QCD

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2nd Lecture

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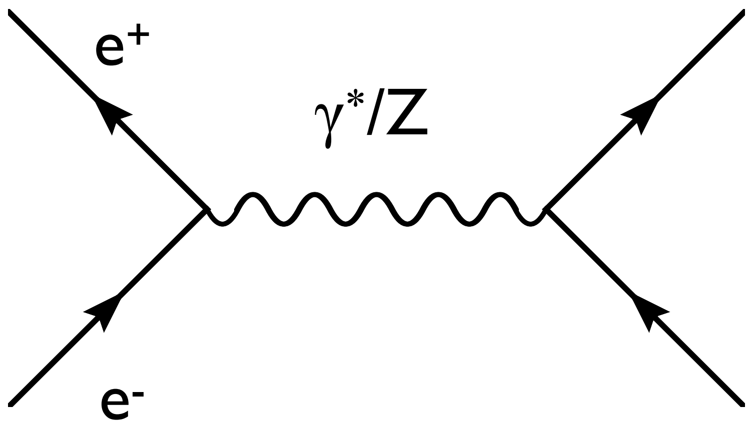
Perturbative expansion of the R-ratio

The R-ratio is defined as



$$R \equiv \frac{\sigma(e^+e^- \to \mathrm{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

At lowest order in perturbation theory (PT)

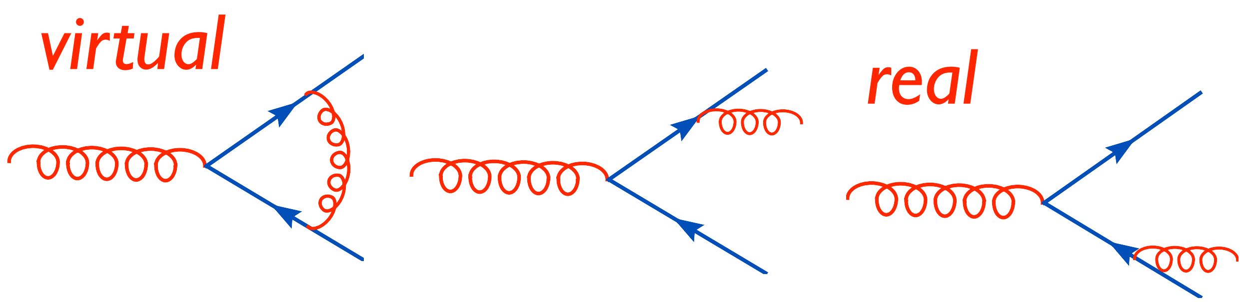
$$\sigma(e^+e^- \to \mathrm{hadrons}) = \sigma_0(e^+e^- \to q\bar{q})$$

Since common factors cancel in numerator/denominator, to lowest order one finds

$$R_0 = \frac{\sigma_0(\gamma^* \to \mathrm{hadrons})}{\sigma_0(\gamma^* \to \mu^+\mu^-)} = N_c \sum_f q_f^2$$

The R-ratio: perturbative expansion

First order correction



Real and virtual do not interfere since they have a different # of particles. The amplitude squared becomes

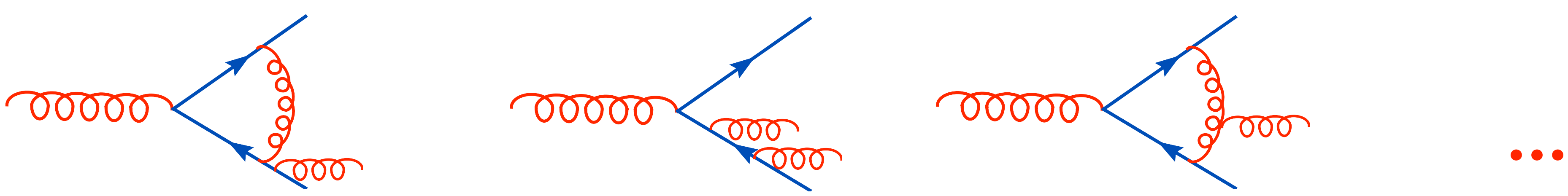
$$|A_1|^2 = |A_0|^2 + \alpha_s \left(|A_{1,r}|^2 + 2\mathrm{Re}\{A_0 A_{1,v}^*\}\right) + \mathcal{O}(\alpha_s^2) \qquad \alpha_s = \frac{g_s^2}{4\pi}$$

Integrating over phase space, the first order result reads

$$R_1 = R_0 \left(1 + \frac{\alpha_s}{\pi}\right)$$

R-ratio and UV divergences

To compute the second order correction one has to compute diagrams like these and many more



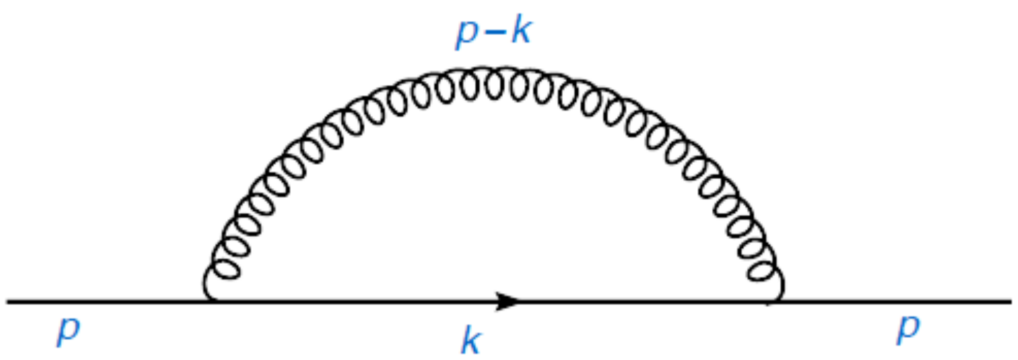
One gets

$$R_2 = R_0 \left(1 + \frac{\alpha_s}{\pi} + \left(\frac{\alpha_s}{\pi}\right)^2 \left(c + \pi b_0 \ln \frac{M_{\rm UV}^2}{Q^2}\right)\right) \qquad b_0 = \frac{11 N_c - 4 n_f T_R}{12\pi}$$

Ultra-violet divergences do not cancel. Result depends on UV cut-off.

Renormalization

Loop corrections in QCD are (often) divergent. Divergences originate from regions of very large momenta



$$\sim g^2 \int_{p^2}^{\infty} d^4k \, \frac{1}{k^2} \frac{1}{(p-k)^2} \to \infty$$

QCD is a renormalizable theory. This means that that one can

1. regularize the divergence (e.g. using dimensional regularization)

$$d^4k \to \mu^{2\epsilon} d^{4-2\epsilon}k$$

2. absorbe all UV divergences into a universal redefinition of a finite number of the bare parameters of QCD

Renormalization and running coupling

For the R-ratio, the divergence is dealt with by renormalization of the coupling constant

$$\alpha_s(\mu) = \alpha_s^{\mathrm{bare}} + b_0 \ln \frac{M_{UV}^2}{\mu^2} \left(\alpha_s^{\mathrm{bare}}\right)^2$$

R expressed in terms of the renormalized coupling is finite

$$R = R_0 \left(1 + \frac{\alpha_s(\mu)}{\pi} + \left(\frac{\alpha_s(\mu)}{\pi}\right)^2 \left(c + \pi b_0 \ln \frac{\mu^2}{Q^2}\right) + \mathcal{O}(\alpha_s^3(\mu))\right)$$

Renormalizability of the theory guarantees that *the same redefinition of the coupling removes all UV divergences from all physical quantities (massless case)*

Renormalization achieved by replacing bare masses and the bare coupling with renormalized ones. Masses and coupling become dependent on the renormalization scale. The dependence is fully predicted in pQCD

- the coupling $\Rightarrow$ β function
- the masses $\Rightarrow$ anomalous dimensions γ_m

The beta-function

$$\boxed{\beta(\alpha_s^{\text{ren}}) \equiv \mu^2 \frac{d\alpha_s(\mu^2)}{d\mu^2}}$$

The renormalized coupling is

$$\alpha_s(\mu) = \alpha_s^{\text{bare}} + b_0 \ln \frac{M_{UV}^2}{\mu^2} \left(\alpha_s^{\text{bare}}\right)^2$$

So, one immediately gets

$$\beta = -b_0 \alpha_s^2(\mu) + \dots$$

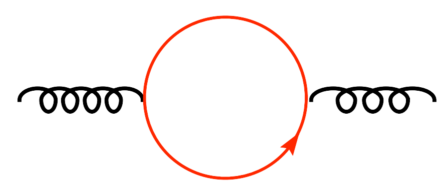
Integrating the differential equation one finds at lowest order

$$\frac{1}{\alpha_s(\mu)} = b_0 \ln \frac{\mu^2}{\mu_0^2} + \frac{1}{\alpha_s(\mu_0)} \qquad \Rightarrow \qquad \alpha_s(\mu) = \frac{1}{b_0 \ln \frac{\mu^2}{\Lambda^2}}$$

More on the beta-function

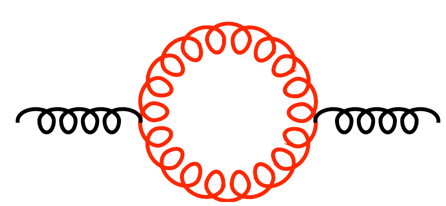
Roughly speaking:

(a) quark loop vacuum polarization diagram gives a negative contribution to $b_0 \sim -2n_f/12\pi$

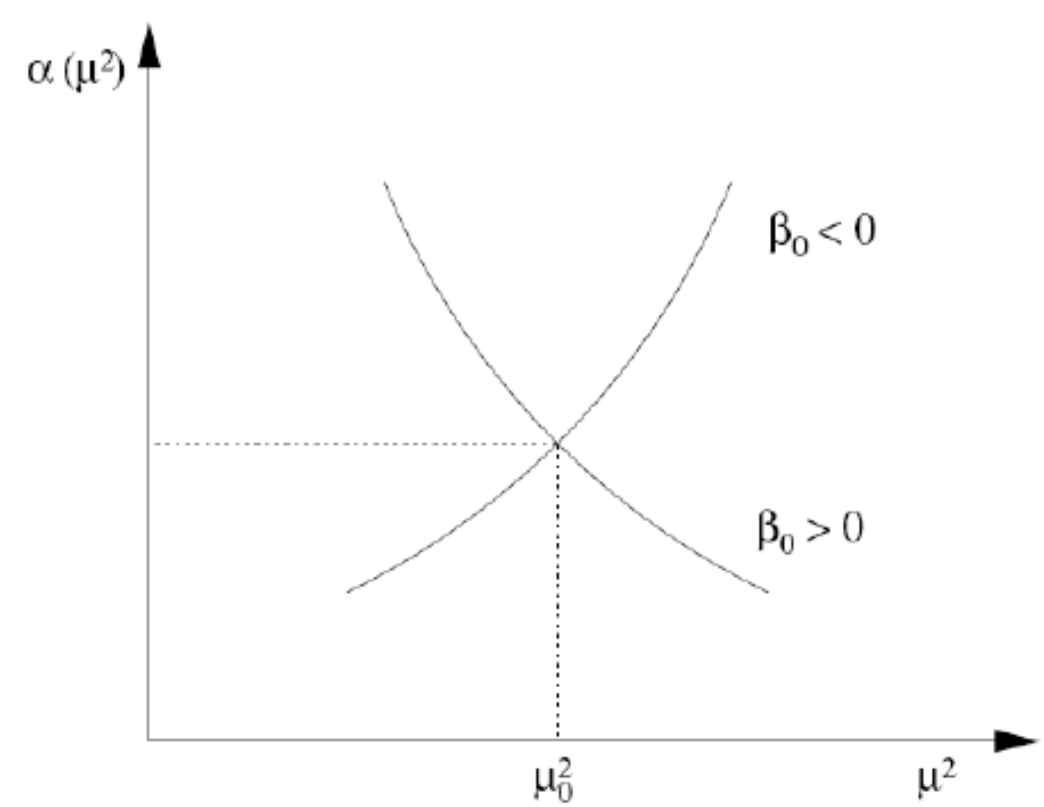


(a)

(b) gluon loop gives a positive contribution to $b_0 \sim 11N_c/12\pi$



(b)



Since (b) > (a) $\Rightarrow$ $b_{0,QCD} > 0$

$\Rightarrow$ overall negative beta-function in QCD

While in QED (b) = 0 $\Rightarrow$ $b_{0,QED} < 0$

$$\beta_{\mathrm{QED}} = \frac{1}{3\pi}\alpha^2 + \ldots$$

More on the beta-function

- QCD: perturbative picture valid for scales $\mu >> \Lambda_{\mathrm{QCD}}$ (about 300 MeV)
- QED: perturbative picture valid for scales $\mu << \Lambda_{\mathrm{QED}}$

Question: why does nobody talk about Λ_{QED}?

Answer:

$$\Lambda_{\mathrm{QED}} = m_e \exp\left\{-\frac{1}{2b_0\alpha(m_e)}\right\} \sim 10^{90}\mathrm{GeV} >> M_{\mathrm{Planck}}$$

(Note that the fact that QED is not a consistent theory up to very high scales implies that it must be an effective theory)

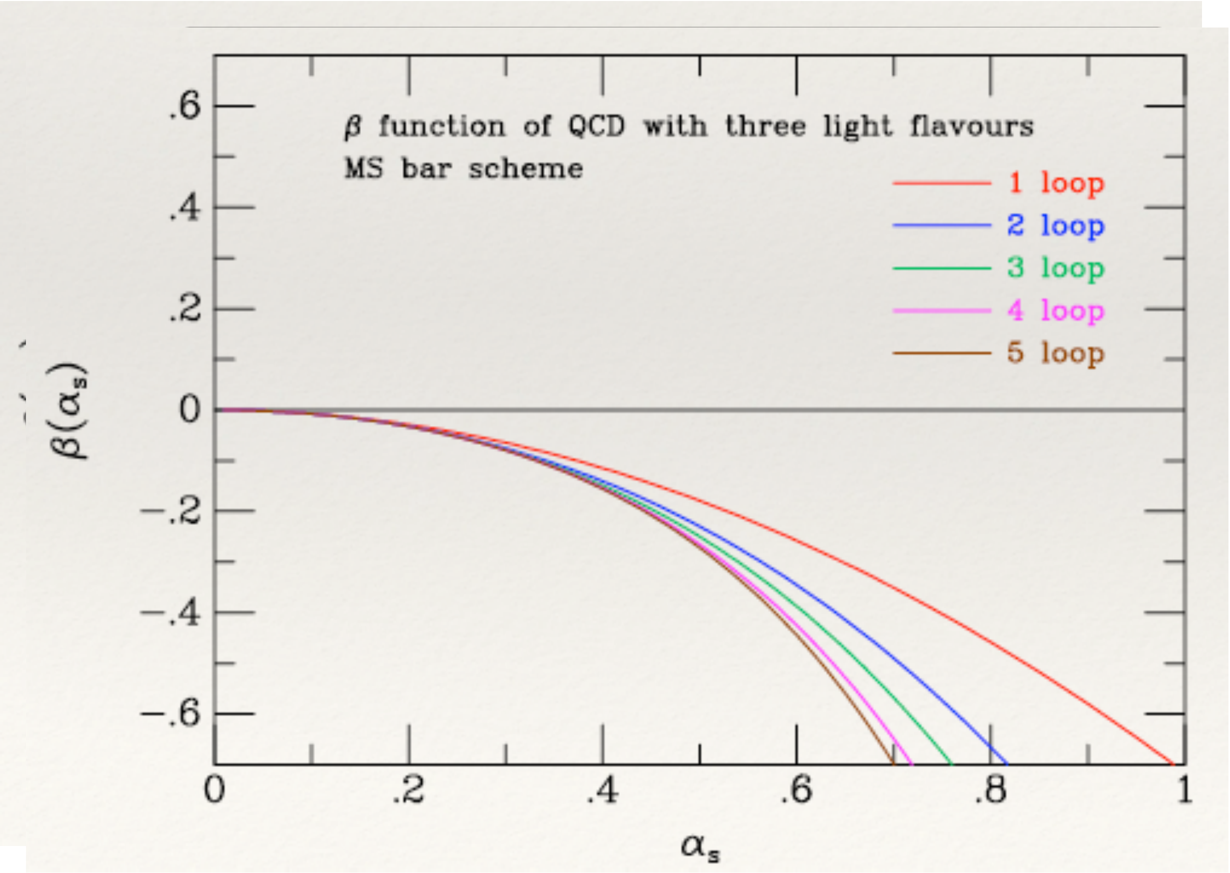
Back to the QCD beta-function

Perturbative expansion of the beta-function:

$$\beta = -\alpha_s^2(\mu) \sum_i b_i \alpha_s^i(\mu)$$

$$b_0 = \frac{11 N_c - 4 n_f T_R}{12\pi}$$

$$b_1 = \frac{17 N_c^2 - 5 N_c n_f - 3 C_F n_f}{24\pi^2}$$

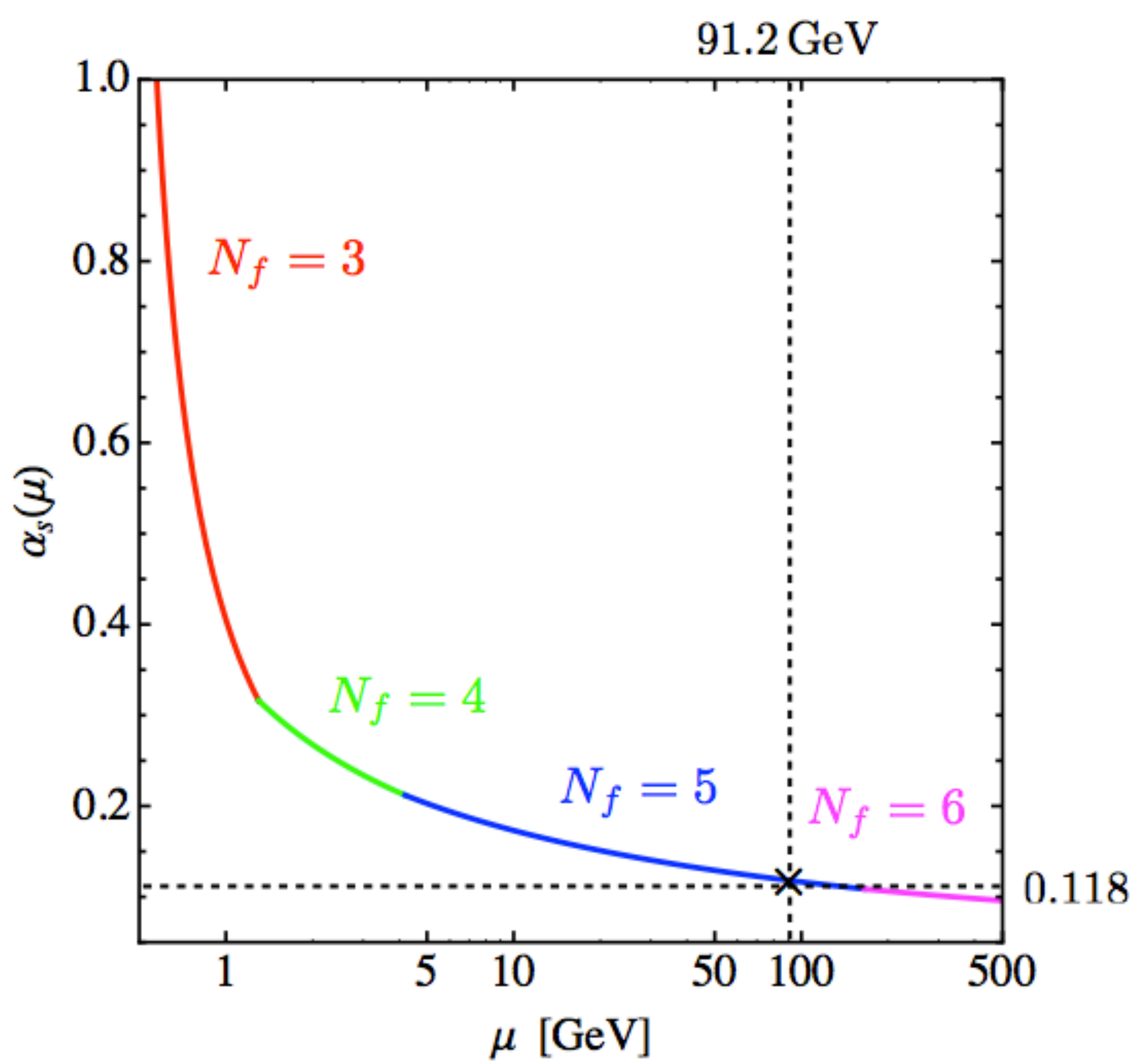


- n_f is the number of active flavours (depends on the scale)
- today, the beta-function known up to five loops, but only first two coefficients are independent of the renormalization scheme

Exercise: proof the above statement [hint: use the fact that at $O(\alpha_s)$ the coupling in two different schemes is related by a finite change]

Active flavours & running coupling

The active field content of a theory modifies the running of the couplings



Constrain New Physics by measuring the running at high scales?

Renormalization Group Equation

Consider a dimensionless quantity A, function of a single scale Q. The dimensionless quantity should be independent of Q. However in quantum field theory this is not true, as renormalization introduces a second scale μ

But the renormalization scale is arbitrary. The dependence on it must cancel in physical observables up to the order to which one does the calculation.

So, for any observable A one can write a renormalization group equation

$$\boxed{\left[\mu^2 \frac{\partial}{\partial \mu^2} + \mu^2 \frac{\partial \alpha_s}{\partial \mu^2} \frac{\partial}{\partial \alpha_s}\right] A\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right) = 0}$$

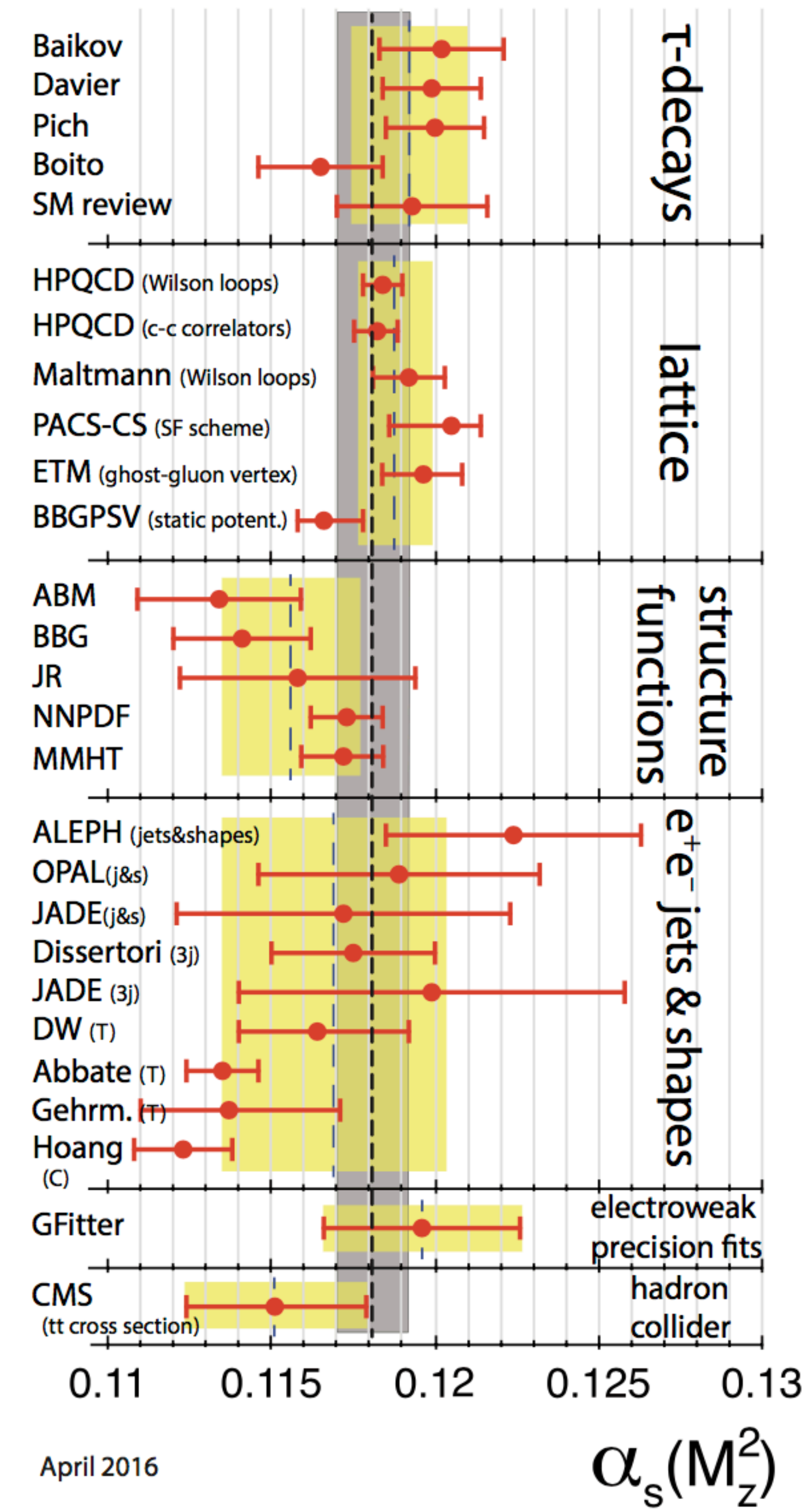
$$\alpha_s = \alpha_s(\mu^2) \qquad \beta(\alpha_s) = \mu^2 \frac{\partial \alpha_s}{\partial \mu^2}$$

Scale dependence of A enters through the running of the coupling: knowledge of $A(1, \alpha_s(Q^2))$ allows one to compute the variation of A with Q given the beta-function

Measurements of the running coupling

<u>Summarizing:</u>

- overall consistent picture: α_s from very different observables compatible
- α_s is not so small at current scales
- α_s decreases slowly at higher energies (logarithmic only)
- higher order corrections are and will remain important



World average

$$\boxed{\alpha_s(M_Z) = 0.1181 \pm 0.0011}$$

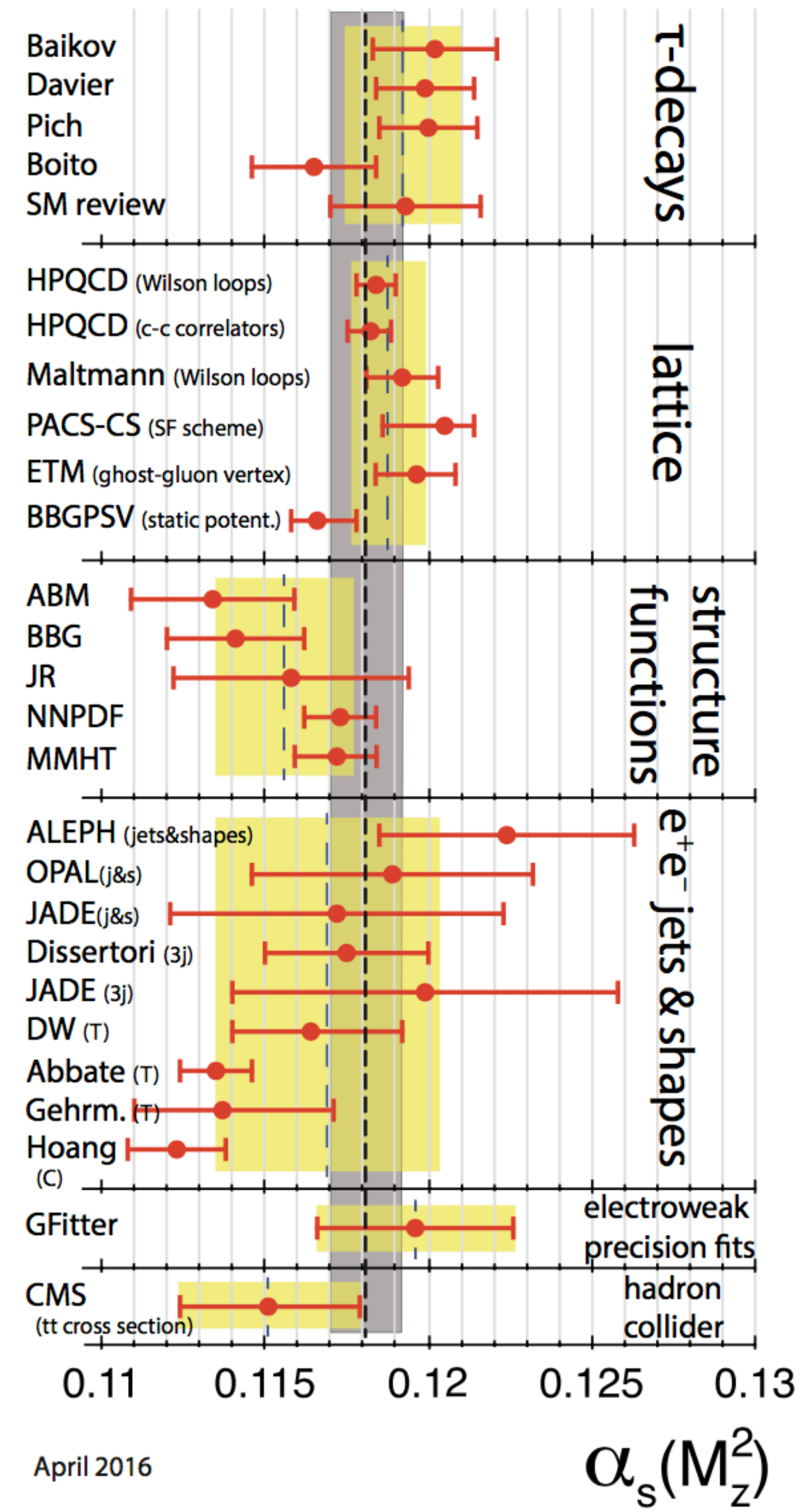
Measurements of the running coupling

Questions:

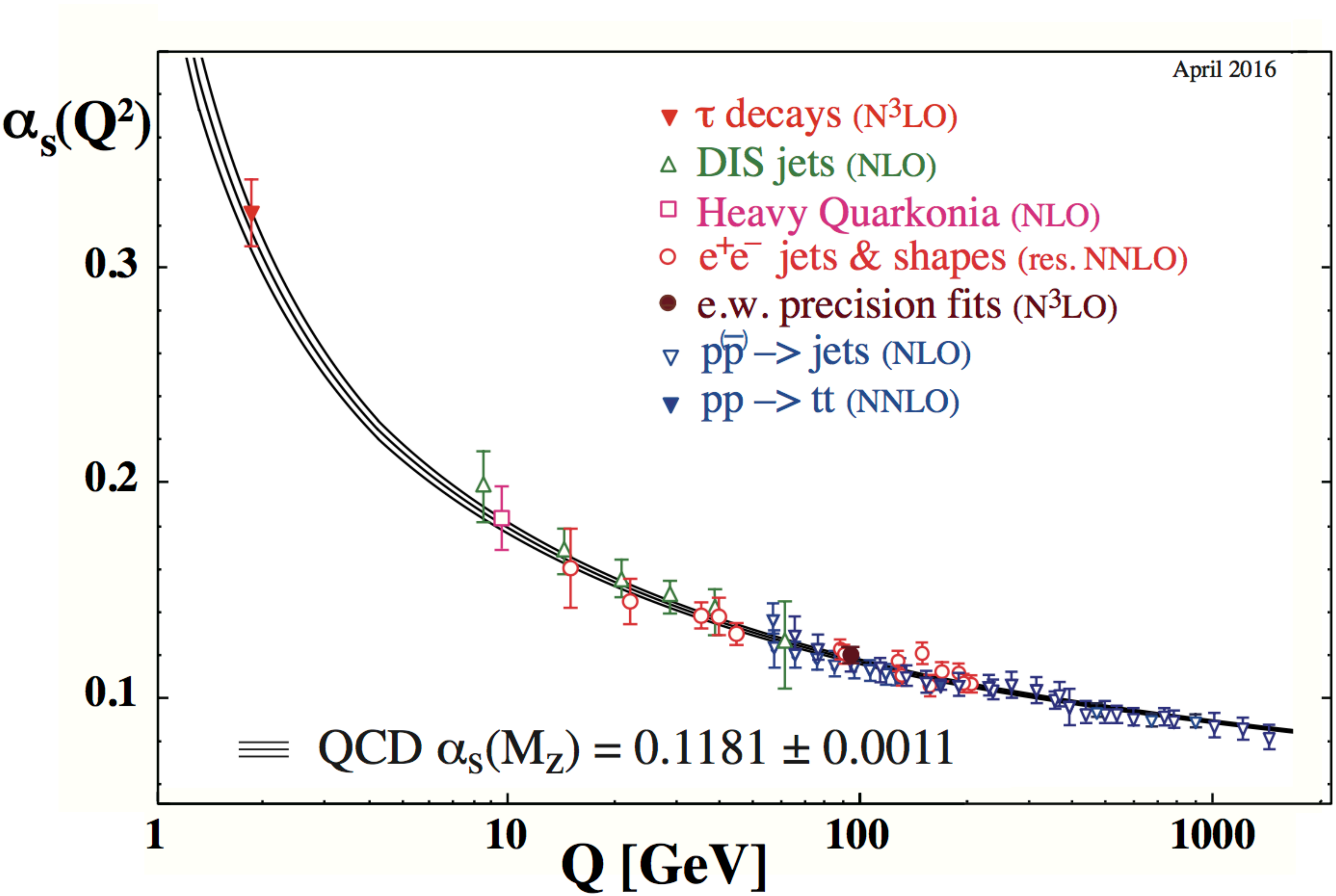
- Why is the determination of α_s from τ-decays so accurate?
- Why is the determination of α_s from the four-jet rate so accurate?

World average

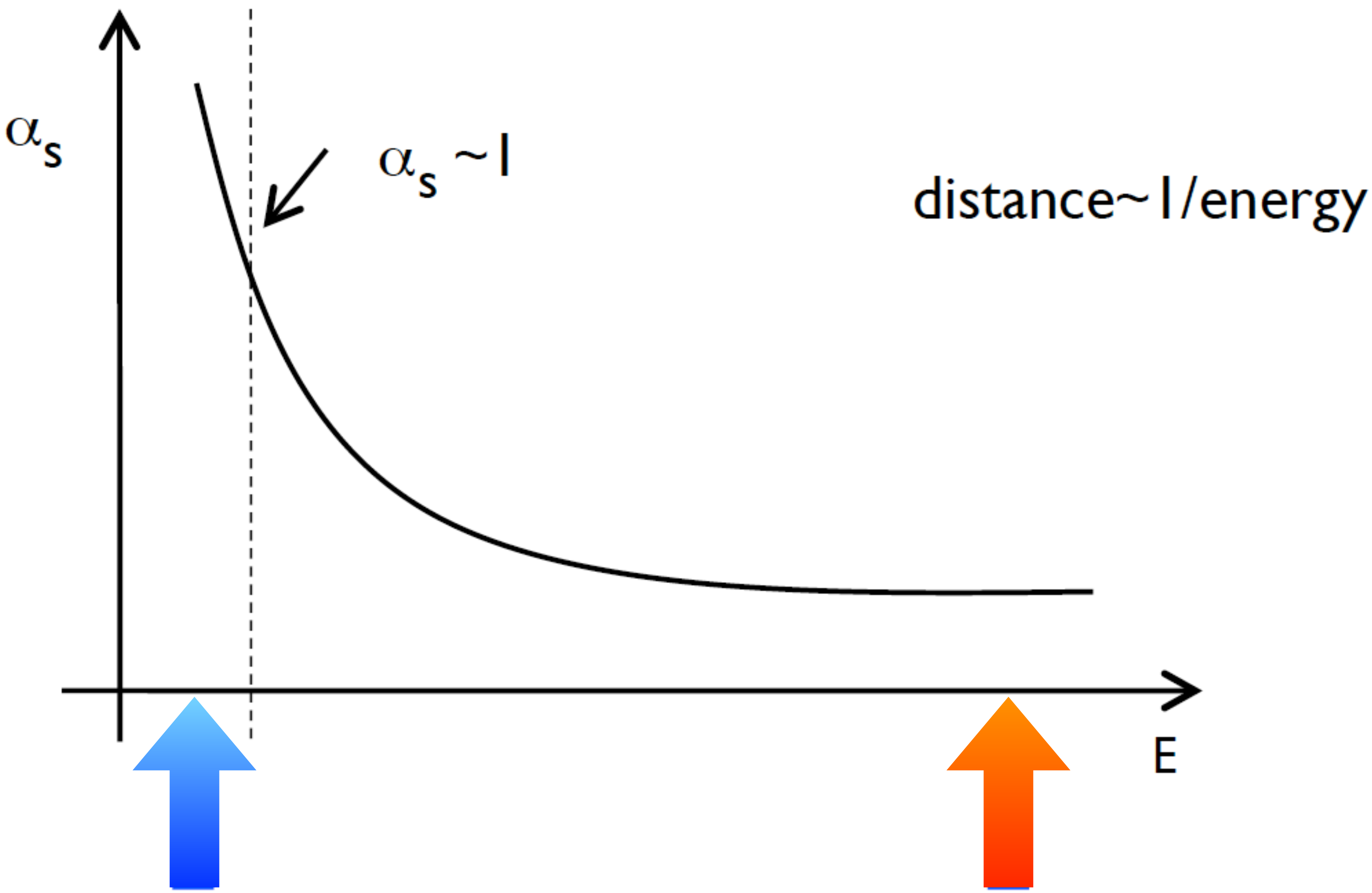
$$\boxed{\alpha_s(M_Z) = 0.1181 \pm 0.0011}$$



Measurements of the running coupling



The two faces of QCD



Confinement
(large distance)

asymptotic freedom
(short distance)

NB: no proof of confinement. We simply never observed quarks as free particles

Recap

We have then discussed the UV behaviour of QCD

- discussed renormalisation of UV divergences
- introduced the running of the coupling constant and the beta-function
- discussed measurements of the coupling constant

As we will see, the perturbative description of QCD is very predictive but we understand much less the regime governed by strong dynamics.

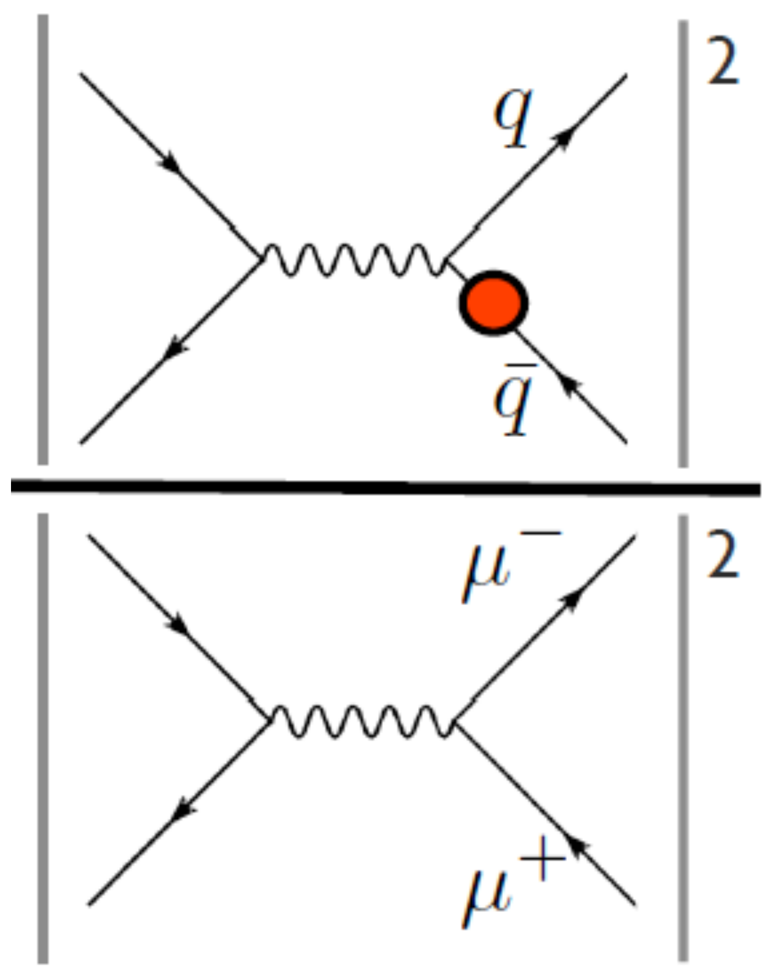
Next

Next we'll discuss generic properties of QCD amplitudes

- Soft-collinear divergences (and how they are dealt with)
- Kinoshita-Lee-Nauenberg theorem
- The concept of infrared finiteness
- Sterman Weinberg jets

The soft approximation

Let's consider again the R-ratio



Leading order result

$$R \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} \approx N_c \sum_q e_q^2$$

Quark-hadron duality

The reliability of parton-level calculations to describe hadron-level observables is known as quark-hadron duality.

This duality relies on the time separation between a hard scattering (partons are produced) and a soft process (quarks hadronize). Since the two processes happen at very different time-scales there is not quantum interference and the soft process does not alter the hard momentum flow "too much"

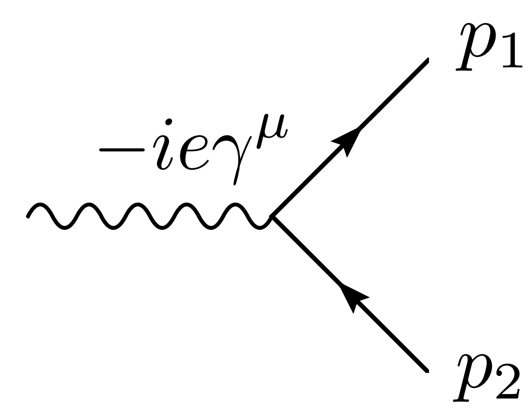
With this in mind, let's apply the parton description and look for a better approximation of R, i.e. let's compute QCD corrections, at least in some approximation

The soft approximation

QCD corrections are only in the final state, i.e. corrections to $\gamma^* \to q\bar{q}$

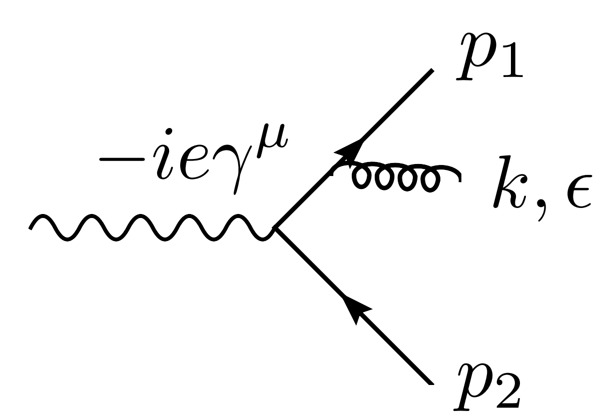
At leading order:

$$M_0^\mu = \bar{u}(p_1)(-ie\gamma^\mu)v(p_2)$$



Emit one gluon:

$$\begin{aligned} M^\mu_{q\bar{q}g} &= \bar{u}(p_1)(-ig_s t^a \not{\epsilon})\frac{i(\not{p}_1 + \not{k})}{(p_1+k)^2}(-ie\gamma^\mu)v(p_2) \\ &+ \bar{u}(p_1)(-ie\gamma^\mu)\frac{i(\not{p}_2 - \not{k})}{(p_2-k)^2}(-ig_s t^a \not{\epsilon})v(p_2) \end{aligned}$$



Consider the soft approximation: $k \ll p_1, p_2$

$$M^\mu_{q\bar{q}g} = \bar{u}(p_1)\left((-ie\gamma^\mu)(-ig_s t^a)v(p_2)\right)\left(\frac{p_1\epsilon}{p_1 k} - \frac{p_2\epsilon}{p_2 k}\right)$$

$\Rightarrow$ factorization of soft part (crucial for resummed calculations)

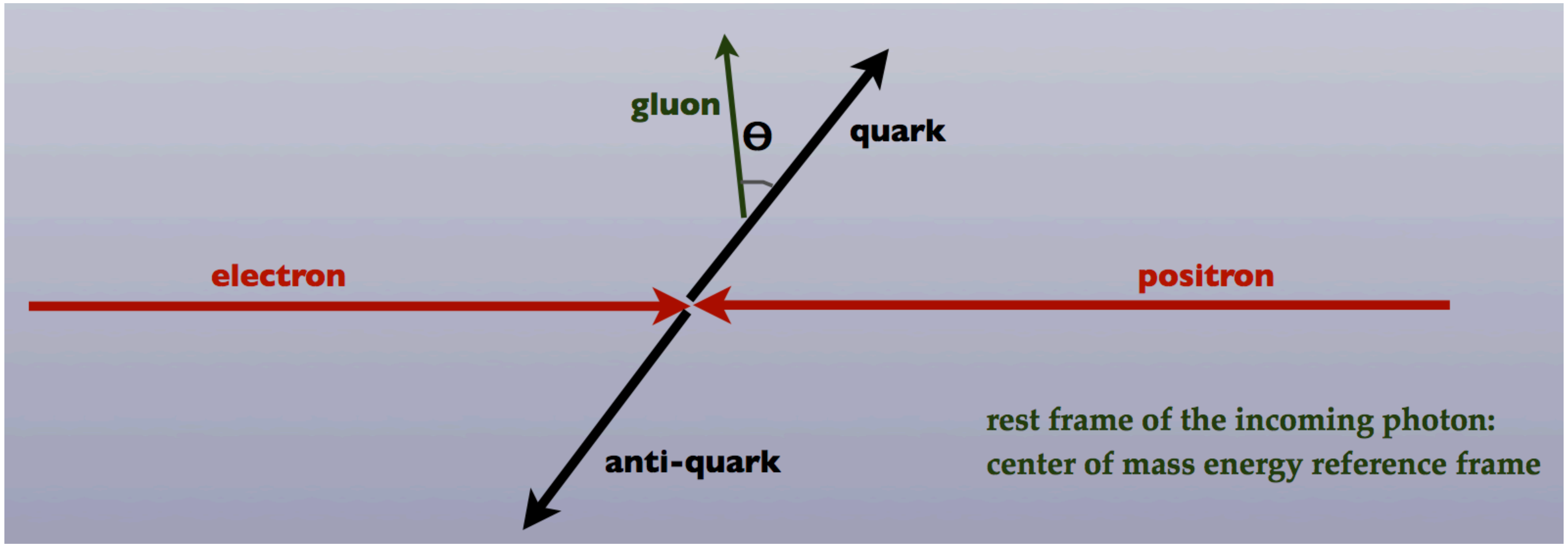
Soft divergences

The squared amplitude becomes

$$|M^{\mu}_{q\bar{q}g}|^2 = \sum_{\mathrm{pol}} \left| \bar{u}(p_1)\left((-ie\gamma^{\mu})(-ig_s t^a)v(p_2)\right)\left(\frac{p_1\epsilon}{p_1 k} - \frac{p_2\epsilon}{p_2 k}\right)\right|^2$$

$$= \quad |M_{q\bar{q}}|^2 C_F g_s^2 \frac{2p_1p_2}{(p_1k)(p_2k)}$$

The above is a Lorentz-invariant amplitude. Go to the centre-of-mass frame:



Soft divergences

The squared amplitude becomes

$$
\begin{aligned}
|M^{\mu}_{q\bar{q}g}|^2 = \sum_{\text{pol}} \left| \bar{u}(p_1)\left((-ie\gamma^{\mu})(-ig_s t^a)v(p_2)\right)\left(\frac{p_1\epsilon}{p_1 k} - \frac{p_2\epsilon}{p_2 k}\right)\right|^2 \\
= \quad |M_{q\bar{q}}|^2 C_F g_s^2 \frac{2p_1 p_2}{(p_1 k)(p_2 k)}
\end{aligned}
$$

Including phase space, in this frame, in terms of energy and angle of the gluon one contains

$$
\begin{aligned}
d\phi_{q\bar{q}g}|M_{q\bar{q}g}|^2 &= d\phi_{q\bar{q}}|M_{q\bar{q}}|^2 \frac{d^3k}{2\omega(2\pi)^3} C_F g_s^2 \frac{2p_1 p_2}{(p_1 k)(p_2 k)} \\
&= d\phi_{q\bar{q}}|M_{q\bar{q}}|^2 \omega d\omega d\cos\theta \frac{d\phi}{2\pi} \frac{2\alpha_s C_F}{\pi} \frac{1}{\omega^2(1-\cos^2\theta)}
\end{aligned}
$$

The differential cross section becomes

$$
d\sigma_{q\bar{q}g} = d\sigma_{q\bar{q}} \frac{2\alpha_s C_F}{\pi} \frac{d\omega}{\omega} \frac{d\theta}{\sin\theta} \frac{d\phi}{2\pi}
$$

Soft & collinear divergences

Cross section for producing a qq̄-pair and a gluon is infinite (IR divergent)

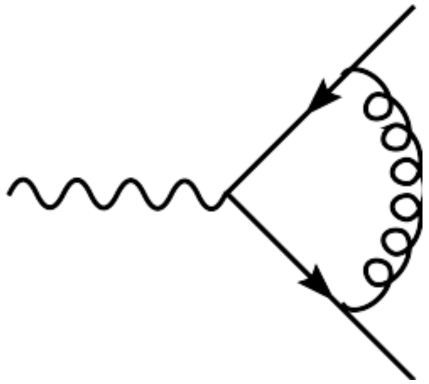
$$d\sigma_{q\bar{q}g} = d\sigma_{q\bar{q}} \frac{2\alpha_s C_F}{\pi} \frac{d\omega}{\omega} \frac{d\theta}{\sin\theta} \frac{d\phi}{2\pi}$$

$\underline{\omega \to 0}$: soft divergence

$\underline{\theta \to 0}$: collinear divergence

But the full O(α_s) correction to R is finite, because one must include a virtual correction which cancels the divergence of the real radiation

$$d\sigma_{q\bar{q},v} \sim -d\sigma_{q\bar{q}} \frac{2\alpha_s C_F}{\pi} \frac{d\omega}{\omega} \frac{d\theta}{\sin\theta} \frac{d\phi}{2\pi}$$



NB: here we kept only soft terms, if we do the full calculation one gets a finite correction of α_s/π

Soft & collinear divergences

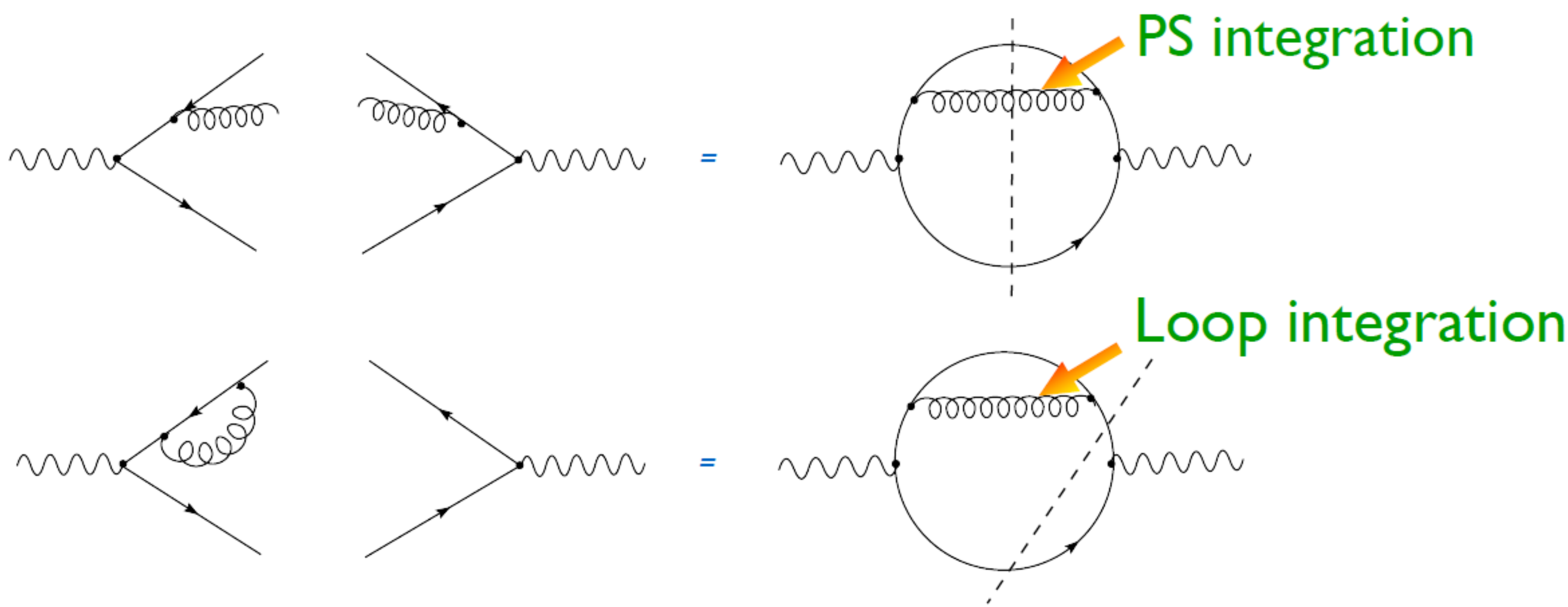
$\underline{\omega \to 0}$ soft divergence: the four-momentum of the emitted particle approaches zero, typical of gauge theories, even if matter (radiating particle) is massive

$\underline{\theta \to 0}$ collinear divergence: particle emitted collinear to emitter. Divergence present only if all particles involved are massless

NB: the appearance of soft and collinear divergences discussed in the specific contect of $e^+e^- \to qq$ are a general property of QCD

Infrared finiteness

Cancellation of IR divergences in R is not a miracle. It follows directly from unitarity provided the measurement is inclusive enough



In the infrared region real and virtual are kinematically equivalent but for a (-1) from unitarity

Compute and regulate real and virtual separately, until a cancelation of divergences is achieved

KLN Theorem

Kinoshita-Lee-Nauenberg theorem: Infrared singularities in a massless theory cancel out after summing over degenerate (initial and final) states

hard   hard + soft gluon   2 collinear partons

Physically a hard parton can not be distinguished from a hard parton plus a soft gluon or from two collinear partons with the same energy. They are degenerate states.

Hence, one needs to add them to get a physically sound observable

Infrared safety (= finiteness)

So, the R-ratio is an infrared safe quantity.

In perturbation theory one can compute only IR-safe quantities, otherwise get infinities, which can not be renormalized away (why not…?)

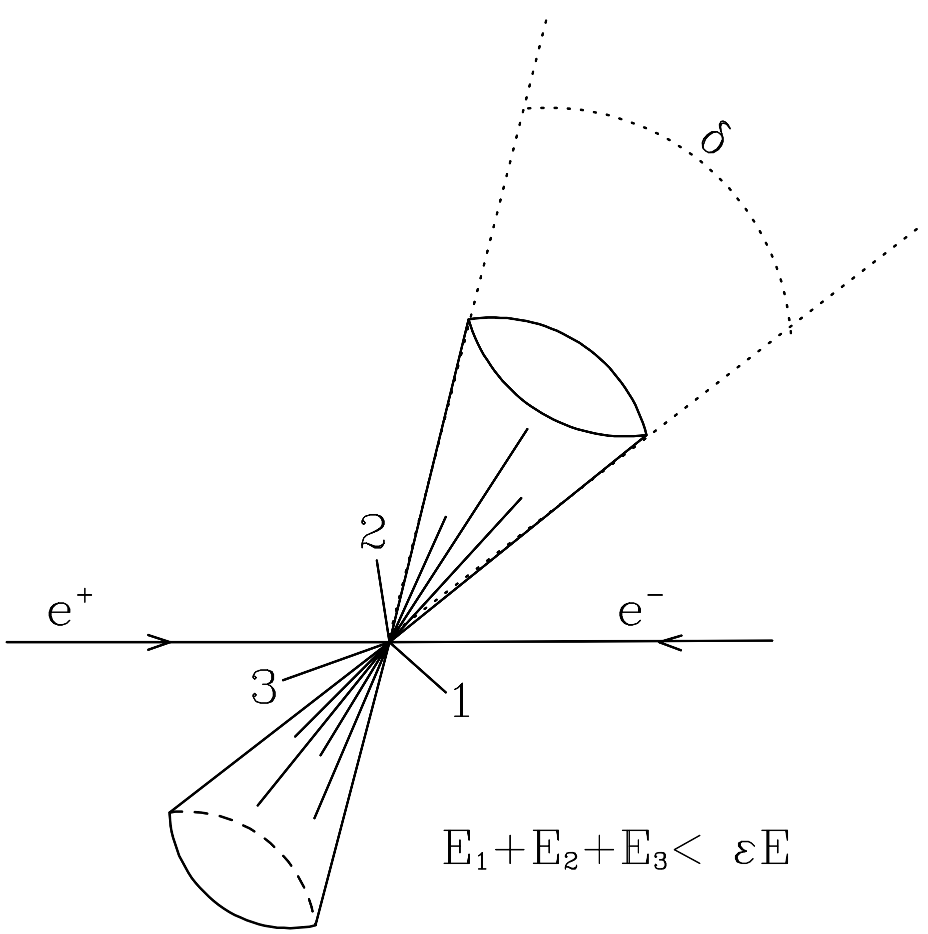
So, the natural questions are:

- are there other IR-safe quantities?
- what property of R guarantees its IR-safety?

Sterman-Weinberg jets

First formulation of cross-sections which are finite in perturbation theory and describe the hadronic final state

Introduce two parameters ε and δ:
a pair of Sterman-Weinberg jets are two cones of opening angle δ that contain all the energy of the event excluding at most a fraction ε

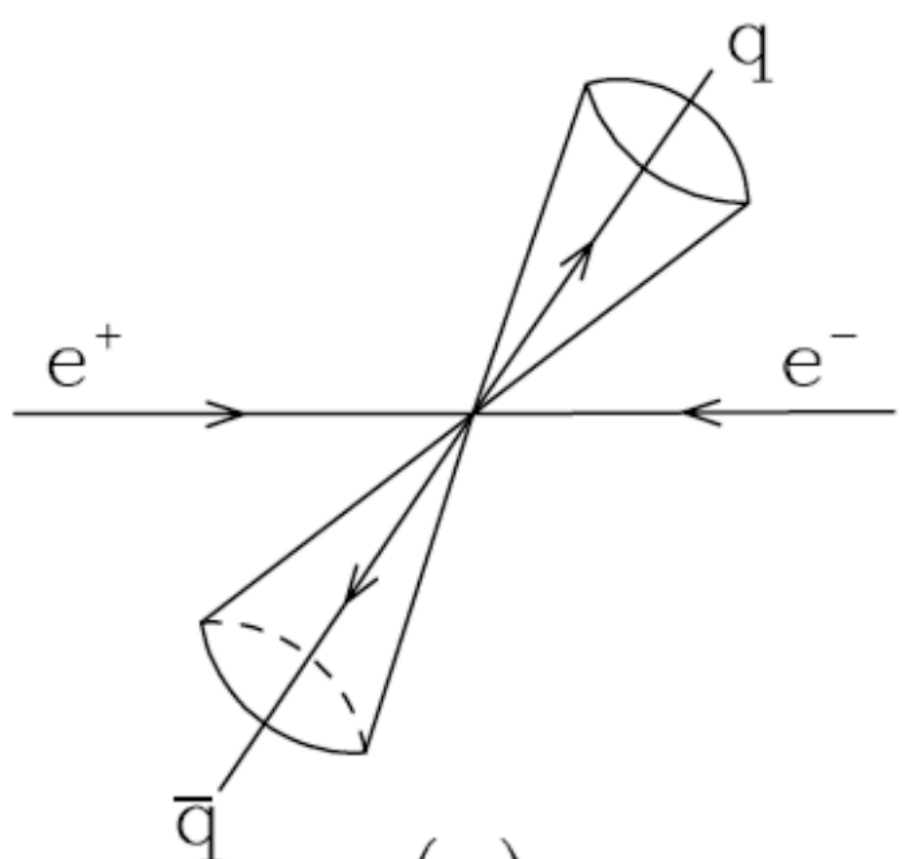


Why finite? the cancelation between real and virtual is not destroyed in the soft/collinear regions

Sterman-Weinberg jets

Let’s compute the O(as) correction to the Sterman-Weinberg jet cross-section in the soft-collinear approximation

a) We have a Born term σ_B which is completely within the Sterman-Weinberg jet definition: since there are only two quarks they keep all the energy inside the cones

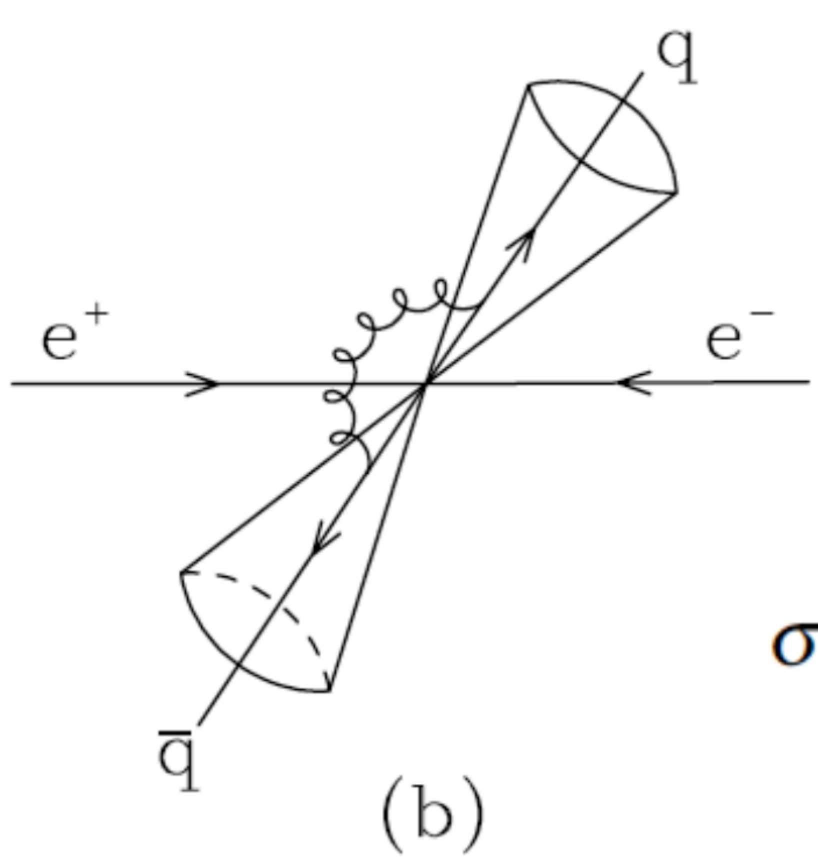


(a)

Sterman-Weinberg jets

Let's compute the O(as) correction to the Sterman-Weinberg jet cross-section in the soft-collinear approximation

b) We have a virtual term which is also completely within the Sterman-Weinberg jet definition (only two quarks)

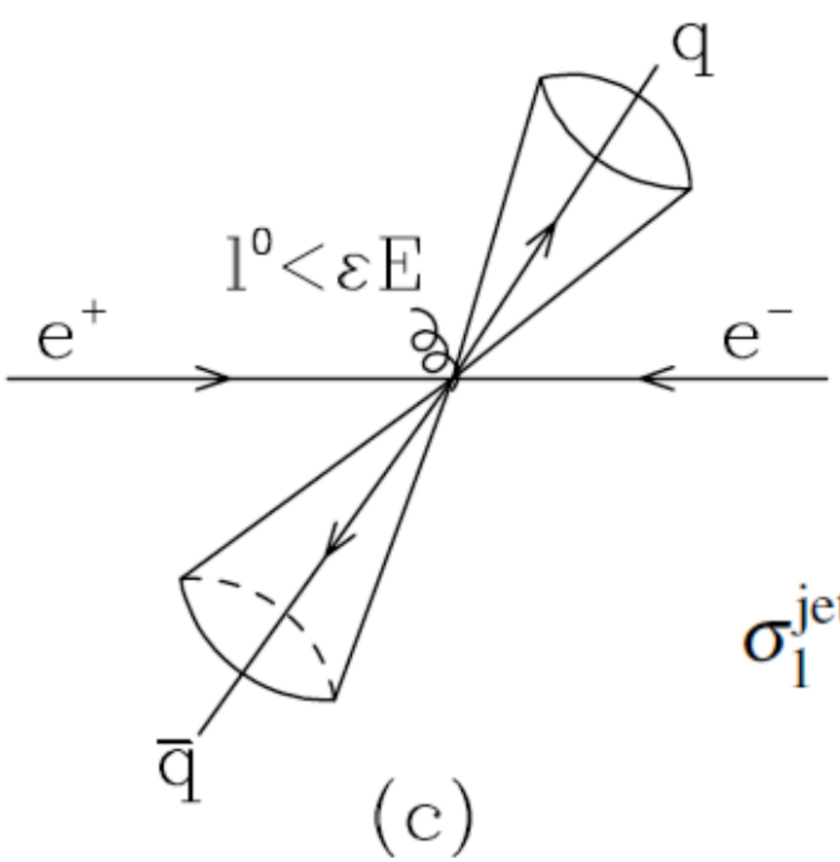


$$\sigma_1^{\text{jet,virt}} = -C_F \frac{g_S^2}{\pi^2} \sigma_{\text{born}} \int_0^E \frac{dl_0}{l_0} \int_0^\pi \frac{d\cos\theta}{1-\cos^2\theta}$$

Sterman-Weinberg jets

Let's compute the O(as) correction to the Sterman-Weinberg jet cross-section in the soft-collinear approximation

c) We have a real term: the emitted gluon can be emitted also outside the jet provided it carries only little energy, or..

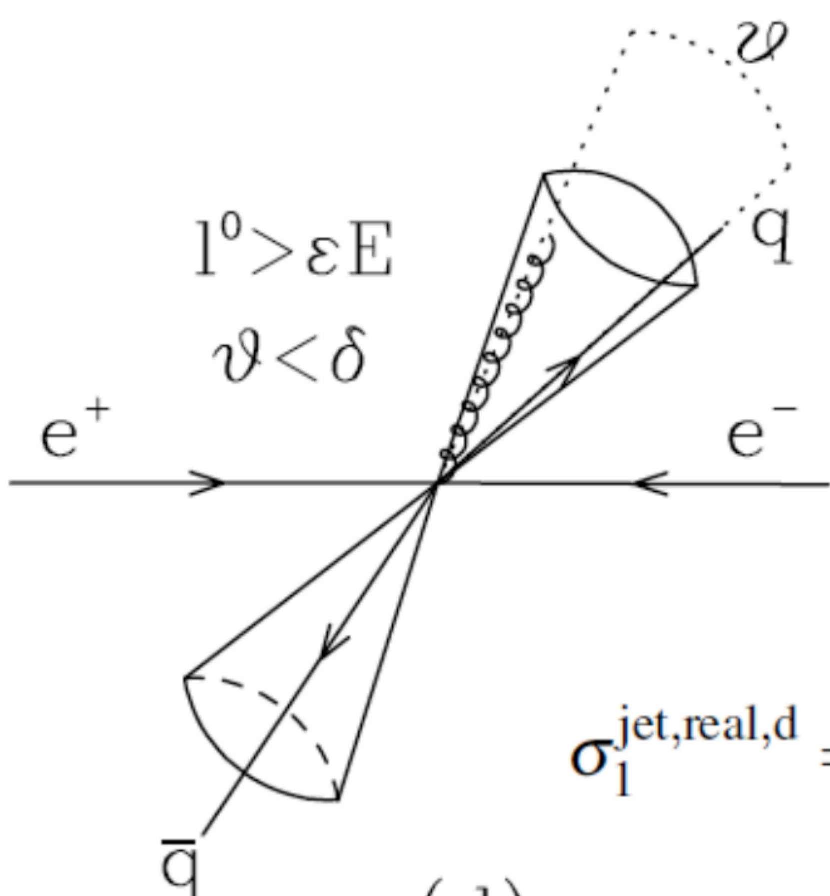


$$\sigma_1^{\text{jet,real,c}} = C_F \frac{g_S^2}{\pi^2} \sigma_{\text{born}} \int_0^{\varepsilon E} \frac{dl_0}{l_0} \left[\int_0^{\pi} \frac{d\cos\theta}{1-\cos^2\theta} \right]$$

Sterman-Weinberg jets

Let's compute the O(as) correction to the Sterman-Weinberg jet cross-section in the soft-collinear approximation

d) .. or it can carry a considerable fraction of energy provided it is emitted inside the cones



(d)

$$\sigma_1^{\text{jet,real,d}} = C_F \frac{g_S^2}{\pi^2} \sigma_{\text{born}} \int_{\varepsilon E}^{E} \frac{dl_0}{l_0} \left[\int_0^{\delta} \frac{d\cos\theta}{1-\cos^2\theta} + \int_{\pi-\delta}^{\pi} \frac{d\cos\theta}{1-\cos^2\theta} \right]$$

Sterman-Weinberg jets

Adding all the contributions, the Sterman-Weinberg jet cross-section up to O(α_s) in the soft-collinear approximation is given by

$$\sigma_1 = \sigma_0 \left(1 + \frac{2\alpha_s C_F}{\pi} \ln \epsilon \ln \delta^2 \right)$$

Effective expansion parameter in QCD is often $\alpha_s C_F/\pi$ not α_s

α_s-expansion enhanced by a double log: left-over from real-virtual cancellation

- if more gluons are emitted, one gets for each gluon
  - a power of $\alpha_s C_F/\pi$
  - a soft logarithm lnε
  - a collinear logarithm lnδ
- if ε and/or δ become too small the above result diverges
- if the logs are large, fixed order meaningless, one needs to resum large infrared and collinear logarithms to all orders in the coupling constant