MARCO PELOSO – LECTURE 4

QFT : FREE SCALAR FIELD QUANTIZATION

MINKOWSKI SPACETIME: $g_{\mu\nu} = \eta_{\mu\nu} = \text{DIAG}(1,-1,-1,-1)$

LAGRANGIAN : $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2$

EULER LAGRANGE EQS: $\partial_\mu\partial^\mu\phi + m^2\phi = 0$

CONJUGATE MOMENTUM: $\Pi = \frac{\partial\mathcal{L}}{\partial\dot\phi} = \dot\phi$

CANONICAL QUANTIZATION: $\left[\phi(t,\vec{x}),\Pi(t,\vec{y})\right] = i\,\delta^{(3)}(\vec{x}-\vec{y})$

FOURIER SPACE: $\phi = \int\frac{d^3k}{(2\pi)^{3/2}}\,e^{i\vec{k}\cdot\vec{x}}\underbrace{\left[\phi_{\vec{k}}(t) + \phi^{+}_{-\vec{k}}(t)\right]}_{\text{WITH THIS CHOICE, } \phi \text{ IS REAL, NAMELY } \phi^{+}=\phi}$

SAME DECOMPOSITION FOR Π

IN THIS WAY WE HAVE ONLY TIME DERIVATIVES, SINCE $\partial_i \rightarrow i k_i$, AND THE ABOVE EQS. BECOME

$$\begin{cases} \dot\phi_k = \Pi_k \\ \dot\Pi_k + (k^2+m^2)\phi_k = 0 \end{cases}$$

SAME EQS. AS SIMPLE HARMONIC OSCILLATOR, WITH FREQUENCY

$\omega^2 = k^2 + m^2$

SO WE QUANTIZE IT IN TERMS OF LOWERING $(\hat{a})$

AND RISING ($\hat{a}^\dagger$) OPERATORS

$$\phi_{\vec{k}}(t) + \phi^\dagger_{-\vec{k}}(t) = \frac{1}{\sqrt{2\omega}}\left[e^{-i\omega t}\,\hat{a}_{\vec{k}} + e^{i\omega t}\,\hat{a}^\dagger_{-\vec{k}}\right]$$

WITH $\left[\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{p}}\right] = \delta^{(3)}(\vec{k}-\vec{p})$

USING THESE RELATIONS WE CAN INDEED SEE THAT

$$\left[\phi(t,\vec{x}), \frac{d\phi(t,\vec{y})}{dt}\right] = i\,\delta^{(3)}(\vec{x}-\vec{y})$$

HAMILTONIAN

$$H = \int d^3x\left[\Pi\dot{\phi} - \mathcal{L}\right] = \frac{1}{2}\int d^3x\left[\Pi^2 + (\partial_i\phi)^2 + m^2\phi^2\right]$$

$$= \ldots = \frac{1}{2}\int d^3k\;\omega_k\left(\hat{a}_k\hat{a}_k^\dagger + \hat{a}_k^\dagger\hat{a}_k\right)$$

NORMAL ORDERING $\quad :H: = \int d^3k\;\omega_k\,\underbrace{\hat{a}_k^\dagger a_k}$

(PLACE DAGGER BEFORE)

COMPUTES THE ENERGY IN THE SYSTEM

SUMS OVER ALL POSSIBLE STATES

ENERGY OF THAT STATE

$\hat{N}_k \equiv \hat{a}_k^\dagger\hat{a}_k$ COUNTS HOW MANY PARTICLES ARE PRESENT IN THE SYSTEM WITH THAT ENERGY.

EX: HOW MANY PARTICLES IN THE VACUUM?

$$\langle 0|\hat{a}_k^\dagger\hat{a}_k|0\rangle = 0 \qquad \left(\text{SINCE } \hat{a}_k|0\rangle = 0\right)$$

EXPANDING UNIVERSE + COUPLING WITH INFLATON

$$S_\chi = \frac{1}{2}\int d^4x \sqrt{-g}\left[g^{\mu\nu}\partial_\mu \chi \partial_\nu \chi - g^2 \varphi^2 \chi^2\right]$$

- WE WORK IN THE SO CALLED CONFORMAL TIME

$$g_{\mu\nu} = a^2(t)\,\eta_{\mu\nu}$$

$$\Rightarrow \quad S_\chi = \frac{1}{2}\int d^4x \left[a^2\left(\partial_0 \chi \partial_0 \chi - \partial_i \chi \partial_i \chi\right) - a^4 g^2 \varphi^2 \chi^2\right]$$

RESCALE $\chi \to \dfrac{\tilde{\chi}}{a}$ AND $\varphi \to \dfrac{\tilde{\varphi}}{a}$

$$S_\chi = \frac{1}{2}\int d^4x \Big[a^2 \frac{1}{a}\left(\dot{\tilde{\chi}} - \frac{\dot{a}}{a}\tilde{\chi}\right)\frac{1}{a}\left(\dot{\tilde{\chi}} - \frac{\dot{a}}{a}\tilde{\chi}\right) - \partial_i \tilde{\chi}\partial_i \tilde{\chi} - g^2 \tilde{\varphi}^2 \tilde{\chi}^2\Big]$$

$$= \frac{1}{2}\int d^4x \Big[\dot{\tilde{\chi}}^2 \underbrace{- \frac{2\dot{a}}{a}\tilde{\chi}\dot{\tilde{\chi}}} + \frac{\dot{a}^2}{a^2}\tilde{\chi}^2 - \partial_i \tilde{\chi}\partial_i \tilde{\chi} - g^2 \tilde{\varphi}^2 \tilde{\chi}^2\Big]$$

$$-\frac{\dot{a}}{a}\frac{d}{dt}\tilde{\chi}^2 \underset{\text{BY PARTS}}{\equiv} \frac{d}{dt}\left[\frac{\dot{a}}{a}\right]\tilde{\chi}^2 = \left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right)\tilde{\chi}^2$$

$$\equiv \frac{1}{2}\int d^4x \left[\dot{\tilde{\chi}}^2 + \frac{\ddot{a}}{a}\tilde{\chi}^2 - \partial_i \tilde{\chi}\partial_i \tilde{\chi} - g^2 \tilde{\varphi}^2 \tilde{\chi}^2\right]$$

- AT THE START THE INFLATON IS A HOMOGENEOUS FIELD, SO WE TAKE $\varphi(t)$ IN OUR COMPUTATIONS. IT IS A CLASSICAL EXTERNAL FIELD, SUCH AS $a(t)$, AND ITS TIME EVOLUTION LEADS TO CREATION OF QUANTA OF X, AS WE WILL NOW SEE.

$\Rightarrow$ IN FOURIER SPACE WE HAVE THE EQUATION

$$\ddot{\tilde{X}}_k + \left[k^2 - \frac{\ddot{a}}{a} + g^2\tilde{\varphi}^2(t)\right]\tilde{X}_k = 0$$

THE PROBLEM HAS BECOME THAT OF A SIMPLE HARMONIC OSCILLATOR WITH TIME DEPENDENT FREQUENCY

$$\omega(t) = \sqrt{k^2 - \frac{\ddot{a}}{a} + g^2\tilde{\varphi}^2(t)} \equiv \sqrt{k^2 + m^2(t)}$$

EFFECTIVE MASS $m^2 = g^2\tilde{\varphi}^2 - \frac{\ddot{a}}{a}$

PRODUCTION FROM OSCILLATING INFLATON

PRODUCTION FROM EXPANDING GEOMETRY
"GRAVITATIONAL PRODUCTION"

ANALOGY: PENDULUM WITH LENGTH THAT CHANGES IN TIME, AS THE PENDULUM IS OCCILLATING. THE CHANGE OF THE LENGTH CREATES EXCITATIONS OF THE PENDULUM = PRODUCTION OF QUANTA

IN FOURIER SPACE $\begin{cases} \dot{\tilde{X}}_k = \Pi_k \\ \dot{\Pi}_k + (k^2+m^2)\tilde{X}_k = 0 \end{cases}$ AS BEFORE

CONVENIENT TO CHANGE VARIABLES, INTRODUCING THE SO CALLED BOGOLIUBOV COEFFICIENTS

$$\tilde{X}_k \equiv \frac{\alpha_k(t)}{\sqrt{2\omega_k}} + \frac{\beta_k(t)}{\sqrt{2\omega_k}} \quad , \quad \Pi_k \equiv \frac{-i\omega\,\alpha_k(t)}{\sqrt{2\omega_k}} + \frac{i\omega\,\beta_k(t)}{\sqrt{2\omega_k}}$$

IN TERMS OF THE NEW VARIABLES, THE ABOVE EQUATIONS BECOME:

$$\boxed{\begin{cases} \dot{\alpha}_k = -i\omega\alpha_k + \dfrac{\dot{\omega}}{2\omega}\beta_k \\ \dot{\beta}_k = i\omega\beta_k + \dfrac{\dot{\omega}}{2\omega}\alpha_k \end{cases}}$$

- IN THE STANDARD CASE, $\dot{\omega} = 0 \Rightarrow \begin{cases} \alpha_k = e^{-i\omega t} \\ \beta_k = 0 \end{cases}$

THIS IS THE STANDARD QFT QUANTIZATION

- FOR $\dot{\omega} \neq 0 \Rightarrow \beta_k \neq 0$. AS WE SHALL NOW SEE, THIS IS ASSOCIATED TO THE PRESENCE OF PARTICLES. TO SEE THIS, WE COMPUTE THE HAMILTONIAN. INSERTING ALL THE RELATIONS ABOVE, THE HAMILTONIAN READS

$$H = \ldots = \frac{1}{2}\int d^3k \; \omega \Big[\left(|\alpha_k|^2 + |\beta_k|^2 \right) \left(\hat{a}_k \hat{a}_k^+ + \hat{a}_k^+ \hat{a}_k \right) + 2\alpha_k \beta_k \hat{a}_k \hat{a}_k + 2\alpha_k^* \beta_k^* \hat{a}_k^+ \hat{a}_k^+ \Big]$$

IN THE STANDARD CASE $\beta_k = 0$ AND WE RECOVER THE SAME HAMILTONIAN AS BEFORE. WE CAN REWRITE

$$H = \frac{1}{2}\int d^3k \; \omega_k \underbrace{\left(\hat{a}_k^+ , \hat{a}_k \right) \begin{pmatrix} \alpha_k^* & \beta_k^* \\ \beta_k & \alpha_k \end{pmatrix}}_{\left(\hat{A}_k , \hat{A}_k^+ \right)} \underbrace{\begin{pmatrix} \alpha_k & \beta_k^* \\ \beta_k & \alpha_k^* \end{pmatrix} \begin{pmatrix} \hat{a}_k \\ \hat{a}_k^+ \end{pmatrix}}_{\begin{pmatrix} \hat{A}_k \\ \hat{A}_k^+ \end{pmatrix}}$$

NAMELY THE HAMILTONIAN CAN BE "DIAGONALIZED" IN TERMS OF NEW, TIME DEPENDENT (SINCE α_k & β_k DEPEND ON TIME) ANNIHILATION & CREATION OPERATORS

$$\boxed{:H: = \int d^3k \; \omega_k \hat{A}_k^+ \hat{A}_k}$$

THE OPERATOR $\hat{N}_k \equiv \hat{A}_k^+ \hat{A}_k$ COUNTS HOW MANY PARTICLES OF FREQUENCY $\omega_k = \sqrt{k^2 + m^2(t)}$ ARE PRESENT AT THE TIME t

HEISENBERG PICTURE, THE OPERATORS EVOLVE IN TIME, $\hat{N}_k(t) = \hat{A}_k^+(t) A_k(t)$, WHILE THE STATES ARE CONSTANT. WE ASSUME THAT, INITIALLY, WE ARE IN THE STANDARD CASE WITH $\beta_k = 0$ AND WITH THE STANDARD VACUUM $\hat{a}_k |0\rangle = 0$

HOW MANY QUANTA ARE PRESENT AT THE TIME t?

$$N_k = \langle 0 | \hat{N}_k | 0 \rangle = \langle 0 | \hat{A}_k^+ \hat{A}_k | 0 \rangle$$

$$= \langle 0 | \left(\beta_k \hat{a}_k + \cancel{\alpha_k^* \hat{a}_k^+} \right) \left(\cancel{\alpha_k \hat{a}_k} + \beta_k^* \hat{a}_k^+ \right) | 0 \rangle$$

$$= |\beta_k|^2 \langle 0 | \hat{a}_k \hat{a}_k^+ | 0 \rangle = |\beta_k|^2 \langle 0 | \hat{a}_k \hat{a}_k^+ - \hat{a}_k^+ \hat{a}_k | 0 \rangle$$

ADDED 0

$$= |\beta_k|^2 \langle 0 | \delta^{(3)}(\vec{k} - \vec{k}) | 0 \rangle = |\beta_k|^2 \underbrace{\delta^{(3)}(0)}_{\text{VOLUME}} \underbrace{\langle 0 | 0 \rangle}_{1}$$

$$\delta(k) = \frac{1}{(2\pi)^3} \int d^3x \, e^{i\vec{k}\cdot\vec{x}}$$

$$\Rightarrow \boxed{\text{NUMBER DENSITY} = |\beta_k(t)|^2}$$

WE HAVE NOW ALL THE EQUATIONS TO COMPUTE THE OCCUPATION NUMBER

$$\boxed{\begin{aligned} \dot{\alpha}_k &= -i\omega\alpha_k + \frac{\dot{\omega}}{2\omega}\beta_k \\ \dot{\beta}_k &= i\omega\beta_k + \frac{\dot{\omega}}{2\omega}\alpha_k \\ \alpha_{in} = 1,\ \beta_{in} &= 0,\ \frac{N_k}{\text{Volume}} = |\beta_k|^2 \end{aligned}}$$

$$\omega_k = \sqrt{k^2 + g^2\bar{\varphi}^2 - \frac{\ddot{a}}{a}}$$

FOR SIMPLICITY, IGNORE THE EXPANSION OF THE UNIVERSE

$$\omega_k = \sqrt{k^2 + g^2\,\underbrace{\bar{\phi}_0^{\,2}\sin^2(mt)}_{\text{INFLATON OSCILLATIONS}}}$$

- MOST OF THE PRODUCTION OCCURS WHEN $\dot{\omega} > \omega^2$

  IN THIS REGIME WE SAY THAT "THE FREQUENCY VARIES NON ADIABATICALLY" (NOTICE BOTH $\dot{\omega}$ AND ω^2 HAVE MASS DIMENSION 2; WHY IT IS $\dot{\omega} > \omega^2$ SHOULD BE CLEAR FROM THE EQS. FOR α_k, β_k)

$\dot{\omega}/\omega^2$ IS MAXIMUM WHENEVER $\phi = 0$

- IN THIS EXAMPLE THE FREQUENCY CHANGES PERIODICALLY $\longrightarrow$ RESONANCE

SLIDE 8
E 9

STRONG PRODUCTION FOR $q \equiv \frac{g^2 \Phi_0^2}{4m^2} > 1$

IN LARGE FIELD INFLATION

$m \simeq 10^{13}\,\mathrm{GeV}$, $\Phi_0 \simeq 10^{18}\,\mathrm{GeV}$

ENOUGH $g \gtrsim 10^{-4}$

ANALYTIC RESULTS

PRODUCTION WHEN $m = g\phi = 0$; EXPAND

$$\phi = \phi_0 \sin(mt) \simeq \phi_0\, m t$$

RESCALE $\tau \equiv \sqrt{g m \phi_0}\, t$, $p \equiv \frac{k}{\sqrt{g m \phi_0}}$

$$\Rightarrow \boxed{\ddot{\chi} + (p^2 + \tau^2)\chi = 0}$$

SOLVED IN TERMS OF PARABOLIC CYLINDER FUNCTIONS

$$\chi = C_1 D_{-\frac{1}{2} - i\frac{p^2}{2}}\left((1+i)\tau\right) + C_2 D_{-\frac{1}{2} + i\frac{p^2}{2}}\left((1-i)\tau\right) \qquad (*)$$

THE SOLUTIONS HAVE AN INTERESTING LIMIT AT $\tau \to \pm\infty$. FOR INSTANCE

$$\lim_{\tau\to\infty} D_{-\frac{1}{2}-i\frac{p^2}{2}}\left((1+i)\tau\right) = \text{const.}\ \frac{e^{-i\left(\frac{\tau^2}{2}+\frac{p^2}{2}\ln\tau\right)}}{\tau^{1/2}}$$

AND

$$\lim \frac{e^{-i\int\omega\, dt}}{\sqrt{2\omega}} \sim \frac{e^{-i\int\sqrt{p^2+\tau^2}\,d\tau}}{\left(\tau^2+p^2\right)^{1/4}} \sim \frac{e^{-i\int\left(\tau+\frac{p^2}{2\tau}\right)d\tau}}{\sqrt{\tau}}$$

→ THE SOLUTION (*) SATISFIES

$$\lim_{\tau\to\pm\infty} \chi = C_1^{\pm}\,\frac{e^{-i\int\omega\, dt}}{\sqrt{2\omega}} + C_2^{\pm}\,\frac{e^{i\int\omega\, dt}}{\sqrt{2\omega}}$$

→ WE SAW THAT $\chi = \dfrac{\alpha(t)}{\sqrt{2\omega}} + \dfrac{\beta(t)}{\sqrt{2\omega}}$

WITH $\alpha = \text{const}\ e^{-i\omega t}$ AND $\beta = \text{const}\ e^{i\omega t}$

WHEN ω IS CONSTANT

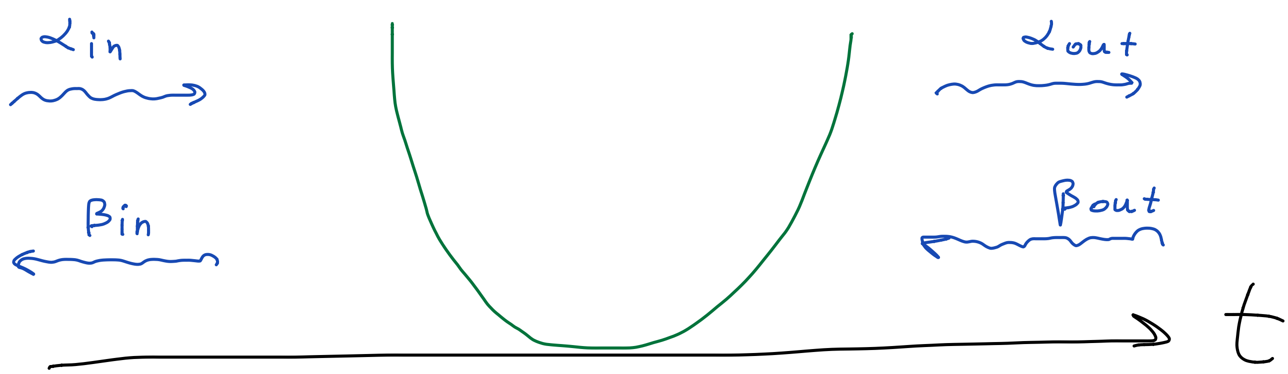
⇒ THE ASYMPTOTIC VALUES C_1^- AND C_2^- PROVIDE THE BOGOLIUBOV COEFFICIENTS $\{\alpha, \beta\}$ BEFORE $m=0$

THE ASYMPTOTIC VALUES C_1^+ AND C_2^+ PROVIDE $\{\alpha, \beta\}$ AFTER $m=0$

* THE EXACT SOLUTION PROVIDES THE RELATION

$$\{\alpha, \beta\}_{BEFORE} \longrightarrow \{\alpha, \beta\}_{AFTER}$$

COMPLETELY ANALOGOUS TO THE THEORY OF SCATTERING, THE PRODUCTION OCCURS AT DISCRETE MOMENTS, AND WE HAVE DEVELOPED THE FORMALISM TO GO IN $\longrightarrow$ OUT STATE



USING THE ABOVE PARABOLIC CYLINDER FUNCTIONS WE FIND, UP TO PHASES,

$$\begin{pmatrix} \alpha_{OUT} \\ \beta_{OUT} \end{pmatrix} = \begin{pmatrix} \sqrt{1+e^{-\pi p^2}} & e^{-\pi p^2/2} \\ e^{-\pi p^2/2} & \sqrt{1+e^{-\pi p^2}} \end{pmatrix} \begin{pmatrix} \alpha_{in} \\ \beta_{in} \end{pmatrix}$$

AT THE FIRST PRODUCTION $\alpha_{in} = 1$, $\beta_{in} = 0$

$$\Rightarrow \quad |\beta_{out}|^2 = e^{-\pi p^2} = \exp\left(-\frac{\pi k^2}{g m \phi_0}\right)$$

SLIDE 10

RESCATTERING & THERMALIZATION

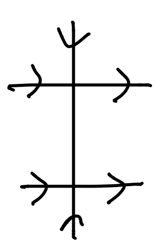
- AS WE SAW IN THE LATTICE SIMULATION, THE PRODUCED X QUANTA RESCATTER AGAINST THE INFLATON CONDENSATE AND FRAGMENT IT.

- THIS DESTROYS THE COHERENCY OF THE INFLATON OSCILLATIONS AND TERMINATES THE RESONANCE

- IN LARGE FIELD INFLATION MODELS, GENERATION OF EXCITATIONS OF QUANTA OF ϕ & X WITH

$\rho_{TOT} \gg E^4_{1\,\text{EXCITATION}}$. VERY FAR FROM THERMAL EQUILIBRIUM . IN A THERMAL BATH $\rho_{TOT} \simeq T^4$

AND $E_{1\,\text{EXCITATION}} \simeq T$ $\Rightarrow$ THERMALIZATION OCCURS VIA PARTICLE FUSION



$4 \to 2$

SLIDES 11 & 12