Large Scale Structure

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(or C.F.T (Cosmological Field theory))

The SM of Cosmology: ΛCDM

6 parameters → agreement with O(10) types of data
(+ some "tensions")

Data:
- Big Bang Nucleosynthesis
- CMB anisotropies
- Large Scale Structure
  - Power Spectrum (BAO)
  - Lyman α
  - Weak Lensing
- SNe Ia
- GW
- 21 cm
- ...

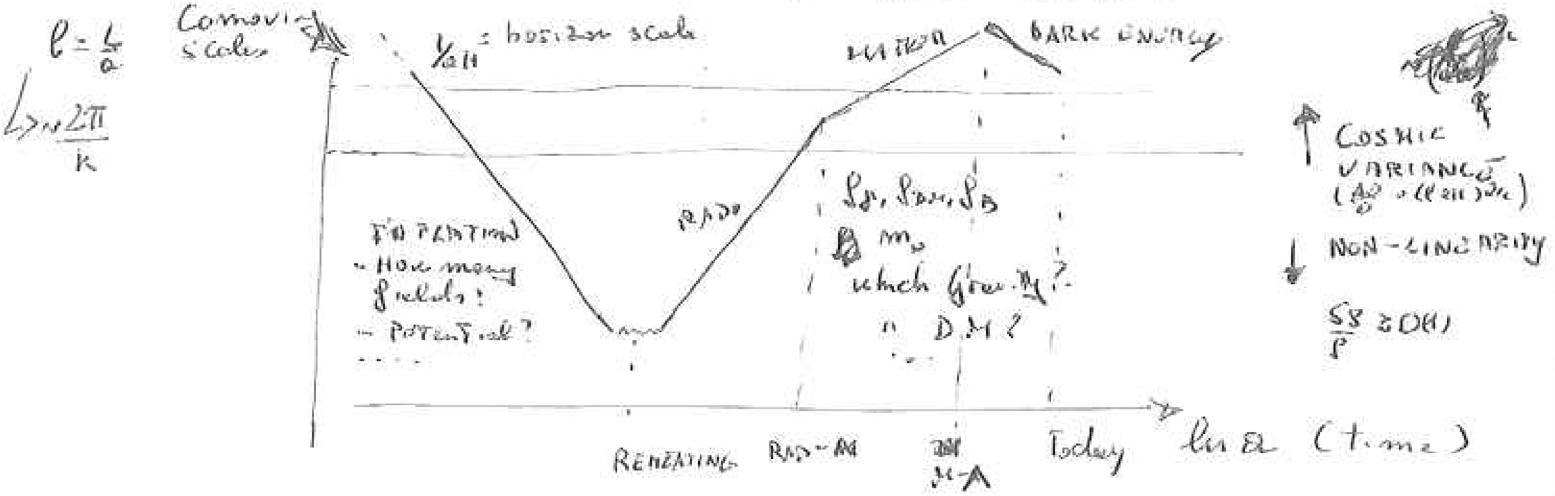
As for "the" SM, we are interested in going beyond:
- ν's (mass, mixings...)
- Primordial Non Gaussianity (Dynamics of Inflation!)
- Modification of General Relativity
- New forms of Dark Matter beyond CDM
- " " " Dark Energy " Λ
- ...

Particle Physics $\to$ Energy frontier

$\searrow$ Intensity frontier

Cosmology $\to$ Measure all possible probes (light, DM, ν, GW, ...)

AT ~~ALL~~ ALL POSSIBLE SCALES.



PLAN OF THE LECTURES

— o — o —

Friedmann-Lemaitre-Robertson-Walker Universe.

0^{th} order $\to$ background level. ($\to$ "~~[illegible]~~ Vacuum theory")

Homogeneity + Isotropy $\to$ $ds^2 = dt^2 - a^2(t) \underbrace{dx^i dx^j \delta_{ij}}_{= d\vec{x}^2}$

+ FLATNESS (observation) $= a^2(\tau)(d\tau^2 - d\vec{x}^2)$

~~[illegible]~~ $a^2(\tau)$ = scale factor

dx^i : comoving coordinates ($dX^i = a(\tau) dx^i$ physical coords.)

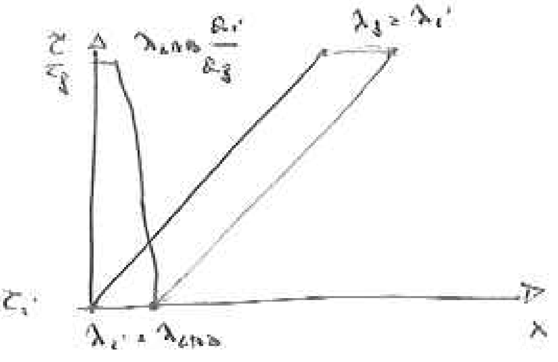
$d\tau = \frac{dt}{a}$: conformal time (dt : physical Time)

FLRW KINEMATICS :

$$ds^2 = a^2(\tau)\,\underbrace{\left(d\tau^2 - d\vec{x}^2\right)}_{\text{Minkowsk}}$$

FLRW is "CONFORMAL" TO MINKOWSKI

Photons Trajectories :



$\ell = \frac{L}{a}$ ← physical

↗ comoving

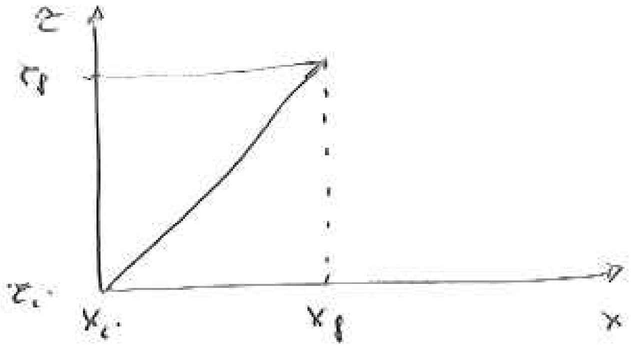
$\Rightarrow \frac{\lambda}{\lambda_{CMB}} \propto a$ cosmological redshift

Today: $a(\tau_0) = a_0 = 1$

$$\boxed{z = \frac{1}{a} - 1}$$

Particle horizon :

Distance covered by a photon from $\tau_i \to \tau_f$:



$$\Delta x = \Delta\tau = \int_{\tau_i}^{\tau_f} d\tau =$$

$$= \int_{a_i}^{a_f} \frac{da}{\dot{a}} = \int_{z_f}^{z_i} \frac{dz}{(1+z)\mathcal{H}(z)}$$

$\mathcal{H} \equiv \frac{\dot{a}}{a}$ (conformal time) Hubble Parameter

depends on the expansion history

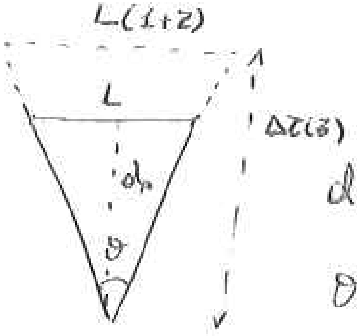
$a_i \to 0$ ($z_i \to \infty$) → $\boxed{\text{PARTICLE HORIZON}}$: $d_H(z) = \int_z^\infty \frac{dz'}{(1+z')\mathcal{H}(z')}$

$$= \int_0^a \frac{da'}{a'\mathcal{H}(a')}$$

if $\mathcal{H} \sim a^{-\frac{(1+3w)}{2}}$ with $w > -\frac{1}{3}$ $\to d_H(a) \sim \frac{1}{\mathcal{H}(a)}$ ← HUBBLE HORIZON

From $\Delta\tau$ we can define observable distances:

ANGULAR DIAMETER DISTANCE : Object of known transverse physical length: L



$$d_A \equiv \frac{L}{\theta}$$

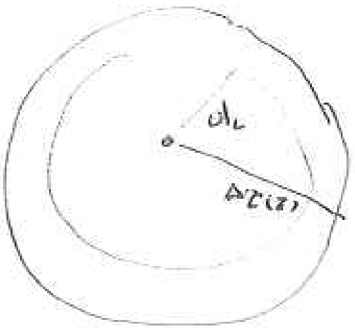
$$\theta = \frac{L(1+z)}{\Delta\tau(z)} \Rightarrow \boxed{d_A(z) = \frac{\Delta\tau(z)}{1+z}}$$

LUMINOSITY DISTANCE Object of known absolute luminosity L

$$d_L = \sqrt{\frac{L}{4\pi F}}$$

L : absolute luminosity

F : Measured flux



$$F = \frac{L/(1+z)^2}{4\pi\,\Delta\tau^2(z)} \quad \begin{matrix} \searrow E_\gamma \sim (1+z)^{-1} \\ \searrow \dfrac{\Delta N_\gamma}{\Delta t} \sim (1+z)^{-1} \end{matrix}$$

$$\Rightarrow \boxed{d_L(z) = \Delta\tau(z)\,(1+z)}$$

CONSISTENCY RELATION : $\boxed{d_L(z) = d_A(z)(1+z)^2}$

No dynamics, JUST FLRW kinematics!

PECULIAR VELOCITY! $v^i = \frac{dx^i}{d\tau}$ ← COMOVING ← CONFORMAL

X^i physical

t "

$$\underset{\substack{\uparrow \\ \text{Physical} \\ \text{velocity}}}{V^i} = \frac{dX^i}{dt} = \frac{d(a x^i)}{a\, d\tau} = \underset{\substack{\uparrow \\ \text{Hubble} \\ \text{flow}}}{\mathcal{H} x^i} + \underset{\substack{\uparrow \\ \text{Peculiar} \\ \text{velocity}}}{v^i}$$

Free massive particle in FLRW: $S = -m\int ds = -m\int\sqrt{1 - \dot{x}^i \dot{x}^i}\, a\, d\tau$

$\mathcal{L} = -m a(\tau) \sqrt{1 - \dot{x}^i \dot{x}^i}$

conj. momentum: $P_i = \frac{\partial \mathcal{L}}{\partial \dot{x}^i} = m \gamma a v^i$

$\dot{P}_i = \frac{\partial \mathcal{L}}{\partial x^i} = 0 \Rightarrow P_i$ is conserved

$$\Rightarrow \boxed{\gamma v^i \sim a^{-1} \xrightarrow[\text{NON REL}]{} v^i \sim a^{-1}}$$

EXPANSION ACTS AS A FRICTION

FLRW DYNAMICS

Einstein eqs.

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$$

Einstein Tensor (GEOMETRY)

Energy-momentum Tensor

Metric: $g_{\mu\nu}$ (EX: FLRW $g_{\mu\nu} = a^2(t)\begin{pmatrix}1 & & & 0\\ & -1 & & \\ & & -1 & \\ 0 & & & -1\end{pmatrix}$)

↓

Christoffel symbol $\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\gamma}(\partial_\alpha g_{\gamma\beta} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta})$

↓

Ricci Tensor $R_{\mu\nu} = \partial_\lambda \Gamma^{\lambda}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}_{\mu\lambda} + \Gamma^{\lambda}_{\lambda\rho}\Gamma^{\rho}_{\mu\nu} - \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\rho}$

\+

Ricci scalar $R = g^{\mu\nu} R_{\mu\nu}$

↓

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

FLRW: $G_{00} = 3\mathcal{H}^2$

(Try!) $G^{\mu}{}_{\mu} = 6a^{-2}(\dot{\mathcal{H}} + \mathcal{H}^2)$

$g^{\mu\nu} = a^{-2}\begin{pmatrix}1 & & & 0\\ & -1 & & \\ & & -1 & \\ 0 & & & -1\end{pmatrix}$

$$T_{\mu\nu} = \int dP_1 dP_2 dP_3 \sqrt{-g}\, \frac{P_\mu P_\nu}{P^0} f(x^i, P_j, \tau)$$

Boltzmann equation

↓

Energy-Momentum Tensor compatible with homogeneity and isotropy:

$$\left.\begin{aligned} T_{00} &= a^2 \rho(t) \qquad T_{0i} = 0 \\ T_{ij} &= a^2 P(t)\, \delta_{ij} \end{aligned}\right\}$$

Perfect fluid in comoving frame!

$\Rightarrow (\ )_{00}$ EQ: $\Rightarrow$ $\mathcal{H}^2 = \frac{8\pi G}{3} \rho a^2$

$(\ )^{\mu}{}_{\mu} - (\ )^{0}{}_{0} \Rightarrow$ $\dot{\mathcal{H}} = -\frac{4\pi}{3} G (\rho + 3P) a^2$

Covariant conservation of the e.m. tensor:

$$T^\mu{}_{\nu;\mu} = \partial_\mu T^\mu{}_\nu + \Gamma^\mu_{\mu\alpha} T_\nu{}^\alpha - \Gamma^\alpha_{\mu\nu} T^\mu{}_\alpha = 0$$

On FLRW → CONTINUITY EQUATION $(\nu = 0)$

$$\hookrightarrow \boxed{\dot\rho = -3\mathcal{H}(\rho + P)}$$

$$\hookrightarrow \begin{cases} \mathcal{H}^2 = \frac{8\pi}{3} G \rho a^2 \\ \frac{d\rho}{d\ln a} = -3(\rho + P) \end{cases}$$

if $P = w\rho$ w : eq of state

$$\begin{array}{ll} \nearrow w = 0 & \text{"MATTER"} \\ \rightarrow w = 1/3 & \text{"RADIATION"} \\ \searrow w = -1 & \text{"}\Lambda\text{"} \end{array}$$

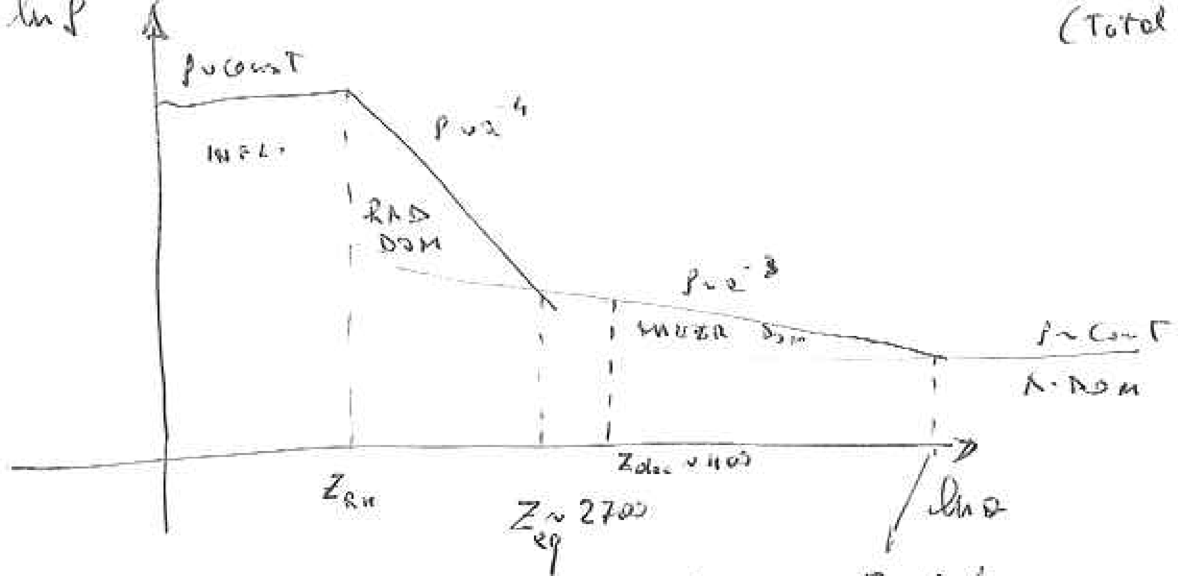
$$\hookrightarrow \rho \sim a^{-3(w+1)} \rightarrow a(\tau) \sim \tau^{\frac{2}{3w+1}}$$

MULTI-FLUIDS : $\rho = \Sigma_i \rho_i = \Sigma_i \rho_i^0 a^{-3(w_i+1)} =$

$$= \rho^0 \Sigma_i \Omega_i a^{-3(w_i+1)}$$

$$\Omega_i = \frac{\rho_i^0}{\rho^0}$$

↖ "critical density" (Total density)



Matter-radiation eq. $\rho_M(z) = \rho_R(z) \Rightarrow \Omega_M (1+z)^3 = \Omega_R (1+z)^4$

$$\rightarrow 1 + z_{eq} = \frac{\Omega_M}{\Omega_R} \simeq \frac{0.23}{8.4 \cdot 10^{-5}}$$

$\boxed{\text{FLRW: } 1^{st} \text{ order perturbations}}$ ($\to$ "tree level")

consider perturbed metric and e.m. Tensors:

$$g_{\mu\nu} = g^{FLRW}_{\mu\nu} + \delta g_{\mu\nu} \quad , \quad T_{\mu\nu} = T^{FLRW}_{\mu\nu} + \delta T_{\mu\nu}$$

$$\swarrow \qquad\qquad\qquad\qquad \swarrow$$

$$a^2\begin{pmatrix} 1 & & 0 \\ & -1 & \\ 0 & & -1 \end{pmatrix} \qquad\qquad \begin{aligned} T_{00} &= \rho a^2 \\ T_{0i} &= 0 \\ T_{ij} &= \delta_{ij} P a^2 \end{aligned}$$

Note: in general $g_{\mu\nu}$ (and $T_{\mu\nu}$) have 10 independent components.

Not all of them are physical. ~~not all of them are couple~~

Not " " " " coupled at first order:

d.o.f.

Scalar - Vector - Tensor (SVT) decomposition $\to$ 4 scalars, $\to$ 4 vectors, $\to$ 2 Tensor

At first order S/V/T do not mix

* Moreover, vectors are damped $V^i \sim a^{-2}$ $\to$ neglect them

* we will Consider only scalar perturbations: $\underset{\delta g_{\mu\nu}}{4} + \underset{\delta T_{\mu\nu}}{4}$

* Not all the 8 scalar d.o.f are physical:

general coord. transformations

$$x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu(\vec{x},\tau)$$

$$\xi^0(\vec{x},\tau) = T(\vec{x},\tau)$$

$$\xi^i(\vec{x},\tau) = \partial^i L(\vec{x},\tau) + \hat{L}^i(\vec{x},\tau)$$

2 scalars

$(\partial_i \hat{L}^i = 0)$

1 vector

$\hookrightarrow$ $\boxed{\text{FIX } T, L \Rightarrow \text{FIX the gauge} \Rightarrow 8-2 = 6 \text{ scalar d.o.f}}$

CONFORMAL NEWTONIAN GAUGE

METRIC : $ds^2 = a^2(\tau)\left(\left(1+2\psi(\vec{x},\tau)\right)d\tau^2 - \left(1-2\phi(\vec{x},\tau)\right)dx^i dx^j \delta_{ij}\right)$

E.M. TENSOR :
$$\begin{cases} T^0{}_0 = \bar{\rho}(\tau) + \delta\rho(\vec{x},\tau) \\ T^0{}_i = \partial_i q = -T^i{}_0 \quad \text{MOMENTUM} \\ T^i{}_j = -\left(\bar{P}(\tau) + \delta P(\vec{x},\tau)\right)\delta^i{}_j - \left(\partial^i\partial_j - \frac{1}{3}\nabla^2\delta^i{}_j\right)\Pi \end{cases}$$
↙ ANISOTROPIC STRESS

ψ, ϕ scalar potentials

$\delta\rho, \delta P, q, \Pi$ matter perturbations.

} 1^{st} order perturbations

$\boxed{\delta T^\mu{}_{\nu;\mu} = 0}$

$\nu = 0 \rightarrow \boxed{\delta\dot{\rho} = -3\mathcal{H}(\delta\rho + \delta P) + 3\dot{\phi}(\bar{\rho}+\bar{P}) - \nabla^2 q}$

continuity equation

$\nu = i \rightarrow \boxed{\dot{q} = -4\mathcal{H}q - (\bar{\rho}+\bar{P})\psi - \delta P - \frac{2}{3}\nabla^2\Pi}$

Euler equation

δP: adiabatic sound speed : $c_s^2 = \dfrac{\delta P}{\delta\rho}$

Π : anisotropic stress : Negligible for DM, B, γ

sourced by ν's → Boltzmann equation

$$\boxed{\delta G^\mu{}_\nu = 8\pi G\, \delta T^\mu{}_\nu}$$

$$(\;)^0{}_0 \Rightarrow \boxed{\nabla^2\Phi - 3\mathcal{H}(\dot\Phi + \mathcal{H}\Psi) = 4\pi G a^2 \delta\rho}$$

Relativistic Poisson eq.

$$(\partial^i\partial_j - \tfrac{1}{3}\delta^i_j\nabla^2)(\;)^j{}_i \Rightarrow \boxed{\Phi - \Psi = 8\pi G a^2 \Pi}$$

No anisotropic stress!

$\Pi = 0 \rightarrow \Phi = \Psi$

$$\left[\begin{aligned} &(\;)^i{}_i \Rightarrow \boxed{\ddot\Phi + 3\mathcal{H}\dot\Phi + (2\dot{\mathcal{H}} + \mathcal{H}^2)\Phi = 4\pi G a^2 \delta P} \\ &(\;^0{}_i) + \text{Poisson} \Rightarrow \boxed{\nabla^2\Phi = 4\pi G a^2 \underbrace{(\delta\rho - 3\mathcal{H} q)}_{\equiv \bar\rho\Delta \,\leftarrow\, \text{comoving density contrast}}} \end{aligned}\right]$$

Notice: multi species ~~case~~ $\Longrightarrow$

$\delta\rho = \Sigma_a \delta\rho_a$

$q = \Sigma_a q_a$ $\qquad a = DM, B, \gamma, \ldots$

$c_s^2 \longrightarrow c_{s,a}^2 = \dfrac{\delta P_a}{\delta\rho_a}$

$\Pi = \Sigma_a \Pi_a$

$\boxed{\text{Evolution of the gravitational potentials } \Psi, \Phi}$

- We will assume $\Pi \simeq 0 \Rightarrow \Phi = \Psi$

$$\ddot{\phi} + 3\mathcal{H}\dot{\phi} + (2\dot{\mathcal{H}} + \mathcal{H}^2)\phi = 4\pi G a^2 \delta P$$

$\nwarrow = c_s^2 \delta\rho$ ← sound speed

* **MATTER DOMINATION:**

- $\mathcal{H}^2 = \frac{8\pi}{3} G \bar{\rho} a^2 = \frac{8\pi}{3} G \bar{\rho}_0 a^{-1}$

$\hookrightarrow \bar{\rho} = \bar{\rho}_0 a^{-3}$

$\Rightarrow 2\dot{\mathcal{H}} = -\mathcal{H}^2$

- $\delta P \simeq \delta P_M = 0$

$\uparrow$ matter

$$\Rightarrow \boxed{\ddot{\phi} + 3\mathcal{H}\dot{\phi} = 0}$$ matter dom $\mathcal{H} = \frac{2}{\tau}$

$$\boxed{\ddot{\phi} + \frac{\cancel{6}}{\tau}\dot{\phi} = 0} \rightarrow \phi \rightarrow \text{const}, \ \searrow \tau^{-5}$$

$y = \ln a \qquad \frac{d}{d\tau} = \mathcal{H}\frac{d}{dy} \qquad \frac{d^2}{d\tau^2} = \dot{\mathcal{H}}\frac{d}{dy} + \mathcal{H}^2\frac{d^2}{dy^2}$

$$= \mathcal{H}^2\left(\frac{d^2}{dy^2} - \frac{1}{2}\frac{d}{dy}\right)$$

$$\Rightarrow \phi'' + \frac{5}{2}\phi' = 0$$

$$\phi \sim e^{cy} \searrow c^2 + \frac{5}{2}c = 0 \ \nearrow c = 0, \ \searrow c = -\frac{5}{2}$$

$\phi \sim \nearrow \text{const}$ ← "growing mode"

$ \searrow a^{-5/2}$ ← "Decaying" mode

NOTE: INDEPENDENT ON SPATIAL SCALE!!

* RADIATION DOMINATION

- $\mathcal{H}^2 = \frac{8\pi G}{3}\rho_R^0 a^{-2}$ $(\rho_R = \rho_R^0 a^{-4})$

  $\Rightarrow \dot{\mathcal{H}} = -\mathcal{H}^2$

- $c_s^2 = \frac{1}{3}$

$$\Rightarrow \ddot{\phi} + 3\mathcal{H}\dot{\phi} - \mathcal{H}^2\phi = \frac{4\pi G}{3}\delta\rho\, a^2 \underset{\substack{\uparrow \\ (\,)^0_0\ \text{eq}}}{=} \frac{1}{3}\left(\nabla^2\phi - 3\mathcal{H}(\dot{\phi} + \mathcal{H}\phi)\right)$$

$$\Rightarrow \boxed{\ddot{\phi} + 4\mathcal{H}\dot{\phi} = \frac{1}{3}\nabla^2\phi}_{\text{RAD DOM}}$$

$$\phi_k(\tau) = \int d^3x\, e^{i\vec{k}\cdot\vec{x}}\,\phi(\vec{x},\tau)$$ FOURIER TRANSFORM

$$\mathcal{H} = \frac{1}{\tau} \Rightarrow \boxed{\ddot{\phi}_k + \frac{4}{\tau}\dot{\phi}_k + \frac{1}{3}k^2\phi_k = 0}$$

select regular sol in $\tau \to 0$

$$\Rightarrow \phi_k(\tau) = 3\,\phi_k(0)\,\underbrace{\frac{\sin y - y\cos y}{y^3}}_{\frac{J_1(y)}{y}}$$

$$\boxed{y = \frac{1}{\sqrt{3}}k\tau = \frac{1}{\sqrt{3}}\frac{k}{\mathcal{H}}}$$

scale-dependent!

- super ~~sub~~-horizon scales: $\frac{k}{\mathcal{H}} \ll 1 \Rightarrow \phi_k(\tau) \simeq \phi_k(0)$

  → CONSTANT

- sub-horizon scales $\frac{k}{\mathcal{H}} \gg 1 \Rightarrow \phi_k(\tau) = -9\phi_k(0) \times \frac{\cos k\tau/\sqrt{3}}{k^2\tau^2}$

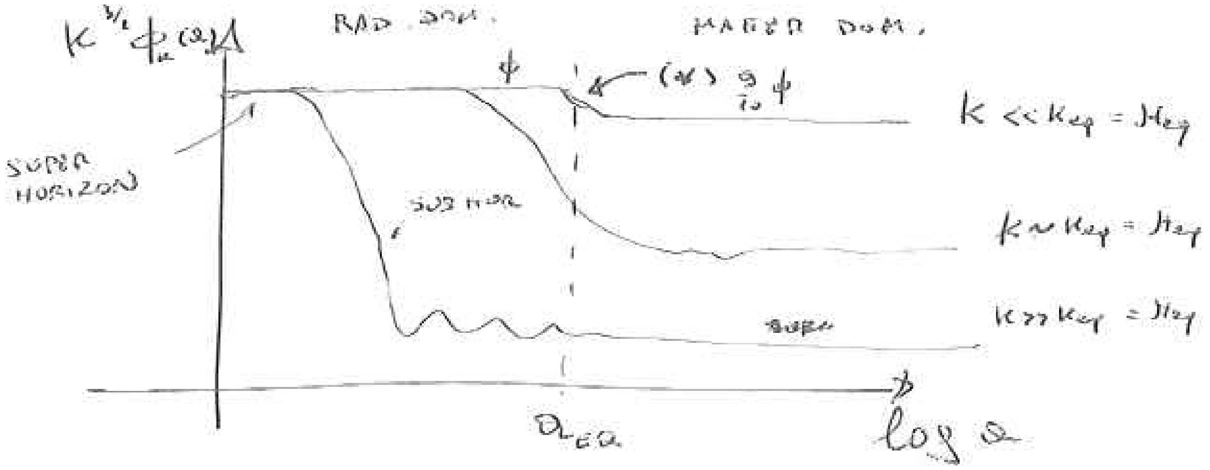
  → Time oscillation with frequency $\frac{k}{\sqrt{3}}$

  → Decaying amplitude as $\sim\frac{1}{\tau^2}\sim\frac{1}{a^2}$

IN SUMMARY

$$\frac{k}{\mathcal{H}} \sim a^{\frac{3w+1}{2}} \begin{cases} \nearrow a & \text{RAD. DOM} \\ \searrow a^{1/2} & \text{MATT. DOM.} \end{cases}$$

$\phi_k(a)$	SUPER HORIZON $k \ll \mathcal{H}$	SUB HORIZON $k \gg \mathcal{H}$
MATTER DOM $\tau \sim a^{1/2}$	CONST	CONST
RAD DOM $\tau \sim a$	CONST	$\sim a^{-2} \cos \frac{k\tau}{\sqrt{3}}$



(*) Note that Φ is not conserved on super horizon scales during a background transition (e.g. from RAD to MAT)

The conserved quantity is $\mathcal{R} = -\phi + \frac{\mathcal{H}}{\rho + P} q$: COMOVING CURVATURE PERTURBATION

To relate $\mathcal{R}$ to q use the $(\ )^0_i$ Einstein eq:

$$\dot{\phi} + \mathcal{H}\phi = -4\pi G a^2 q \xrightarrow[k \ll \mathcal{H}]{} q = -\frac{\mathcal{H}\phi}{4\pi G a^2}$$

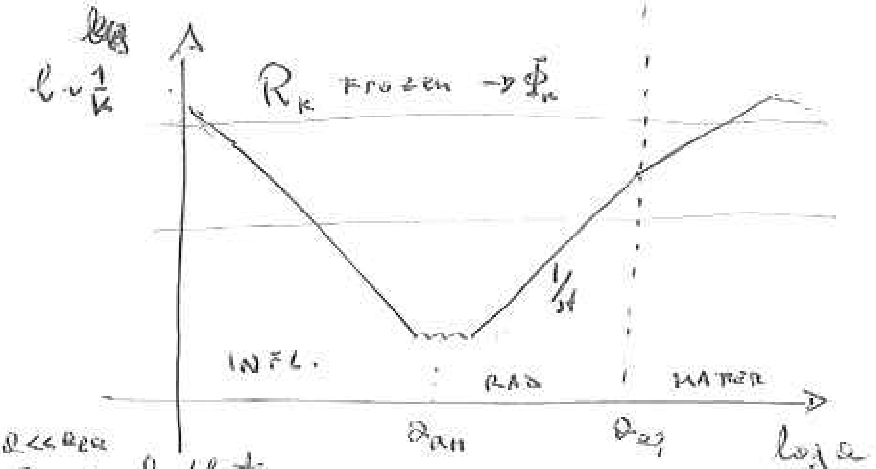
$$\Rightarrow \boxed{\mathcal{R} = -\phi \, \frac{5+3w}{3(1+w)}}$$

RAD DOM : $w = \frac{1}{3}$ $\Rightarrow$ $\mathcal{R} = -\frac{3}{2}\phi_{RAD}$

MAT DOM : $w = 0$ $\Rightarrow$ $\mathcal{R} = -\frac{5}{3}\phi_{MAT}$

$$\Rightarrow \boxed{\phi_{MAT} = \frac{9}{10}\phi_{RAD}}$$

Initial conditions



Initial conditions given at $\overset{a \ll a_{eq}}{a}$ such that all interesting scales are $\frac{k}{\mathcal{H}} \ll 1$ (superhorizon)

$\Leftrightarrow$ SUPER HORIZON + RAD DOMINATION $\quad \Phi_k = -\frac{2}{3} R_k$

MATCHING WITH INITIAL CONDITIONS FROM INFLATION !!

$\delta_a = \frac{\delta\rho_a}{\rho_a}$ dimensionless density contrast

$\delta\rho = \sum_a \rho_a \delta_a \underset{\text{RAD. DOM.}}{\simeq} \rho_{RAD}\,\delta_{RAD}$ (because all the δ_a's are of the same order, see below)

Rel. Poisson eq $\left(^0_0\right) \xrightarrow[\frac{k}{\mathcal{H}} \ll 1]{} \quad -3\mathcal{H}^2\phi \simeq \underbrace{4\pi G a^2 \rho_{RAD}}_{\frac{3}{2}\mathcal{H}^2}\delta_{RAD} \Rightarrow \boxed{\delta_{RAD,\vec{k}} = -2\phi_k = \frac{4}{3}R_k}$

For the remaining species, we impose ADIABATIC INITIAL CONDITIONS :

$$\boxed{\frac{\delta_{a,k}}{1+w_a} = \frac{\delta_{RAD,k}}{\frac{4}{3}} = R_k \qquad \forall a}$$

All the perturbations are due to the same local time shift :

$\tau \to \tau + \Delta\tau(\vec{x}) \Rightarrow \rho_a(\tau) \to \rho_a(\tau + \Delta\tau(x)) \simeq \rho_a(\tau) + \dot{\rho}_a(\tau)\cdot\Delta\tau(x)$

$$\delta_a(\vec{x},\tau) = \frac{\rho_a(\tau+\Delta\tau(x)) - \rho_a(\tau)}{\rho_a(\tau)} = \frac{\dot{\rho}_a(\tau)}{\rho_a(\tau)}\Delta\tau(\vec{x}) = -3\mathcal{H}\frac{(\rho_a + P_a)(\tau)}{\rho_a}\Delta\tau(\vec{x})$$

$$\Rightarrow \frac{\delta_a(\vec{x},\tau)}{1+w_a} = -3\mathcal{H}\Delta\tau(\vec{x}) \quad \forall a$$

Adiabatic initial condition are a prediction of single-clock inflationary models (1 field). OK with CMB observations!

Orthogonal possibility: ISO-CURVATURE PERTURBATIONS

$$S_{ij} = \frac{\delta_i}{1+w_i} - \frac{\delta_j}{1+w_j} \qquad (= 0 \text{ in the following})$$

$$\boxed{\text{OBSERVABLES}} \longrightarrow \boxed{\text{CORRELATION FUNCTIONS}}$$

$$\xi^N(\vec{x}_1, \dots \vec{x}_N) \equiv \langle \delta(\vec{x}_1) \dots \delta(\vec{x}_N) \rangle$$

(we will consider only $\tau_1 = \tau_2 \dots = \tau_N$)

$$= \int d\delta_1 \dots d\delta_N \, P_N(\delta_1 \dots \delta_N) \, \delta_1 \dots \delta_N$$

Average over realizations identified by a PDF

P.D.F. cosmology-dependence

$N = 2$ → "correlation function"

$$\xi(\vec{x}_1, \vec{x}_2) = \langle \delta(\vec{x}_1) \delta(\vec{x}_2) \rangle = \frac{\langle \rho(\vec{x}_1) \rho(\vec{x}_2) \rangle}{\langle \rho \rangle^2} - 1$$

- excess of "clustering" with respect to a random field ($\langle \rho(x_1) \rho(x_2) \rangle = \langle \rho \rangle^2$)
- translation invariance → $\xi(\vec{x}_1, \vec{x}_2) \to \xi(\vec{x}_1 - \vec{x}_2)$
- rotation " → $\xi(\vec{x}_1 - \vec{x}_2) = \xi(|\vec{x}_1 - \vec{x}_2|)$

$$\boxed{-1 \leq \xi < \infty} \quad \text{as} \quad \int_0^\infty dr \, \xi(r) = 0$$

GAUSSIAN FIELD

$$P_N(\delta_1,\ldots\delta_N)=\frac{1}{(2\pi)^{N/2}\sqrt{\det\Sigma}}\exp\left(-\frac{1}{2}\Delta^T\cdot\Sigma^{-1}\cdot\Delta\right)$$

$$\Delta=\begin{pmatrix}\delta_1\\ \vdots\\ \delta_N\end{pmatrix}\qquad \Sigma_{ij}=\langle\delta_i\cdot\delta_j\rangle$$

$$\Rightarrow\begin{cases}\langle\delta_1\cdots\delta_{2N+1}\rangle=0\\ \langle\delta_1\cdots\delta_{2N}\rangle=\langle\delta_1\delta_2\rangle\cdots\langle\delta_{2N-1}\delta_{2N}\rangle+\text{Permutations}\end{cases}$$

Fourier Space:

$$\delta_k(\tau)=\int d^3x\, e^{i\vec{k}\cdot\vec{x}}\,\delta(\vec{x},\tau)\qquad\left(\delta^*_{\vec{k}}=\delta_{-\vec{k}}\right)$$

$$\delta(\vec{x},\tau)=\int\frac{d^3k}{(2\pi)^3}\,e^{-i\vec{k}\cdot\vec{x}}\,\delta_k(\tau)$$

$$\langle\delta_{\vec{k}}(\tau)\,\delta_{\vec{k}'}(\tau)\rangle=(2\pi)^3\delta_D(\vec{k}+\vec{k}')\,P(k,\tau)\ \leftarrow\text{Power Spectrum}$$

$$\boxed{P(k,\tau)=\int d^3r\, e^{i\vec{k}\cdot\vec{r}}\,\xi(r,\tau)}\qquad \downarrow\ \boxed{\langle|\delta_{\vec{k}}|^2\rangle=(2\pi)^3\underbrace{\delta_D(\vec{k}=0)}_{Vol}P(k)}\ (*)$$

$$\xi(0)=\langle\delta^2(\vec{x})\rangle=\int\frac{d^3k}{(2\pi)^3}P(k)=\int\frac{dk}{k}\underbrace{\frac{k^3P(k)}{2\pi^2}}_{\text{Dimension-less Power Spectrum}}$$

higher orders:

$$\langle\delta_{\vec{k}_1}\delta_{\vec{k}_2}\delta_{\vec{k}_3}\rangle=(2\pi)^3\delta_D(\vec{k}_1+\vec{k}_2+\vec{k}_3)\,B(k_1,k_2,k_3;\tau)$$

"Bispectrum"

(= 0 for a gaussian field)

$$\langle\delta_{\vec{k}_1}\delta_{\vec{k}_2}\delta_{\vec{k}_3}\delta_{\vec{k}_4}\rangle=(2\pi)^6\delta_D(\vec{k}_1+\vec{k}_2)\delta_D(\vec{k}_3+\vec{k}_4)\,P(k_1,\tau)P(k_2,\tau)$$

$$+(2+3)(4+1)$$

$$+(3+1)(2+4)$$

$$+(2\pi)^3\,T(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)\ \leftarrow\text{"Trispectrum"}$$

↖ connected, Non gaussian part.

. . .

the linear matter PS

~~Rel~~ From $\nabla^2\phi = \frac{3}{2}\mathcal{H}^2\Delta$ → $\boxed{\Delta_k(a) = -\frac{2}{3}\frac{k^2}{\mathcal{H}^2}\phi_k(a)}$

$$\Delta = \delta - 3\mathcal{H}\frac{q}{\rho} = \delta + 3(1+w)\frac{\mathcal{H}^2}{k^2}\frac{\theta}{\mathcal{H}} \xrightarrow[\delta = -\frac{\theta}{\mathcal{H}} \text{ in linear th.}]{k^2 \gg \mathcal{H}^2} \simeq \delta\left(1 + O\left(\frac{\mathcal{H}^2}{k^2}\right)\right)$$

↑ $-kq = (1+w)\rho\theta$

↑ $\vec\nabla\cdot\vec v$ ↑ peculiar velocity

Remember the evolution for $\phi_k(a)$: Consider $a \gg a_{eq}$, $a_{in} \ll a_{eq}$ $\forall k$

$\mathcal{H}_{eq} = \mathcal{H}(a_{eq}) \equiv k_{eq}$, a_{eq} equivalence

$\mathcal{H}_k$: $\mathcal{H}_k = k$, a_k epoch at which k enters the horizon.

1) $k < k_{eq}$ ($\Rightarrow a_k > a_{eq}$): scale enters the horizon after equivalence.

$$\underbrace{\phi_k(a) = \phi_k(a_{eq})}_{MD} \underbrace{=}_{RD} \frac{9}{10}\phi(a_{in}) = \frac{9}{10}\left(-\frac{2}{3}\right)R_k = -\frac{3}{5}R_k$$

2) $k > k_{eq}$

$$\phi_k(a) = \phi_k(a_{eq}) = \underbrace{\left(\frac{a_k}{a_{eq}}\right)^2\phi_k(a_k)}_{RD,\ \text{sub horizon}} = -\frac{3}{5}\left(\frac{k_{eq}}{k}\right)^2 R_k$$

↓ $\mathcal{H}^2 \sim a^{-2}$ RAD DOM

$$\left(\frac{a_k}{a_{eq}}\right)^2 = \left(\frac{\mathcal{H}_{eq}}{\mathcal{H}_k}\right)^2 = \left(\frac{k_{eq}}{k}\right)^2$$

$$\Rightarrow \boxed{\Delta_k(a) = \frac{2}{5}\frac{k^2}{\mathcal{H}^2}\,\cancel{R_k}\,T(k)\,R_k}$$

$\frac{2}{5}\frac{k^2}{\mathcal{H}^2}$ ↙ Growth FACTOR $\frac{1}{\mathcal{H}^2} \sim a$ (MATTER DOMINATION)

$T(k)$ ↓ $T \to 1$ $k \ll k_{eq}$, $T \to \left(\frac{k_{eq}}{k}\right)^2$ $k \gg k_{eq}$ TRANSFER FUNCTION

R_k ↑ initial condition (inflation)

① Time-dependence.

Before, we have considered the eqs. for $\dot{\Phi}, \ddot{\Phi}$ in a simplified background (that is, either $\rho = \rho_{RAD}$ or $\rho = \rho_{MATTER}$). We now consider

$$\rho = \rho_{RAD} + \underset{\substack{\uparrow \\ DM+B}}{\rho_{MATTER}} + \rho_\Lambda \quad \text{for } \underline{\underline{z < z_{dec}}} \quad (\delta_B \sim \delta_{DM})$$

CONTINUITY EQ FOR MATTER: $w = c_s^2 = 0$

↑ EULER

$$\rightarrow \begin{cases} \dot{\delta}_M = -(\theta_M - 3\dot{\phi}) \\ \dot{\theta}_M = -\mathcal{H}\theta_M + k^2\phi \end{cases} \qquad \theta = \frac{-k^2 q}{\rho(1+w)} \equiv \vec{\nabla}\cdot\vec{v}$$

$$\hookrightarrow \ddot{\delta}_M + \mathcal{H}\dot{\delta}_M = -k^2\phi + 3(\ddot{\phi} + \mathcal{H}\dot{\phi}) \underset{k^2 \gg \mathcal{H}^2}{\simeq} -k^2\phi = \frac{3}{2}\mathcal{H}^2\Omega_M\delta_M$$

$$\rho_{TOT}\delta_{TOT} \simeq \rho_M\delta_M = \rho_{TOT}\Omega_M\delta_M$$

$$\uparrow \; \delta_\gamma, \delta_\Lambda \ll \delta_M$$

$$\ddot{\delta}_M + \mathcal{H}\dot{\delta}_M = \frac{3}{2}\underbrace{\mathcal{H}^2\Omega_M}\delta_M(k,a)$$

↑ DAMPING $\mathcal{H}^2 \sim \rho_{TOT}$

GRAV. FORCE $\propto \rho_{TOT}\Omega_M$

$$\delta_M(a) = D(a)\,\delta_M(a_0)$$

↑ growth factor

$$k \gg \mathcal{H} \rightarrow D(a) \begin{cases} \ln\frac{a}{a_{eq}} & \text{RAD DOM} \rightarrow P(k) \overset{k \gg k_{eq}}{\sim} k^{n_s-4} \times \ln^2\left(1+\frac{k}{k_{eq}}\right) \\ a & \text{MATT. DOM} \\ \text{CONST} & \Lambda\text{-DOM} \end{cases}$$

Λ-DOM → Pert. growth ~~stops~~ slows down in recent times.

↓

$$\Omega_M \simeq \frac{\rho_M}{\rho_M + \rho_\Lambda} = \frac{1}{1 + \frac{\Omega_\Lambda}{\Omega_M}a^3} \rightarrow 0$$

If Dark Energy $\neq \Lambda$ $\Rightarrow w_{DE} > -1$

$$\Omega_M \sim \frac{1}{1+\frac{\Omega_{DE}}{\Omega_M} a^{-3w_{DE}}} \Rightarrow \text{growth stops earlier}$$

② Baryonic Acoustic Oscillations

Prior to decoupling, baryons and photons can be described as a single fluid: $\delta_\gamma = \frac{4}{3}\delta_b$, $q = q_\gamma + q_b$ conserved MOMENTUM

$$\hookrightarrow \ddot{\delta}_\gamma + \underset{\text{DAMPING}}{\frac{\mathcal{H}R}{1+R}\dot{\delta}_\gamma} + \underset{\text{PRESSURE!}}{c_s^2 k^2 \delta_\gamma} = \underset{\text{gravity}}{-\frac{4}{3}k^2\phi} + O\left(\frac{\mathcal{H}^2}{k^2}\right)$$

$$R \equiv \frac{3}{4}\frac{\rho_b}{\rho_\gamma} = 0.6\left(\frac{\Omega_b h^2}{0.02}\right)\left(\frac{a}{10^{-3}}\right) \quad , \quad c_s^2 \equiv \frac{1}{3(1+R)} \quad \text{sound speed}$$

NOTICE!

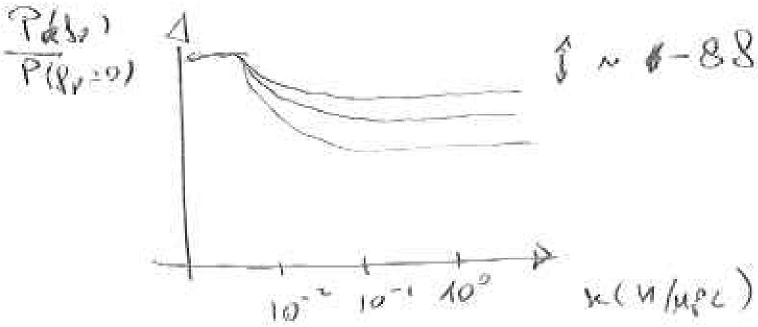
before decoupling $\delta_b \sim \delta_\gamma \sim \Phi(k)\cos(c_s k \tau)$

while $\delta_{DM} \propto D(\tau)$ grows!

This is the main reason for NON-BARYONIC DM!!

At $\tau = \tau_{REC}$ oscillations $\cos(k\, c_s(\tau_{rec})\cdot \tau_{rec})$ are frozen in the baryon δ_b. $\rightarrow$ the scale $r_s = c_s(\tau_{rec})\cdot\tau_{rec} = \frac{\tau_{rec}}{\sqrt{3(1+R(\tau_{rec}))}}$ is imprinted in the fluctuations $\nearrow$ CMB

$\searrow$ LSS



$\updownarrow \sim \cancel{6}-8f_\nu$

$$\cancel{f_\nu} \approx \frac{\sum_i m_i}{93.14\,h^2\,\mathrm{eV}} = \Omega_\nu$$

$$= f_\nu(1-\Omega_\Lambda) \quad \cancel{\approx \frac{1}{16}}\,\Omega_\Lambda$$

$$\text{if } \sum_i m_\nu \approx 0.1\,\mathrm{eV} \rightarrow \frac{\Delta\rho}{\rho} \sim -8f_\nu \simeq -0.0\cancel{8}$$

Notice: Unlike $m_\nu = 0$ case, the growth of the perturbation becomes <u>scale-dependent</u>

④ Features of Physics Beyond ΛCDM

- WARM DM
- FUZZY DM
- QUINTESSENCE / DYNAMICAL DE
- MODIFIED GR

⋮

⟹ "Discussion session!!"

③ The effect of massive neutrinos. $\Gamma_w \sim G_F^2 T^5 \lesssim H \sim g_*^{1/2} \frac{T^2}{M_{PL}}$

- SM Neutrinos decouple at $T \lesssim$ MeV $\rightarrow$ relativistic!

$$T_\nu(z) = \left(\frac{4}{11}\right)^{1/3} T_\gamma(z) \qquad \langle P \rangle = 3.15\, T_\nu \quad \text{(Relativistic Fermi-Dirac)}$$

$$\frac{\langle P \rangle}{m_\nu} \propto T_\nu(z) \propto T_\nu^0(1+z) \qquad \frac{\langle P \rangle}{m} \lesssim 1 \rightarrow \text{NON-RELATIVISTIC TRANSITION}$$

$\hookrightarrow T = T_{NR},\ \mathcal{H}_{NR},\ K_{NR}$

if $\underset{\uparrow \atop 3.15\left(\frac{4}{11}\right)^{1/3}T_0}{5.28\cdot 10^{-4}\,\text{eV}} \lesssim m_\nu \lesssim \underset{\uparrow \atop 3.15\left(\frac{4}{11}\right)^{1/3}T_{eq}}{3.1\,\text{eV}}$ $\rightarrow$ ν becomes NR between today and equivalence

- $m_\nu < O(1\,\text{eV})$ Neutrinos become NR during Matter Domination
- At most 1 SM ν is still relativistic today.
(~~$\Delta m^2_{atm} \simeq 10^{-3}\,\text{eV}^2$~~)
- Relativistic neutrinos : $c_\nu \simeq 1$, $\lambda_{fs} = \frac{2\pi}{k_{fs}} = 2\pi \frac{c_\nu}{\mathcal{H}(z)} \left(\sqrt{\frac{2}{3}}\right)$
$\uparrow$ free-streaming scale (comoving)

$\hookrightarrow \delta_\nu \ll \delta_{CDM}$ (as for ~~decoupled~~ photons)

- Non-relativistic neutrinos : $c_\nu \sim a^{-1}$

$$k_{NR} = k_{fs}(\tau_{NR}) \simeq 0.0178\, \Omega_M^{1/2} \left(\frac{m_\nu}{\text{eV}}\right)^{1/2} h/\text{Mpc}$$

$\hookrightarrow$ same equation as for CDM (and decoupled baryons)

but with ~~a~~ initial condition $\delta_\nu \ll \delta_{CDM}$

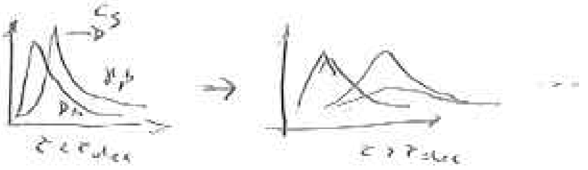
$$\hookrightarrow \delta_{C+B+\nu} = f_C \delta_C + f_B \delta_B + f_\nu \delta_\nu \underset{\overset{\nearrow}{\delta_B \sim \delta_C \gg \delta_\nu}}{\simeq} (f_C + f_B)\delta_C = (1 - f_\nu)\delta_C \quad \text{for } k \gg k_{NR}$$

$$f_\nu = \frac{\rho_\nu}{\rho_B + \rho_C + \rho_\nu}, \ldots$$

$\delta_{C+B+\nu} \simeq \delta_C$ for $k \ll k_{NR}$

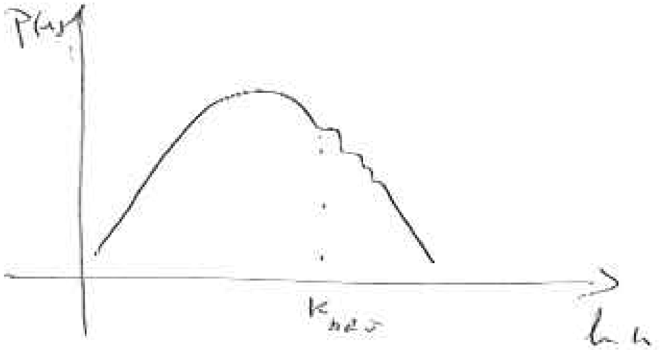
BARYONS $\longrightarrow \phi \longrightarrow$ DARK MATTER
+
BARYON

BAO cartoon in real space.



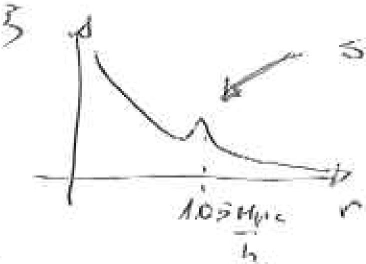
$$\rightarrow \quad P(k) \rightarrow P(k)\left(1 + A(k)\sin(k r_s)\right)$$

$A(k)$ ↳ $\frac{\Omega_b}{\Omega_m} \times e^{-(k/k_d)^2}$ damping due to photon mean free path



$r_s = 105 \, \cancel{Mpc} \, \frac{Mpc}{h}$ $\quad k_{BAO} = \frac{2\pi}{r_s} \simeq 0.04 \, h/Mpc$

in configuration space → peak in the corr. function



standard rod !

Measurements of angular diameter distance at different redshifts → Test ΛCDM

→ show plot

angular diameter distance

$$\theta = \frac{r_s}{\Delta z} = \frac{r_s}{(1+z)\, d_a(z)} = \frac{r_s}{\int_z^{z_{dec}} \frac{dz}{(1+z)H(z)}}$$

↖ COSMOLOGY DEPENDENCE

Show plots of ξ, P

Getting closer to real observations

1.) Bias (LINEAR!)

We do not observe the DM field directly. We observe galaxies, galaxy clusters, Ly α, 21 cm ... All these are "biased Tracers".

Ex: galaxy distribution: $n_g(\vec{x})$

DM " : $\rho(\vec{x})$

Assume $n_g(\vec{x}) = A \cdot \rho(\vec{x})$ $\quad A = \text{constant}$

$$\delta_g(\vec{x}) = \frac{n_g(\vec{x}) - \bar{n}_g}{\bar{n}_g} = \frac{A(\rho(\vec{x}) - \bar{\rho})}{A\bar{\rho}} = \delta(\vec{x}) \rightarrow \delta(\vec{x}) = \delta(\vec{x})$$

un-biased tracer.

On the other hand, we find, at large scales

$$\delta_g(\vec{x}, z) \approx b_g(z)\, \delta(\vec{x}, z)$$

$b_g(z)$: linear bias coefficient
- galaxy type dependent
- time-dependent

Large scales → linear th. Eq. principle

$$\delta = -\frac{k^2}{H^2}\frac{2}{3}\phi \qquad \delta_g = b_1 \nabla^2\phi + b_2 \left(\frac{\nabla^2\phi}{H^2}\right)^2 + \dots$$

$\propto \nabla^2\phi$

Kaiser '84 → Galaxies form on peaks of the DM density field

$\delta_g \propto \delta$ if we look at scales >> galaxy formation scales

scale indep

$\Leftrightarrow \delta_g(\vec{x}) \sim b_g\, \delta(x)$

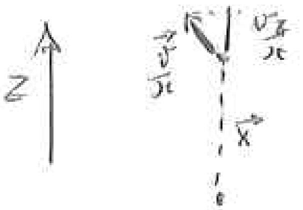
② Redshift-space distortions

$d = \frac{V}{H}$ Hubble law $\frac{V}{c} \sim z$

Hubble velocity ↓ ↓ Peculiar velocity

Physical velocity of a distant galaxy: $\vec{V} = H\vec{x} + \vec{v}$

We infer the comoving distance as $\vec{s} = \frac{\vec{V}}{H} = \underset{\uparrow \text{ True distance}}{\vec{x}} + \underset{\uparrow \text{ effect of peculiar velocity}}{\frac{\vec{v}}{H}}$



Distant observer approximation

$$v_{\parallel} = v_z \rightarrow \vec{s} = \vec{x} + \frac{v_z}{H}\hat{z}$$

What is the effect on the Power spectrum?

$$\rho^g_{RS}(\vec{s}) = \int d^3\vec{x}\, \rho(\vec{x})\, \delta_D\left(\vec{s} - \vec{x} - \frac{v_z(x)}{H}\hat{z}\right)$$

Notice: ρ not δ

$$\Rightarrow \bar{\rho}_{RS} = \frac{1}{V}\int d^3s\, \rho_{RS}(\vec{s}) = \frac{1}{V}\int d^3x\, \rho(\vec{x}) = \bar{\rho}$$

$$\rho^g(\vec{x}) = \bar{\rho}\left(1 + \delta(\vec{x})\right) \qquad \rho^g_{RS}(\vec{s}) = \bar{\rho}\left(1 + \delta_{RS}(\vec{s})\right)$$

$$(2\pi)^3\delta_D(\vec{k}) + \delta_{RS,g}(\vec{k}) = \int d^3s\, e^{i\vec{k}\cdot\vec{s}}\left(1 + \delta_{RS,g}(\vec{s})\right) =$$

$$= \int d^3\vec{x}\, e^{i\vec{k}\cdot\left(\vec{x} + \frac{v_z(x)}{H}\hat{z}\right)}\left(1 + \delta_g(\vec{x})\right)$$

$$\hookrightarrow e^{i\vec{k}\cdot\vec{x}}\left(1 + \frac{i k_z v_z(x)}{H} + \dots\right)$$

$$\Rightarrow \delta_{RS,g}(\vec{k}) = \delta_g(\vec{k}) + \frac{i k_z v_z(\vec{k})}{H} \underset{\uparrow}{=} \delta_g(\vec{k}) - \frac{k_z^2}{k^2}\frac{\theta(\vec{k})}{H}$$

$$\theta(x) = \vec{\nabla}\cdot\vec{v} \rightarrow v_z(k) = \frac{i k_z \theta(k)}{k^2}$$

CONTINUITY EQ: $\left(w = c_s^2 = 0 ,\ k/\mathcal{H} \gg 1 \right)$

$$\Leftrightarrow \quad \dot{\delta}(k) = -\theta(k)$$

Linear th: $\delta(k,\tau) = D(\tau)\,\delta(k,\tau_0)$

↑ growth factor

$$\frac{d}{d\tau} = \mathcal{H}\,\frac{d\ln D}{d\ln a}\,\frac{d}{d\ln D} \qquad \frac{d\ln D}{d\ln a} \equiv f$$

$$\Rightarrow \dot{\delta} = \mathcal{H} f \delta = -\theta \quad \Rightarrow \boxed{\delta = -\frac{\theta}{\mathcal{H} f}}$$

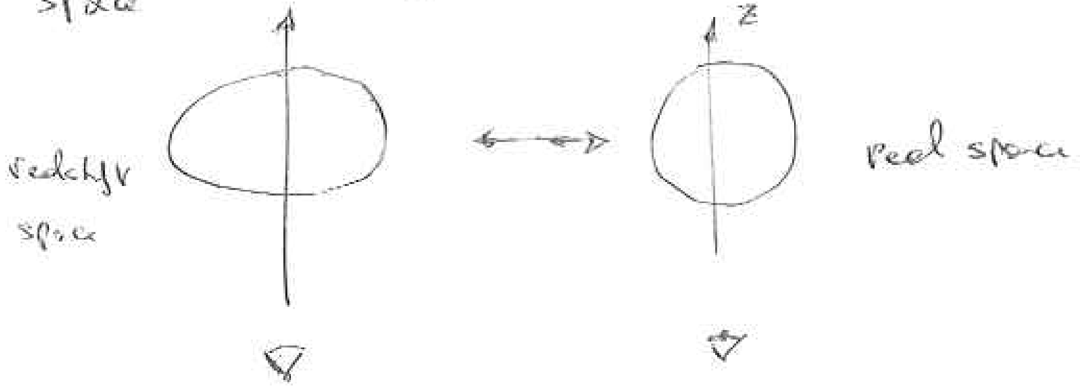
MATTER OVERDENSITY : $\delta = \frac{\delta_g}{b_g}$

$$\Rightarrow \delta_{RS,g}(\vec{k}) = \delta_g(\vec{k})\left(1 + \mu^2 \frac{f}{b_g}\right) \qquad \mu \equiv \hat{k}\cdot\hat{z}$$

2D

$$\Rightarrow P_g^{RS}(k,\mu) = \underbrace{\left(1 + \mu^2 \frac{f}{b_g}\right)^2}_{\text{Kaiser effect}} P_g(k)$$

↑ redshift space ↑ real space



- direction dependent enhancement of the PS
- ~~Possible~~ Sensitive to f/b_g (f: growth function)

  + adiabatic in. cond.

- NOTICE: ASSUMES THE Equivalence principle: galaxies and DM fall with the same velocity $\vec{v}$.

③ Finite Volume → Cosmic Variance

$$P(k)\, \underbrace{(2\pi)^3 \delta_D(\vec{k})}_{\text{Volume} = L^3} = \langle |\delta(\vec{k})|^2 \rangle \qquad (\delta^*(\vec{k}) = \delta(-\vec{k}))$$

$$P(\vec{k}) = \frac{\langle |\delta_{\vec{k}}|^2 \rangle}{L^3}$$

Number of k-modes in bin Δk, in a volume L^3.

~~$N_k = \frac{4\pi k^2 \Delta k}{(2\pi)^3} L^3 = \frac{4\pi k^2 \Delta k}{k_0^3}$~~ $\qquad k_0 \leq 2\pi$

$k_0 = \frac{2\pi}{L}$ $\qquad N_k = \frac{4\pi k^2 \Delta k}{k_0^3}$. N° of modes in the bin $k, k+\Delta k$

Fundamental scale ↗

Gaussian error on $P(k)$ → $\frac{\Delta P_k}{P_k} \sim \frac{1}{N_k^{1/2}} = \sqrt{\frac{k_0^3}{4\pi k^2 \Delta k}} = \sqrt{\frac{2\pi^2}{k^3 L^3}}$

EX: ~~$k_0 \approx 0.02\, h/Mpc$~~ Measure BAO's at % level

$k \sim \Delta k \sim 0.05\, h/Mpc = k_{BAO}$ $\quad L \gtrsim \frac{1}{k_{BAO}} (10^4 \cdot 2\pi^2)^{1/3} \sim 1200\, \frac{Mpc}{h}$

$(k_0 \leq 0.005\, \frac{Mpc}{h})$

④ Discrete tracers of the density field → Shot noise

Correlation function from a discrete set of points

$$\xi_d(|\vec{x} - \vec{y}|) = \frac{\langle m(\vec{x})\, m(\vec{y}) \rangle}{\bar{m}^2} - 1 \qquad m(x) = \bar{m}(1 + \delta(\vec{x}))$$

$\bar{m} = \langle m(x) \rangle$

Assume $\delta = 0$ $\vec{x} = \vec{y}$ → $\frac{\langle m^2 \rangle}{\bar{m}^2} - 1 = (\bar{m})^{-1}$

$\langle m^2 \rangle = \langle m \rangle^2 + \langle m \rangle$

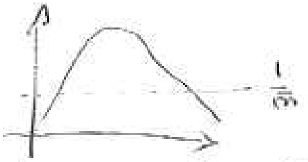
Poisson Statistics

$$\Rightarrow \quad \xi_d(|\vec{x} - \vec{y}|) = \xi(|\vec{x} - \vec{y}|) + \frac{1}{\bar{m}} \delta_D(\vec{x} - \vec{y})$$

↑ continuous cor. function.

$$\Rightarrow P_d(k) = P(k) + \frac{1}{\bar{m}}$$

↖ shot-Noise



Ex. Euclid $\bar{m}(z) \sim (0.8 \div 2.0) \cdot 10^{-3} \left(\frac{h}{Mpc}\right)^3$

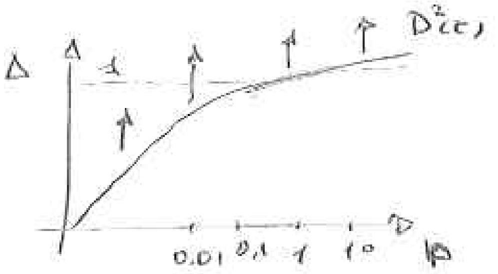
The Non-linear Universe (= Loop corrections + resummations + Non-perturbative effects)

Different sources of non-linearity:

- **Matter non-linearity**: "$\delta \ll 1$" not valid at all scales

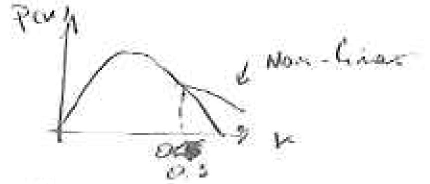
$$\langle \delta(x)^2 \rangle_\Lambda = \int^\Lambda \frac{d^3p}{(2\pi)^3} P(p) = \int^\Lambda \frac{dp}{p} \underbrace{\frac{p^3}{2\pi^2} P(p)}$$

→ Dimension-less PS $\Delta(p)$



$P(p) \sim p^{n_s-4} \ln p$
large p
$\Delta(p) \sim p^{n_s-1} \ln p$

- At $z=0$ $k \gtrsim 0.8-1$ h/Mpc are non-linear
- At higher z k_{NL} grows.
- If you need % accuracy → trust linear theory only for $k \lesssim 0.03$ h/Mpc

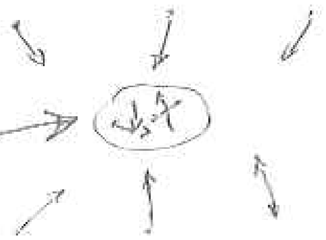


- **Non-Linear bias**

$$\delta_g(x,t) = b_g(z)\,\delta(x,z) \longrightarrow F[\partial_i\partial_j\Phi](x) \qquad \left(\nabla^2\Phi = \tfrac{3}{2}\mathcal{H}^2\delta\right)$$

- Non-linear
- Non-local
- Non-deterministic

- Redshift - space - distortions



Non-Linear

Fingers Of God

F.O.G.

↓

Depends on small scale
potential fluctuations

Linear effect → Kaiser
\+
PT corrections
(depends on large scale
potential $\partial_r \phi$)

- Baryon feedback
  - Hydrodinamic codes
  - small scales effects

All the dominant non-linear effects involve SMALL SCALES
and late times (low z's z_g, $Z_{obs} \ll Z_{dec}$)

$\Rightarrow$ background made by MATTER + Λ ↑ or DE

$\Rightarrow$ $K/_{\mathcal{H}} \gg 1$ $\longrightarrow$ Newtonian limit of GR

- Run a Boltzmann Linear code (e.g. CAMB, CLASS)
  down to $1 \ll Z_{in} \ll Z_{dec}$ (Full GR + Linear approx)
- Use the $PS(Z_{in})$ as the initial condition for

Newtonian equations from $Z_{in} \rightarrow Z = 0$ (or Z_{obs})

Equation of motion for CDM (and B)

$\mathcal{H}^2 = \frac{8\pi}{3} G \rho_{TOT}$ $\qquad \rho_{TOT} = \rho_M + \rho_{DE}$

$$\underbrace{\nabla^2 \Phi = \frac{3}{2} \mathcal{H}^2 \Omega_m \delta}_{\text{Poisson eq.}} \qquad \rho_M(\vec{x},\tau) = \bar{\rho}\left(1 + \delta(\vec{x},\tau)\right)$$

No linear approximation, only $\frac{k}{\mathcal{H}} \gg 1$

CDM: (and decoupled baryons) $\rightarrow$ Non-relativistic particles interacting only through gravity

$$\begin{cases} \vec{p} = a m \dot{\vec{x}} \\ \dot{\vec{p}} = -a m \vec{\nabla} \Phi \end{cases} \quad \text{eqs. of motion}$$

$$+ \quad \nabla^2 \Phi = \frac{3}{2} \mathcal{H}^2 \Omega_m \delta$$

SYMMETRY:
$$\begin{cases} \vec{x} \rightarrow \vec{x} + \vec{d}(\tau) \\ \vec{p} \rightarrow \vec{p} + a m \dot{\vec{d}}(\tau) \\ \vec{\nabla}\Phi \rightarrow \vec{\nabla}\Phi - \ddot{\vec{d}}(\tau) - \mathcal{H} \dot{\vec{d}}(\tau) \end{cases}$$

time-dep., uniform shift

$\Rightarrow$ EQUIVALENCE PRINCIPLE

EXACT Consequences on the structure of EXACT correlators

↳ consistency Relations (see later)
(Ward Identities)

Eulerian framework: study the evolution of the distribution function: $f(\vec{x},\vec{p},\tau)$

No collisions
$$0 = \frac{d}{d\tau} f(\vec{x},\vec{p},\tau) = \left(\frac{\partial}{\partial\tau} + \frac{p^i}{a m}\frac{\partial}{\partial x^i} - a m \frac{\partial\phi}{\partial x^i}\frac{\partial}{\partial p^i}\right) f(\vec{x},\vec{p},\tau) = 0$$

(Collision term)

$$+ \quad \nabla^2\phi = \frac{3}{2}\mathcal{H}^2\Omega_m\,\delta$$

$\hookrightarrow 1+\delta(\vec{x},\tau) = \frac{1}{\bar{n}}\int d^3p\, f(\vec{x},\vec{p},\tau)$

comoving number density

Vlasov - Poisson System

$$f(\vec{x},\vec{p},\tau) = \lim_{N\to\infty}\sum_{i}^{N} \delta_D(\vec{x}-\vec{x}_i(\tau))\,\delta_D(\vec{p}-\vec{p}_i(\tau))$$

It is more physical to consider a volume-averaged (coarse-grained) distribution function:
- finite resolution
- isolation of UV behavior

$$\bar{f}(\vec{x},\vec{p},\tau) \equiv \int d^3y\, W\left[\frac{y}{R}\right] f(\vec{x}-\vec{y},\vec{p},\tau)$$

smoothing function ex: $W\left[\frac{x}{R}\right] = \frac{1}{(2\pi)^{3/2}}\frac{e^{-\frac{x^2}{2R^2}}}{R^3}$

$$\Rightarrow \boxed{\begin{aligned} &\left(\frac{\partial}{\partial\tau} + \frac{p^i}{a m}\frac{\partial}{\partial x^i} - a m\frac{\partial\bar{\phi}(x)}{\partial x^i}\frac{\partial}{\partial p^i}\right)\bar{f}(\vec{x},\vec{p},\tau) = \\ &= a m\left[\overline{\frac{\partial\phi}{\partial x^i}\frac{\partial}{\partial \vec{p}^i} f} - \frac{\partial\bar{\phi}}{\partial x^i}\frac{\partial}{\partial p^i}\bar{f}\right](\vec{x},\vec{p},\tau)\end{aligned}}$$

← large scales evolution ($l \gg R$)

← short scales effects ($l \ll R$)

COARSE-GRAINED
VLASOV EQUATION

$\left[\overline{(\ldots)}(\vec{x}) \equiv \int d^3y\, W\left(\frac{y}{R}\right)(\ldots)(\vec{x}-\vec{y})\right]$

MOMENTS :

$$\frac{1}{\bar m}\int d^3p\, \bar f(\vec x, p, \tau) = 1 + \bar\delta(\vec x, \tau) \qquad \text{density} \qquad (T^0{}_0)$$

$$\frac{1}{\bar m}\int d^3p\, \frac{p^i}{a m}\, \bar f(\cdots) = (1+\bar\delta(\vec x,\tau))\,\bar v^i(\vec x,\tau) \qquad \text{momentum} \qquad (T^0{}_i)$$

($\bar v^i$: peculiar velocity)

$$\frac{1}{\bar m}\int d^3p\, \frac{p^i}{a m}\frac{p^j}{a m}\, \bar f(\cdots) = (1+\bar\delta(\vec x,\tau))(\bar v^i \bar v^j + \underset{\uparrow}{\bar\sigma^{ij}}(\vec x,\tau)) \qquad (-T^i{}_j)$$

velocity dispersion

MOMENTS OF THE C.G. Vlasov EQ:

$$\frac{\partial}{\partial\tau}\bar\delta(\vec x,\tau) + \frac{\partial}{\partial x^i}\left[\underbrace{(1+\bar\delta(\vec x,\tau))\,\bar v^i(\vec x,\tau)}\right] = 0 \qquad \text{continuity eq}$$

NON-LINEAR TERMS

$$\frac{\partial}{\partial\tau}\bar v^i + \mathcal{H}\bar v^i + \underbrace{\bar v^k\frac{\partial}{\partial x^k}\bar v^i} = -\frac{\partial}{\partial x^i}\bar\Phi \qquad \text{Euler eq.}$$

$$\left.\begin{aligned} &-\frac{1}{1+\bar\delta}\frac{\partial}{\partial x^k}\left[(1+\bar\delta)\bar\sigma^{ki}\right] \\ &-\frac{1}{1+\bar\delta}\left[\overline{(1+\delta)\frac{\partial}{\partial x^i}\phi} - (1+\bar\delta)\frac{\partial}{\partial x^i}\bar\Phi\right]\end{aligned}\right\}\ \text{short distance effects}$$

$$\frac{\partial}{\partial\tau}\bar\sigma^{ij} + 2\mathcal{H}\bar\sigma^{ij} + \cdots$$

$$\vdots$$

M.P. Mangano, Saviano, Viel
1108.5203

Carrasco, Hertzberg, Senatore
1206.2976

To close the hierarchy, we need some info/assumptions on the short-distance effects

"STANDARD" PERTURBATION THEORY

- Neglect short-distance effects
- Neglect vorticity : $\vec{v} = \vec{\nabla}\varphi + \vec{\nabla}\times\vec{w}$ $\quad (\vec{\nabla}\cdot\vec{w} = 0)$

  vorticity → $\vec{w} = 0$ $\qquad \left(\frac{\partial \vec{w}}{\partial \tau} + \mathcal{H}\, \vec{w} = 0 \ \text{linear P.T.} \right)$

  $\theta = \vec{\nabla}\cdot\vec{v} = \nabla^2\varphi$ velocity divergence

- Iteratively solve the eqs. of motion

<u>Notice</u>: it amounts to assuming distribution function of the form

$$f(\vec{x},\vec{p},\tau) = \bar{n}\left(1+\delta(\vec{x},\tau)\right)\delta_D\left(\vec{p} - a m \vec{\nabla}\varphi(\vec{x},\tau)\right)$$

"SINGLE STREAM APPROXIMATION"

$(\sigma^{ij} = 0)$

$$\Rightarrow \begin{cases} \dot{\delta}_{\vec{k}}(\tau) + \theta_k(\tau) = I_{\vec{k},\vec{q}_1\vec{q}_2}\, \alpha(\vec{q}_1,\vec{q}_2)\, \theta_{\vec{q}_1}(\tau)\, \delta_{\vec{q}_2}(\tau) \\ \dot{\theta}_{\vec{k}}(\tau) + \mathcal{H}\theta_{\vec{k}}(\tau) - k^2\Phi_k(\tau) = I_{\vec{k};\vec{q}_1,\vec{q}_2}\, \beta(\vec{q}_1,\vec{q}_2)\, \theta_{q_1}(\tau)\, \theta_{q_2}(\tau) \\ k^2\Phi_k = -\frac{3}{2}\mathcal{H}^2\Omega_m\delta_k \end{cases}$$

$$I_{\vec{k};\vec{q}_1\ldots\vec{q}_n} \equiv \int\frac{d^3q_1}{(2\pi)^3}\cdots\frac{d^3q_n}{(2\pi)^3}(2\pi)^3\,\delta_D\Big(\vec{k}-\sum_1^n\vec{q}_i\Big)$$

$$\alpha(\vec{q}_1,\vec{q}_2) \equiv 1 + \frac{\vec{q}_1\cdot\vec{q}_2}{q_1^2}, \quad \beta(\vec{q}_1,\vec{q}_2) = \frac{|\vec{q}_1+\vec{q}_2|^2\,\vec{q}_1\cdot\vec{q}_2}{2q_1^2q_2^2}$$

$\boxed{\text{NON-LINEAR COUPLINGS}}$

SPT → expansion in α, β.

Linear order : $\alpha = \beta = 0$ ("Tree-level")

$$\begin{cases} \dot{\delta}_k + \theta_k = 0 \\ \dot{\theta}_k + \mathcal{H}\theta_k = -\frac{3}{2}\mathcal{H}^2\Omega_m\delta_k \end{cases} \rightarrow \ddot{\delta} + \mathcal{H}\dot{\delta} = \frac{3}{2}\mathcal{H}^2\Omega_m\delta \qquad (*)$$

Linear GR equation for $\frac{k}{\mathcal{H}} \gg 1$
(gauge-independent)

solution

$$\hookrightarrow \delta_k^{(1)}(\tau) = -\frac{\theta_k^{(1)}(\tau)}{\mathcal{H}f_+(\tau)} = \frac{D_+(\tau)}{D_+(\tau_{in})}\delta_k(\tau_{in})$$

↖ initial condition ($1 \ll \tau_{in}$

$$f_+ = \frac{d\ln D_+}{d\ln a}$$ growth function (growing mode)

($\Omega_M = 1 \rightarrow f_+ = 1, \ D_+ \propto a$)

(general case (ΛCDM) $\rightarrow$ solve $(*)$ find D_+ and D_-)

$$\boxed{P_{\delta\delta}(k,\tau) = \langle|\delta_k(\tau)|^2\rangle' = \left(\frac{D_+(\tau)}{D_+(\tau_{in})}\right)^2 P_{\delta\delta}(k;\tau_{in}) = P_{\delta\theta}(k,\tau) = P_{\theta\theta}(k,\tau)}$$

$\swarrow$

$\langle\delta_k\left(-\frac{\theta}{\mathcal{H}f}\right)_{-k}\rangle$

$\left(\langle\cdots\rangle' \equiv \frac{\langle\ \rangle}{(2\pi)^3\delta_D(k-\dots)}\right)$

growing mode : $\delta_k = -\frac{\theta_k}{\mathcal{H}f}$

COMPACT NOTATION

$$\eta = \ln\frac{D(\tau)}{D(\tau_0)} \qquad \varphi_a(\vec{k},\tau) = \begin{pmatrix} \delta_k(\tau) \\ -\frac{\theta_k(\tau)}{\mathcal{H}f_+} \end{pmatrix} e^{-\eta}$$

$\varphi_a(\vec{k},\tau)$ = const in linear th.
$= u_a\,\varphi(\vec{k},\tau)$
$u_a = \binom{1}{1}$

$$\gamma_{112}(\vec{q},\vec{p}) = \gamma_{121}(\vec{p},\vec{q}) = \frac{1}{2}\alpha(\vec{p},\vec{q})$$

$$\gamma_{222}(\vec{q},\vec{p}) = \beta(\vec{q},\vec{p})$$

and zero otherwise

vertex ↙

$$\Rightarrow \boxed{\underbrace{\left(\delta_{ab}\partial_\eta + \Omega_{ab}(\eta)\right)}_{= g^{-1}_{ab}(\eta)}\varphi_b(\vec{k},\eta) = e^{\eta}\,I_{\vec{k},\vec{q}_1,\vec{q}_2}\,\gamma_{abc}(\vec{q}_1,\vec{q}_2)\,\varphi_b(\vec{q}_1,\eta)\,\varphi_c(\vec{q}_2,\eta)}$$

↗ inverse "propagator"

(summed over repeated indices b, c)

$$\Omega_{ab}(\eta) = \begin{pmatrix} 1 & -1 \\ -\frac{3}{2}\frac{\Omega_m}{f_+^2} & +\frac{3}{2}\frac{\Omega_m}{f_+^2} \end{pmatrix}$$

$g_{ab}(\eta)$ propagator : $\left(\delta_{ab}\partial_\eta + \Omega_{ab}(\eta)\right) g_{bc}(\eta) = \delta_{ac}\,\delta_D(\eta)$

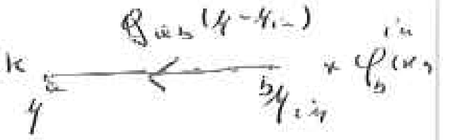
with causal b.c. $g_{ab}(\eta) \propto \Theta(\eta)$

Formal solution of the e.o.m.

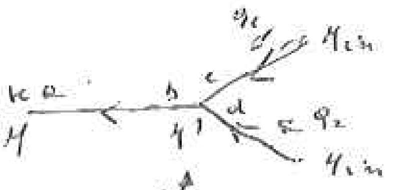
$$\Rightarrow \boxed{\varphi_a(k,\eta) = \int_0^\eta d\eta'\, g_{ab}(\eta-\eta')\, e^{\eta'}\, I_{k;q_1,q_2}\, \gamma_{bcd}(q_1,q_2)\, \varphi_c(q_1,\eta')\,\varphi_d(q_2,\eta')}$$

P.T. solution

1st order $\quad \varphi_a^{(1)}(k,\eta) = g_{ab}(\eta-\eta_{in})\,\varphi_b^{in}(k)$

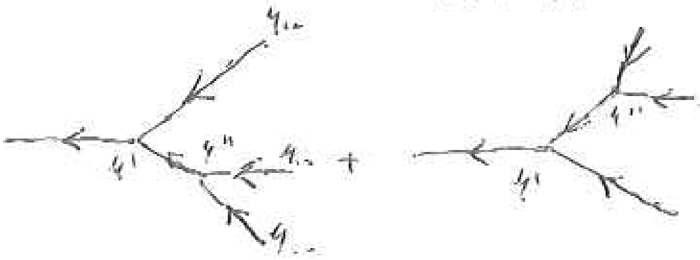


2nd order



$e^{\eta'}\,\gamma_{bcd}(\vec{q}_1,\vec{q}_2)$ mode-coupling

3rd order



⋮

$$\varphi_a^{(n)}(k,\eta) = \underbrace{e^{(-1+n)\eta}}_{\text{Time}}\ \underbrace{I_{\vec{k};\vec{q}_1\ldots\vec{q}_n}}_{\text{space}}\ F_a^{(n)}(\vec{q}_1,\ldots\vec{q}_n)\,\varphi^{in}(\vec{q}_1)\cdots\varphi^{in}(\vec{q}_n)$$

N.B. Time × SPACE Factorization is possible

because $D(\eta)$ is SCALE-INDEPENDENT!

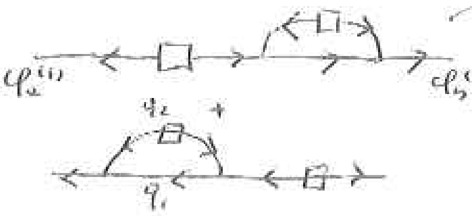
$\rightarrow \nu'$
$\rightarrow$ Modified GR $\Big\}$ scale-dep growth

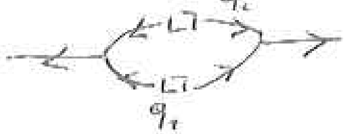
POWER-SPECTRUM:

$$P_{ab}(k,\eta) = \langle \varphi_a(\vec{k},\eta)\varphi_b(-\vec{k},\eta)\rangle' = \langle \varphi_a^{(1)}\varphi_b^{(1)}\rangle +$$

$$+ \langle \varphi_a^{(1)}\varphi_b^{(3)}\rangle + \langle \varphi_a^{(3)}\varphi_b^{(1)}\rangle +$$

$$+ \langle \varphi_a^{(2)}\varphi_b^{(2)}\rangle + \alpha$$





$$+ O(P_{in}^{3})$$

↑ 2-loop

integrate over $\vec{q}_1, \vec{q}_2$ → 1-loop diagrams.

Notice:

- these are NOT quantum loops/effects, ~~but~~ as the momentum integral is over the initial statistics (PS)

- $\langle \varphi^{(1)}\varphi^{(2)}\rangle = 0$ if initial conditions are gaussian

$\boxed{\text{1-loop PS}}$ $P_{ab}(k,\eta) = P_{in}(k)\, u_a u_b + e^{2\eta}\left(P^{13}_{ab}(k) + P^{22}_{ab}(k) + P^{31}_{ab}(k)\right)$

$$\begin{cases} P^{13}_{ab}(k) = ~~[illegible]~~\; 3\, \mathbb{I}_{\vec{k},\vec{q}_1,\vec{q}_2}\, F_b^{(3)}(\vec{k},\vec{q}_1,\vec{q}_2)\, P_{in}(\vec{q}_1)\, P_{in}(\vec{k})\, u_a \\ P^{31}_{ab}(\,) = \dots\, F_a^{(3)}\, u_b \\ P^{22}_{ab} = 2\, \mathbb{I}_{\vec{k},\vec{q}_1,\vec{q}_2}\, F_a^{(2)}(\vec{q}_1,\vec{q}_2)\, \vec{F}_b^{(2)}(\vec{q}_1,\vec{q}_2)\, P_{in}(q_1)\, P_{in}(q_2) \end{cases}$$

2-dim integrals (P^{22}) over a non-power law function ($P_{in}(q)$)

↪ ∃ Speedy method: FFTLog! (discussion?)

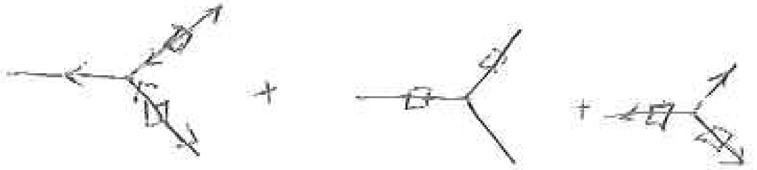
- Tree-Level bispectrum:

$$\langle \varphi_a(\vec{k}_1)\varphi_b(\vec{k}_2)\varphi_c(\vec{k}_3)\rangle' = B_{abc}(k_1,k_2,k_3)$$

Trans + rot. invariance
↓
(depends only on moduli)

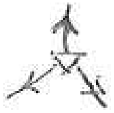
= 0 at linear order if initial condition are GAUSSIAN

Lowest order =>



ex: $B_{111}(k_1,k_2,k_3) = 2\,P^{L}(k_1)\,P^{in}(k_2)\,F_1^{(2)}(\vec{k}_1,\vec{k}_2)$ + 2 permutations

IF NON-GAUSSIAN INITIAL CONDITIONS (ex f_{NL} from inflation) → $\langle \varphi_{in}\varphi_{in}\varphi_{in}\rangle' \neq 0$ →



New vertex

Measurements of f_{NL} from LSS bispectrum:

→ disentangle early NG from late-time NG