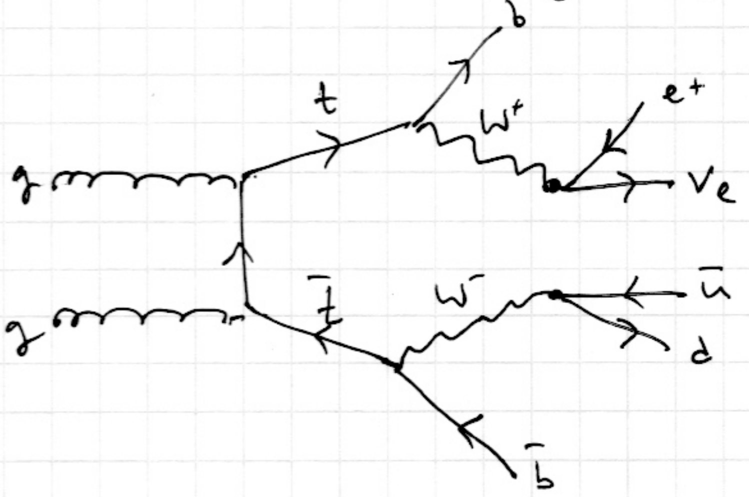
Because factorization is so essential for making collider predictions, I want to walk you through an example of factorization where you can see all of the moving parts.

Our example will be the $pp \to t\bar{t}$ cross section. We don't see the top quark, but using the gg parton channel, we can compute:



Of course, we can't measure this, but for a suitable observable, we can find a reasonable proxy.

But there is an immediate confusion. The top quark is unstable, so shouldn't we actually compute this?



plus similar diagrams for other $W^{+/-}$ decay modes.

Yes, as long as we don't ask detailed questions about beams and use a suitable jet algorithm to reconstruct $b, \bar{b}, \bar{u}, d$, this is fine.

But wait! Order (i.e. number of gauge couplings) of these diagrams is very different. Which is closer to the "right" answer?

Three seemingly very!? different approaches:

$$\underset{\substack{\uparrow \\ \text{whatever} \\ \text{that is}}}{\sigma_{t\bar{t}}} = \sum_{ij} \int d\hat{s}\, \mathcal{L}_{ij}(\hat{s}) \frac{1}{2\hat{s}} \int d\Phi_2 \left|M_{ij \to t\bar{t}}\right|^2 \underset{\substack{\uparrow \\ \text{Some kind of} \\ \text{acceptance function.}}}{f_{t\bar{t}}}(\Phi_2)$$

or . . .

$$\sum_{ij} \int d\hat{s}\, \mathcal{L}_{ij}(\hat{s}) \frac{1}{2\hat{s}} \int d\Phi_4 \left|M_{ij \to bW^+\bar{b}W^-}\right|^2 f_{bW^+\bar{b}W^-}(\Phi_4)$$

or . . . .

$$\sum_{ij} \int d\hat{s}\, \mathcal{L}_{ij}(\hat{s}) \frac{1}{2\hat{s}} \int d\Phi_6 \sum_{klmn} \left|M_{ij \to b\bar{b}klmn}\right|^2 f_{b\bar{b}klmn}(\Phi_6)$$

Which is it?, For appropriate choice of f, <u>all</u> of these are reasonable approximations to $pp \to t\bar{t}$.

How can that be?! Factorization! Decay of top and W boson factorizes from hard scattering process (approximately). We will be able to see this explicitly via:

Narrow Width Approximation.

Before we get to the NWA, we need to do some basic kinematics, since it relies on properties of Lorentz-invariant N-body phase space.

Kinematics

$$\underset{\substack{\uparrow \\ \text{a 4-vector}}}{p^\mu} = (E, p_x, p_y, p_z) = (E, \vec{p})$$

$$p^\mu p_\mu \equiv p^2 = E^2 - p_x^2 - p_y^2 - p_z^2$$

Collider physics uses mostly minus metric.

Lorentz-invariant phase space for a single particle:

$$\int \underbrace{\frac{d^4p}{(2\pi)^4}}_{\substack{\text{manifestly} \\ \text{Lorentz invariant}}} \underbrace{(2\pi)\,\delta(p^2-m^2)}_{\text{on-shell}} \underbrace{\Theta(E)}_{\substack{\text{positive} \\ \text{energy}}}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3}\,\frac{1}{2E} \qquad \text{where } E = \sqrt{m^2 + \vec{p}^{\,2}}$$

↑ not so manifestly Lorentz invariant

Total energy-momentum conservation

$$(2\pi)^4\,\delta^{(4)}\Big(p_{CM} \underbrace{- p_1 - p_2 \cdots - p_n}_{\text{n particle final state}}\Big)$$

For example, (total) decay width for a particle of mass M

$$\Gamma_A = \frac{1}{2M} \sum_{n=2}^{\infty} \int d\Phi_n \left| \mathcal{M}_{A\to 12\cdots n} \right|^2$$ (has dimensions of mass)

where
$$\int d\Phi_n = \int \prod_{i=1}^{n} \frac{d^4 p_i}{(2\pi)^4} (2\pi)\, \delta\left(p_i^2 - m_i^2\right) \Theta\left(E_i\right)$$
$$\times (2\pi)^4 \delta^{(4)}\left(p_{CM} - p_1 - p_2 - \cdots - p_n\right)$$

Counting degrees of freedom:

$4n$	degrees of freedom off shell
$-n$	on-shell constraints
-4	energy-momentum constraints
$3n-4$	physical degrees of freedom

n	2	3	4	5	6	etc...
d.of.	2	5	8	11	14	

+3 +3 +3 +3

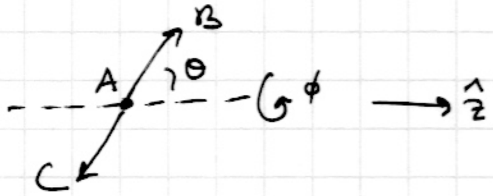
Checking our dimension counting.

$$\Gamma = \frac{1}{2M}\sum_{n=2}^{\infty}\int \underset{\uparrow}{d\Phi_n}\,\Big|\underset{\uparrow}{\mathcal{M}_{A\to 12\cdots n}}\Big|^2$$

mass dimension: -1 $\qquad \underset{\substack{\uparrow \\ \text{Why not 3?} \\ \text{b/c of } \delta(p^2-m^2)}}{2n-4} \qquad \underset{\substack{\uparrow \\ \text{squared} \\ \text{amplitude.}}}{2\cdot(4-n-1)} \qquad \Rightarrow \quad +1$

Let's do the simple case of a $1\to 2$ decay (which is all we're going to need for our NWA example.)

$$A \to BC$$



$$p_A = (m_A,\ 0,\ 0,\ 0)$$
$$p_B = \left(\sqrt{m_B^2+k^2},\ k\sin\theta\cos\phi,\ k\sin\theta\sin\phi,\ k\cos\theta\right)$$
$$p_C = \Big(\sqrt{m_C^2+k^2},\ \underbrace{-k\sin\theta\cos\phi,\ -k\sin\theta\sin\phi,\ -k\cos\theta}_{\text{automatically conserves 3-momentum}}\Big)$$

Subject to constraint:

$$m_A = \sqrt{m_B^2+k^2} + \sqrt{m_C^2+k^2}$$

It's a good exercise to show:

$$k = \frac{m_A}{2}\sqrt{1-\frac{(m_B+m_C)^2}{m_A^2}}\sqrt{1-\frac{(m_B-m_C)^2}{m_A^2}}$$

$$\int d\Phi_2 = \int \frac{d\Omega}{(2\pi)^2}\frac{k}{4m_A} = \int \frac{d\Omega}{32\pi^2}\frac{2k}{m_A}$$

Note that $d\Phi_2 \to 0$ as $k \to 0$ ("no phase space for decay")

For a typical amplitude: $|\mathcal{M}|^2 = \Big|$ 

$\Big|^2$

$$\simeq |g\, m_A|^2 \qquad \left(m_B \sim m_C \sim 0 \Rightarrow \frac{2k}{m_A} \to 1\right)$$

$$\Gamma \sim \frac{g^2}{64\pi^2} m_A \underbrace{\int d\Omega}_{4\pi} \sim \frac{g^2}{16\pi} m_A$$

This is a typical (partial) width, and to get total width, have to add up multiple decay channels.

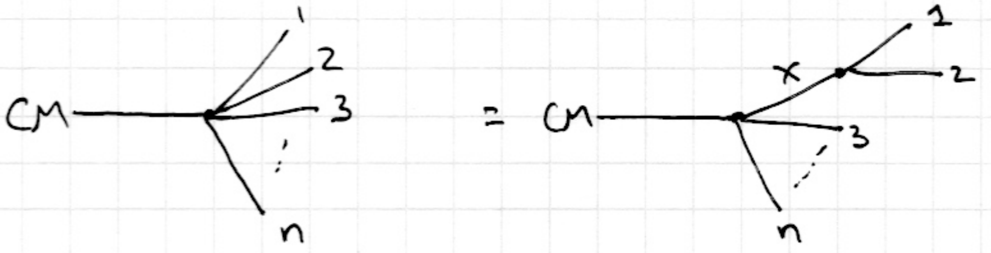
As we'll see, branching ratios are given by

$$Br(A\to X) = \frac{\Gamma_{A\to X}}{\Gamma^A_{total}} \quad \text{where } \Gamma^A_{total} = \sum_X \Gamma_{A\to X}$$

This will be a key step in NWA.

Finally, we will need the following useful decomposition of n-body phase space:

$$d\Phi_n\left(p_{CM}\to p_1 p_2 \cdots p_n\right) = d\Phi_{n-1}\left(p_{CM}\to p_x\, p_3 \cdots p_n\right) \times \frac{dm_x^2}{2\pi} \times d\Phi_2\left(p_x \to p_1\, p_2\right)$$



where m_x ranges over all allowed values

This derivation is straightforward if you insert a dummy 4-vector with dummy mass:

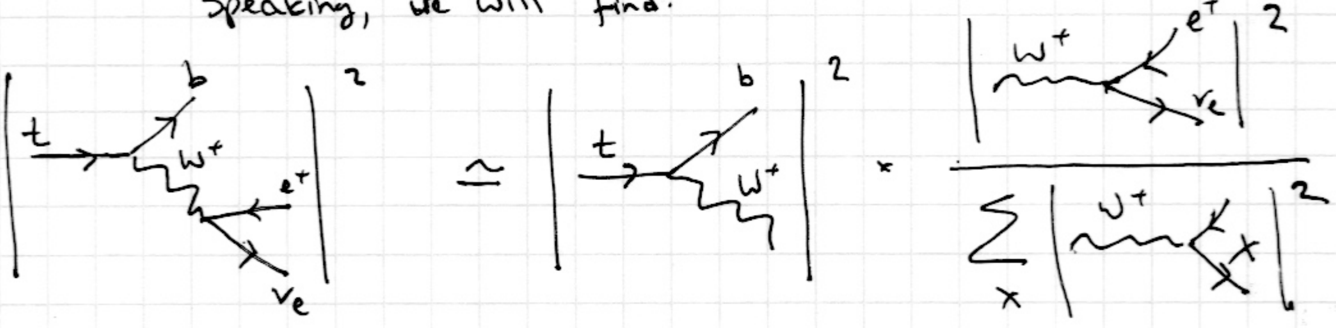
$$\underbrace{\int \frac{d^4 p_x}{(2\pi)^4}\,(2\pi)^4\,\delta^{(4)}\left(p_x - p_1 - p_2\right)}_{\text{Fancy way to write } 1}\;\underbrace{\int \frac{dm_x^2}{2\pi}\,(2\pi)\,\underbrace{\delta\left(p_x^2 - m_x^2\right)}_{\text{putting dummy on-shell}}\,\Theta(E_x)}_{\text{Fancy way to write } 1}$$

These will be essential to go from Feynman diagrams to cross sections, especially for NWA.

The Narrow Width Approximation

This is the workhorse of cascade decay analyses and an essential approximation for BSM searches.

We are going to study it for top decays, Roughly speaking, we will find:



This is an important (but simple) example of factorization.

Key to the NWA :
$$\underbrace{\left|\frac{1}{q^2-\underset{\substack{\uparrow\\ \text{mass}}}{M^2}+iM\underset{\substack{\uparrow\\ \text{total width}}}{\Gamma}}\right|^2}_{\text{propagator}} \simeq \underset{\substack{\uparrow\\ \text{on-shell condition.}}}{\delta\left(q^2-M^2\right)}\cdot\frac{\pi}{M\Gamma}$$

<u>If</u> you can ignore finite width effects, including quantum mechanical interference, then this is a good approximation,

Have to beware of cases where it doesn't work though!

The derivation of the NWA is straightforward:

$$d\Phi_3(p_t \to p_b p_e p_\nu) = d\Phi_2(p_t \to p_b p_W) \frac{dm_W^2}{2\pi} d\Phi_2(p_W \to p_e p_\nu)$$

This just the phase space decomposition, no approximations.

$$|\mathcal{M}_{t\to be\nu}|^2 = \left| \frac{\sum_\varepsilon \mathcal{M}^\varepsilon_{t\to bW} \mathcal{M}^\varepsilon_{W\to e\nu}}{q^2 - M_W^2 + iM_W\Gamma_W} \right|^2$$

where ε are the W boson polarizations. Again, no approximations yet.

Now the approximation:

$$\simeq \sum_\varepsilon |\mathcal{M}^\varepsilon_{t\to bW}|^2 |\mathcal{M}^\varepsilon_{W\to e\nu}|^2 \delta(q^2 - M_W^2) \frac{\pi}{\Gamma_W M_W}$$

In addition to replacing the propagator, we got rid of interference between different polarizations.

Reassembling the pieces:

$$\Gamma_{t\to be\nu} = \frac{1}{2m_t} \int d\Phi_3 |\mathcal{M}_{t\to be\nu}|^2$$

$$\simeq \frac{1}{2m_t} \sum_\varepsilon \int d\Phi_2 |\mathcal{M}^\varepsilon_{t\to bW}|^2$$

$$\cdot \int \frac{dm_W^2}{2\pi} \frac{\pi}{\Gamma_W M_W} \delta(q^2 - M_W^2) \int d\Phi_2 |\mathcal{M}^\varepsilon_{W\to e\nu}|^2$$

After the dust settles, and ignoring polarizations (which can be easily summed/averaged as appropriate) we have:

$$\Gamma_{t\to be\nu} = \Gamma_{t\to bW} \cdot \underbrace{\frac{\Gamma_{W\to e\nu}}{\Gamma_W^{\text{total}}}}_{= Br(W\to e\nu)} \longleftarrow \text{did you see how I did that??}$$

The full 3-body top decay factorizes into a 2-body decay times a $1\to 2$ branching fraction.

This is why $gg\to t\bar{t} \simeq gg\to b\bar{b}e^+\nu\bar{u}d$ up to branching ratio effects. Of course, we implicitly assumed that measurement function also factorized.

To summarize: the master formula says we have to calculate

$$pp \to \text{(quasi-) stable particles}$$

With PDFs, you can replace

$$pp \quad \text{with} \quad \sum_{ij} \underbrace{i \; j}_{\text{partons}}$$

With NWA, we can often group final states into "bundles" where cross section is dominated by on-shell intermediate resonances.

Have to think very carefully if this is true.

Next lectures, we'll start to extend this logic to QCD radiation.

For reference, here are branching ratios for heavy resonances in the standard model.

W^+ W boson $m_W = 80.4$ GeV

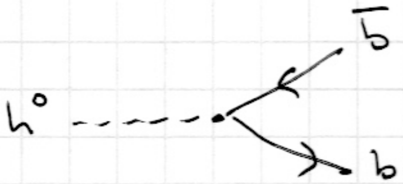
11% $\rightarrow e^+ \nu_e$
11% $\rightarrow \mu^+ \nu_\mu$
11% $\rightarrow \tau^+ \nu_\tau$
67% $\rightarrow$ hadrons (two jets)

Z^0 Z boson $m_Z = 91.2$ GeV

3.4% $\rightarrow e^+ e^-$
3.4% $\rightarrow \mu^+ \mu^-$
3.4% $\rightarrow \tau^+ \tau^-$
20.5% $\rightarrow$ neutrinos
70% $\rightarrow$ hadrons (two jets)

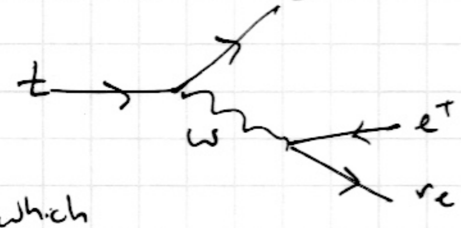
h^0 Higgs boson $m_h = 125$ GeV

60% $\rightarrow b \bar{b}$ (two b-tagged jets)
21% $\rightarrow W W^*$ (see above)
9% $\rightarrow g g$
5% $\rightarrow \tau^+ \tau^-$
2.5% $\rightarrow c \bar{c}$
2.5% $\rightarrow Z Z^*$ (see above)
0.2% $\rightarrow \gamma \gamma$
0.15% $\rightarrow Z \gamma$



t top quark $m_t = 172$ GeV

99% $\rightarrow bW$ (see above)



Large branching fractions to jets, which is why hadronic final states are so important.