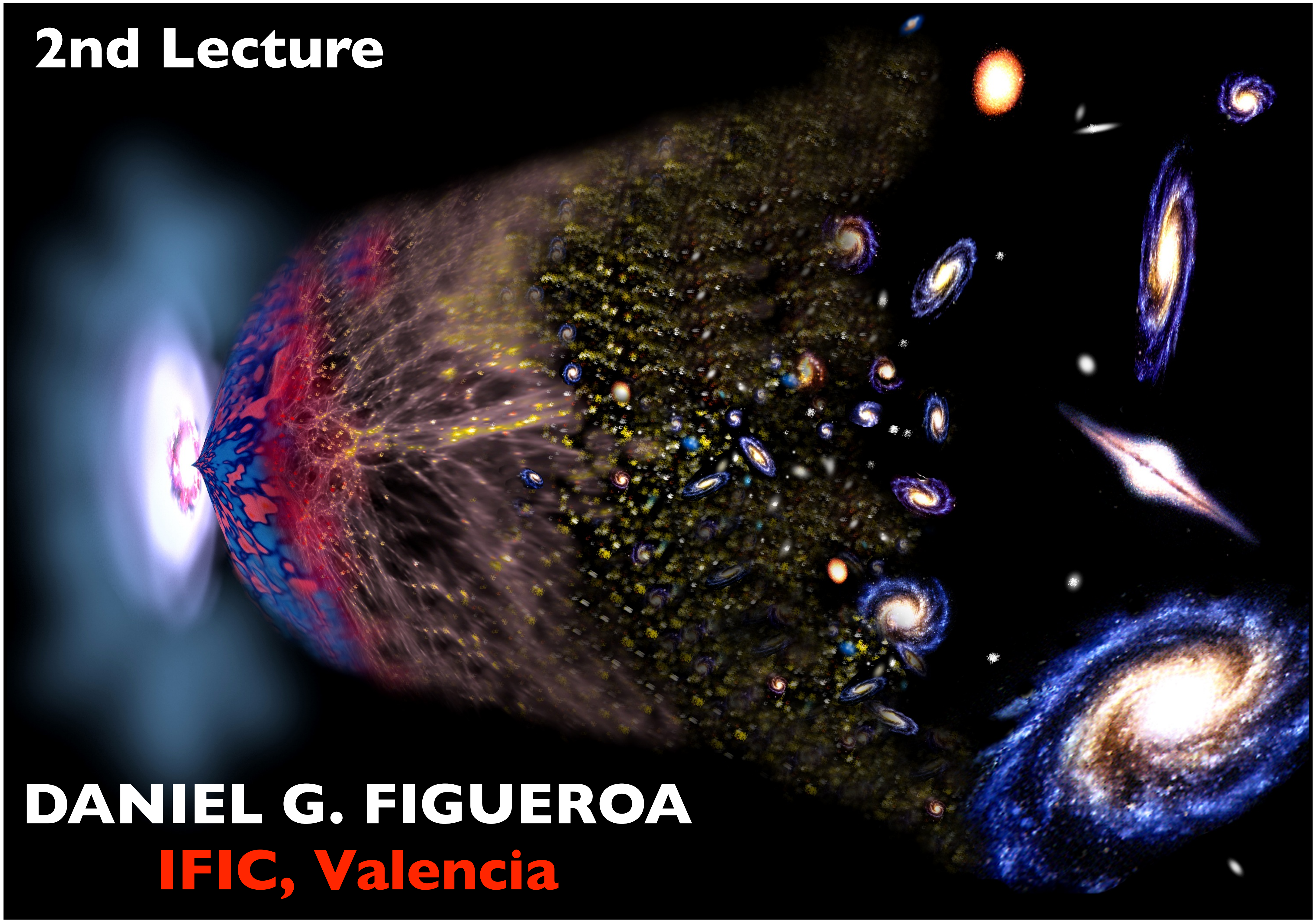
GRAVITATIONAL WAVE
— BACKGROUNDS —



GGI LECTURES ON THE THEORY OF FUNDAMENTAL INTERACTIONS — 2022 Program (3rd week)

OUTLINE

1st Bloc

1) Cosmology/GR + GWs

$\frac{1}{2}$) Cosmo/GR ✓

$\frac{2}{2}$) GW definition

2) GWs from Inflation

3) GWs from Preheating

4) GWs from Phase Transitions

5) GWs from Cosmic Defects

OUTLINE

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A PRIMER ON GRAVITATIONAL WAVES

Gravitational Framework

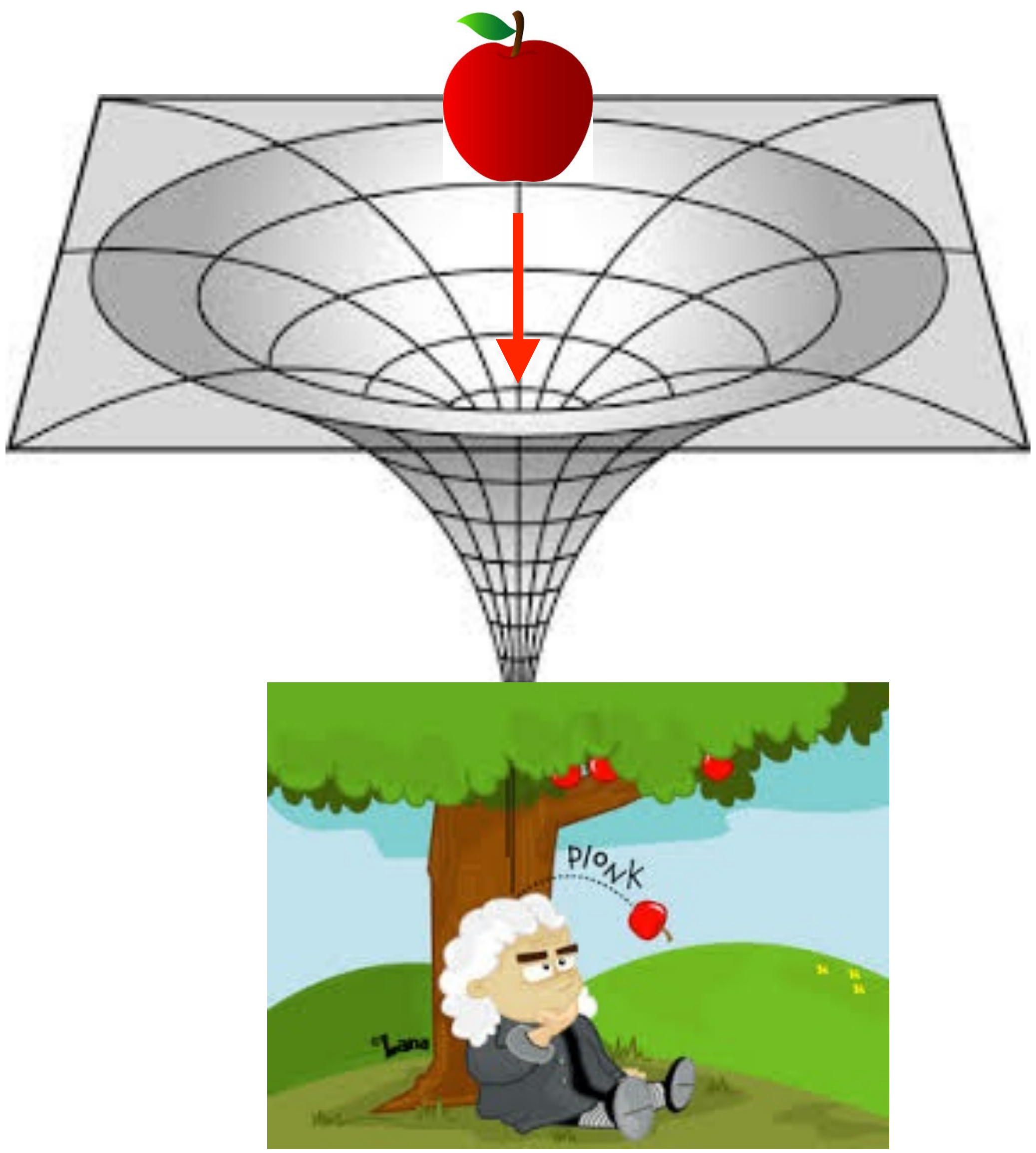
General Relativity (GR)

$$G_{\mu\nu} = \frac{1}{m_p^2} T_{\mu\nu}$$

geometry matter

$$\left[m_p = (8\pi G)^{-1/2} = 2.44 \cdot 10^{18}\,\mathrm{GeV} \right]$$

Reduced Planck mass



$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu$$

$$\mathrm{DIFF}: \quad x^\mu \to x'^\mu(x)$$

symmetry

Gravitational Framework

General Relativity (GR)

$$G_{\mu\nu} = \frac{1}{m_p^2} T_{\mu\nu}$$

geometry matter

metric

$$G_{\mu\nu} \equiv \mathcal{D}[g_{\alpha\beta}] \quad = \quad m_p^{-2} T_{\mu\nu}(\mathrm{Matt}, \mathrm{Rad}, \mathrm{Top.Defects}, \mathrm{DarkEnergy}, \ldots)$$

source

2nd order, non-Linear

Gravitational Framework

General Relativity (GR)

$$G_{\mu\nu} = \frac{1}{m_p^2} T_{\mu\nu}$$

geometry matter

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2nd order, non-Linear

source

$$\begin{tikzcd} & T^{[U]}_{\mu\nu} \arrow[dd, bend left] \\ \textbf{Universe ?} \arrow[ur] \arrow[dr] & \\ & g^{[U]}_{\mu\nu} \end{tikzcd}$$

Cosmological Principle

Background metric and matter

Homogeneous & Isotropy

FLRW expanding Universe !

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = a^2(t)\left(-dt^2 + d\mathbf{x}^2\right)$$

Gravitational Framework

General Relativity (GR)

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geometry matter

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$$G_{\mu\nu} \equiv \mathcal{D}[g_{\alpha\beta}] \quad = \quad m_p^{-2} T_{\mu\nu}(\mathrm{Matt}, \mathrm{Rad}, \mathrm{Top.Defects}, \mathrm{DarkEnergy}, ...)$$

2nd order, non-Linear

source

How do we define GWs ?

Gravitational Framework

General Relativity (GR)

$$G_{\mu\nu} = \frac{1}{m_p^2} T_{\mu\nu}$$

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source

expand in perturbations

$$g_{\alpha\beta} = \bar{g}_{\alpha\beta} + \delta g_{\alpha\beta}$$

How do we define GWs ?

Gravitational Framework

General Relativity (GR)

$$G_{\mu\nu} = \frac{1}{m_p^2} T_{\mu\nu}$$

geometry matter

metric

$$G_{\mu\nu} \equiv \mathcal{D}[g_{\alpha\beta}] \quad = \quad m_p^{-2} T_{\mu\nu}(\mathrm{Matt}, \mathrm{Rad}, \mathrm{Top.Defects}, \mathrm{DarkEnergy}, \ldots)$$

expand in perturbations

source of GWs

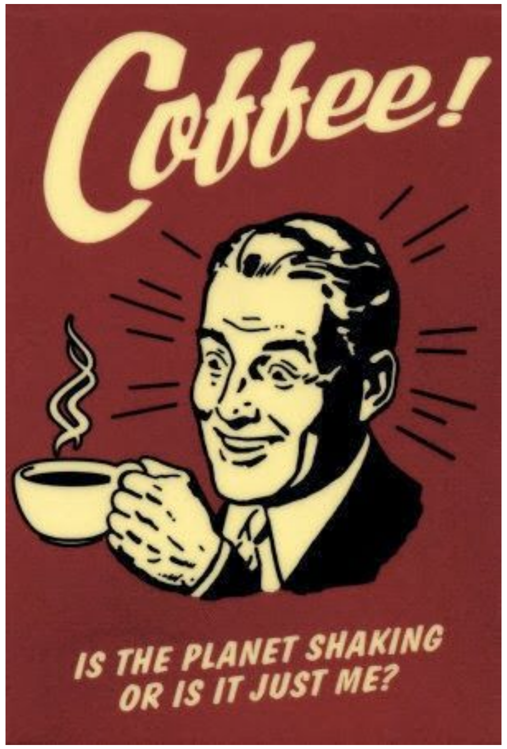
$$g_{\alpha\beta} = \bar{g}_{\alpha\beta} + \delta g_{\alpha\beta}$$

How do we define GWs ?

$$g_{\alpha\beta} = \bar{g}_{\alpha\beta} + \delta g_{\alpha\beta}$$

Let's continue this approach…

I hope you took a good load of coffee ('cause you are gonna need it)



Definition of GWs
1st approach

Gravitational Wave Definition

Minkowski

1st approach to GWs

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

$$(\ |h_{\mu\nu}| \ll 1 \ )$$

Gravitational Wave Definition

LINEARIZED GRAVITY

$$g_{\mu\nu} = \overset{\text{Minkowski}}{\eta_{\mu\nu}} + h_{\mu\nu}(x)$$

$$(\ |h_{\mu\nu}| \ll 1\ )$$

Gravitational Wave Definition

1st approach to GWs

$$g_{\mu\nu} = \underbrace{\eta_{\mu\nu}}_{\text{Minkowski}} + h_{\mu\nu}(x)$$

$$(\ |h_{\mu\nu}| \ll 1\ )$$

Gravitational Wave Definition

1st approach to GWs

Minkowski

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

fixed frame

$$(\ |h_{\mu\nu}| \ll 1\ )$$

Gravitational Wave Definition

Minkowski

1st approach to GWs

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

fixed frame

$$(\ |h_{\mu\nu}| \ll 1 \ )$$

$$\mathrm{DIFF}: \quad x^\mu \to x'^\mu(x)$$

symmetry **?**

Gravitational Wave Definition

Minkowski

1st approach to GWs

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

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$$\mathrm{DIFF}: \quad \cancel{x^\mu \to x'^\mu}(x)$$

Gravitational Wave Definition

Minkowski

1st approach to GWs $$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$ **fixed frame**

$$(\ |h_{\mu\nu}| \ll 1\ )$$

$$\mathrm{DIFF}:\quad \cancel{x^\mu \to x'^\mu(x)}$$

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x)$$

$$\boxed{(\ |\partial_\mu \xi_\nu(x)| \lesssim |h_{\mu\nu}|\ )}$$

residual symm.

$$\boxed{h_{\mu\nu}(x) \to h'_{\mu\nu}(x') = h_{\mu\nu}(x) - \partial_{(\mu}\xi_{\nu)}}$$

Gravitational Wave Definition

1st approach to GWs

Minkowski

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

$$(\ |h_{\mu\nu}| \ll 1\ )$$

fixed frame

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$$\boxed{h_{\mu\nu}(x) \to h'_{\mu\nu}(x') = h_{\mu\nu}(x) - \partial_{(\mu}\xi_{\nu)}}$$

residual symm.

Notation:

$$\begin{cases} \partial_{(\mu}\xi_{\nu)} \equiv \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \\ \partial_{[\mu}\xi_{\nu]} \equiv \partial_\mu \xi_\nu - \partial_\nu \xi_\mu \end{cases}$$

Gravitational Wave Definition

Minkowski

1st approach to GWs

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

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Let's expand Einstein Equations !

Gravitational Wave Definition

Minkowski

1st approach to GWs

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

fixed frame

$$(\ |h_{\mu\nu}| \ll 1 \ )$$

Trace-reversed

$$\boxed{\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}}$$

Gravitational Wave Definition

Minkowski

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$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

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$$\boxed{\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}} \Rightarrow \partial^\alpha \partial_\alpha \bar{h}_{\mu\nu} + \eta_{\mu\nu} \partial^\alpha \partial^\beta \bar{h}_{\alpha\beta} - \partial^\alpha \partial_{(\mu} \bar{h}_{\alpha\nu)} = -\frac{2}{m_p^2} T_{\mu\nu}$$

Gravitational Wave Definition

Minkowski

1st approach to GWs

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

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Trace-reversed

$$\boxed{\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}} \Rightarrow \underbrace{\partial^\alpha\partial_\alpha\bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial^\alpha\partial^\beta\bar{h}_{\alpha\beta} - \partial^\alpha\partial_{(\mu}\bar{h}_{\alpha\nu)}}_{\mathcal{O}(h_{**})\ \text{Einstein tensor expanded}} = -\frac{2}{m_p^2}T_{\mu\nu}$$

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$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

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residual symm.

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residual symm.

$$\boxed{\partial^\nu \bar{h}_{\mu\nu} = 0}$$

Lorentz gauge

Gravitational Wave Definition

Minkowski

1st approach to GWs

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$\mathcal{O}(h_{**})$ **Einstein tensor expanded**

residual symm.

$$\boxed{\partial^\nu \bar{h}_{\mu\nu} = 0}$$

Lorentz gauge

Technical Note: If $\partial^\mu \bar{h}_{\mu\nu} \neq 0$

Gravitational Wave Definition

Minkowski

1st approach to GWs

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

fixed frame

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$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x)$$

$$h_{\mu\nu}(x) \to h'_{\mu\nu}(x') = h_{\mu\nu}(x) - \partial_{(\mu} \xi_{\nu)}$$

Gravitational Wave Definition

Minkowski

1st approach to GWs

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Lorentz gauge

Technical Note: If $\partial^\mu \bar{h}_{\mu\nu} \neq 0$

$$\partial'^\mu \bar{h}'_{\mu\nu}(x') = \underbrace{\partial^\mu \bar{h}_{\mu\nu}(x)}_{\equiv f(x) \neq 0} - \Box \xi_\nu = 0$$

Gravitational Wave Definition

1st approach to GWs

Minkowski

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

fixed frame

$$(\; |h_{\mu\nu}| \ll 1 \;)$$

Trace-reversed

$$\boxed{\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}}$$

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Gravitational Wave Definition

Minkowski

1st approach to GWs

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x) \qquad (\; |h_{\mu\nu}| \ll 1 \;)$$

fixed frame

Trace-reversed

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$$\underbrace{\partial^\alpha \partial_\alpha \bar{h}_{\mu\nu} + \eta_{\mu\nu} \partial^\alpha \partial^\beta \bar{h}_{\alpha\beta} - \partial^\alpha \partial_{(\mu} \bar{h}_{\alpha\nu)}}_{\mathcal{O}(h_{**})\ \text{Einstein tensor expanded}} = -\frac{2}{m_p^2} T_{\mu\nu}$$

residual symm.

$$\boxed{\partial^\nu \bar{h}_{\mu\nu} = 0}$$

Lorentz gauge

Technical Note: If $\partial^\mu \bar{h}_{\mu\nu} \neq 0$

$$\boxed{\partial'^\mu \bar{h}'_{\mu\nu}(x')} = \underbrace{\partial^\mu \bar{h}_{\mu\nu}(x)}_{\equiv f(x) \neq 0} - \Box \xi_\nu \boxed{= 0} \iff \Box \xi_\nu = f(x)$$

(solution always!)

Gravitational Wave Definition

Minkowski

1st approach to GWs

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

fixed frame

$$(\ |h_{\mu\nu}| \ll 1\ )$$

Trace-reversed

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Minkowski

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$$\boxed{\partial^\nu \bar{h}_{\mu\nu} = 0}$$

Lorentz gauge

Gravitational Wave Definition

Minkowski

1st approach to GWs

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fixed frame

$$(\; |h_{\mu\nu}| \ll 1 \;)$$

Trace-reversed

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Lorentz gauge

Gravitational Wave Definition

Minkowski

1st approach to GWs

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fixed frame

$$(\ |h_{\mu\nu}| \ll 1 \ )$$

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Lorentz gauge

$$\partial_\alpha \partial^\alpha \bar{h}_{\mu\nu} = -\frac{2}{m_p^2} T_{\mu\nu}$$

Gravitational Wave Definition

1st approach to GWs

Minkowski

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fixed frame

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residual symm.

$$\boxed{\partial^\nu\bar{h}_{\mu\nu} = 0}$$

Lorentz gauge

$$\boxed{\partial_\alpha\partial^\alpha\bar{h}_{\mu\nu} = -\frac{2}{m_p^2}T_{\mu\nu}}$$

(10 - 4 = 6 d.o.f.)

Gravitational Wave Definition

Minkowski

1st approach to GWs

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

fixed frame

$$(\ |h_{\mu\nu}| \ll 1 \ )$$

Is that all ?

Gravitational Wave Definition

1st approach to GWs

Minkowski

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

fixed frame

$$(\ |h_{\mu\nu}| \ll 1 \ )$$

Is that all ? Not really …

Gravitational Wave Definition

Minkowski

1st approach to GWs

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

fixed frame

$$(\ |h_{\mu\nu}| \ll 1\ )$$

$$x'^{\mu} = x^{\mu} + \xi^{\mu}(x)$$

with $\partial_\alpha \partial^\alpha \xi_\mu = 0$

(further residual gauge)

Gravitational Wave Definition

Minkowski

1st approach to GWs

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

fixed frame

$$(\ |h_{\mu\nu}| \ll 1 \ )$$

$$x'^{\mu} = x^{\mu} + \xi^{\mu}(x)$$

with $\partial_\alpha \partial^\alpha \xi_\mu = 0$

(further residual gauge)

$$(\partial^\mu \bar{h}_{\mu\nu} = 0 \quad \rightarrow \quad \partial'^\mu \bar{h}'_{\mu\nu} = 0)$$

(Lorentz preserving)

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$$x'^{\mu} = x^{\mu} + \xi^{\mu}(x)$$

with $\partial_\alpha \partial^\alpha \xi_\mu = 0$

(further residual gauge)

IF $T_{\mu\nu} = 0$

Outside Source

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with $\partial_\alpha \partial^\alpha \xi_\mu = 0$

(further residual gauge)

IF $T_{\mu\nu} = 0$

Outside Source

$$\boxed{h^{0\mu} = 0\,, \qquad h^i_i = 0\,, \qquad \partial_j h_{ij} = 0}$$

(transverse-traceless gauge)

Gravitational Wave Definition

Minkowski

1st approach to GWs

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fixed frame

$$(\ |h_{\mu\nu}| \ll 1 \ )$$

$$x'^{\mu} = x^{\mu} + \xi^{\mu}(x)$$

with $\partial_\alpha \partial^\alpha \xi_\mu = 0$

(further residual gauge)

IF $T_{\mu\nu} = 0$

Outside Source

$$h^{0\mu} = 0\,, \qquad h^i_i = 0\,, \qquad \partial_j h_{ij} = 0$$

$$\boxed{\partial_\mu \partial^\mu h_{ij} = 0}$$

(**transverse-traceless gauge**)

(6 - 4 = 2 d.o.f.)

Gravitational Wave Definition

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1st approach to GWs

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

fixed frame

$$(\ |h_{\mu\nu}| \ll 1 \ )$$

$$x'^{\mu} = x^{\mu} + \xi^{\mu}(x)$$

with $\partial_\alpha \partial^\alpha \xi_\mu = 0$

(further residual gauge)

IF $T_{\mu\nu} \cancel{\neq} 0$

Inside Source !

$$\boxed{h^{0\mu} = 0\,, \qquad h^i_i = 0\,, \qquad \partial_j h_{ij} = 0}$$

?

$$\boxed{\partial_\alpha \partial^\alpha \bar{h}_{\mu\nu} = -\frac{2}{m_p^2} T_{\mu\nu}}$$

(transverse-traceless gauge)

6 - 4 = 2 d.o.f. ?

?

Gravitational Wave Definition

Minkowski

1st approach to GWs

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

fixed frame

$$(\; |h_{\mu\nu}| \ll 1 \;)$$

$$x'^{\mu} = x^{\mu} + \xi^{\mu}(x)$$

with $\partial_\alpha \partial^\alpha \xi_\mu = 0$

(further residual gauge)

IF $T_{\mu\nu} \neq 0$

Inside Source !

$$\cancel{h^{0\mu} = 0\,, \quad h^i_i = 0\,, \quad \partial_j h_{ij} = 0}$$

$$\boxed{\partial_\alpha \partial^\alpha \bar{h}_{\mu\nu} = -\frac{2}{m_p^2} T_{\mu\nu}}$$

~~**(transverse-traceless gauge)**~~

6 - 4 = 2 d.o.f. ?

Gravitational Wave Definition

Minkowski

1st approach to GWs

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

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$$(\ |h_{\mu\nu}| \ll 1 \ )$$

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with $\partial_\alpha \partial^\alpha \xi_\mu = 0$

(further residual gauge)

IF $T_{\mu\nu} \neq 0$

Inside Source !

Cannot make $h_{*0} = 0$

$$\boxed{\partial_\alpha \partial^\alpha \bar{h}_{\mu\nu} = -\frac{2}{m_p^2} T_{\mu\nu}}$$

(6 - 4 = 2 d.o.f.)

Yet there are still only 2 radiative *dof* !

Gravitational Wave Definition

(TT gauge: 6 - 4 = 2 d.o.f.)

1st approach to GWs

$$\boxed{h^{0\mu} = 0\,, \qquad h^i_i = 0\,, \qquad \partial_j h_{ij} = 0}$$

Outside Source

Gravitational Wave Definition

(TT gauge: 6 - 4 = 2 d.o.f.)

1st approach to GWs

$$\boxed{h^{0\mu} = 0\,, \qquad h^i_i = 0\,, \qquad \partial_j h_{ij} = 0}$$

Outside Source

$$\boxed{\partial_\mu \partial^\mu h_{ij} = 0}$$

Wave Eq. $\longrightarrow$ **Gravitational Waves !**

Gravitational Wave Definition

(TT gauge: 6 - 4 = 2 d.o.f.)

1st approach to GWs

$$\boxed{h^{0\mu} = 0\,, \quad h^i_i = 0\,, \quad \partial_j h_{ij} = 0}$$

Outside Source

$$\boxed{\partial_\mu \partial^\mu h_{ij} = 0}$$

Wave Eq. → Gravitational Waves !

can GW be 'gauged away' ?

Gravitational Wave Definition

(TT gauge: 6 - 4 = 2 d.o.f.)

1st approach to GWs $\boxed{h^{0\mu} = 0\,, \quad h^i_i = 0\,, \quad \partial_j h_{ij} = 0}$

Outside Source

$\boxed{\partial_\mu \partial^\mu h_{ij} = 0}$ **Wave Eq. ⟶ Gravitational Waves !**

can GW be 'gauged away' ? No !

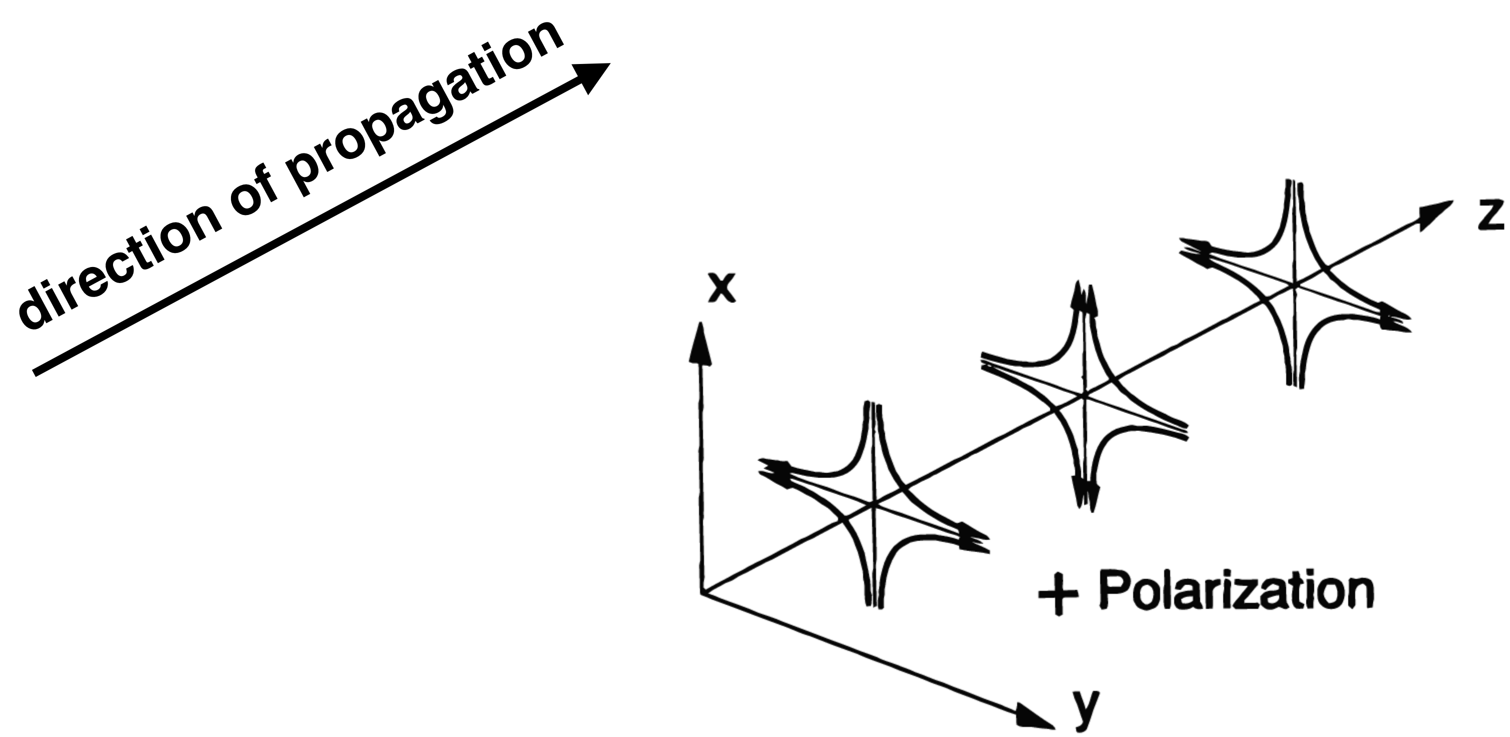
Gravitational Wave Definition

(TT gauge: 6 - 4 = 2 d.o.f.)

1st approach to GWs $\boxed{h^{0\mu} = 0\,, \quad h^i_i = 0\,, \quad \partial_j h_{ij} = 0}$ Outside Source

$\boxed{\partial_\mu \partial^\mu h_{ij} = 0}$ **Wave Eq. → Gravitational Waves !**

can GW be 'gauged away' ? No !



Transverse
(& Traceless)

Gravitational Wave Definition

(TT gauge: 6 - 4 = 2 d.o.f.)

1st approach to GWs

$$\boxed{h^{0\mu} = 0\,, \qquad h^i_i = 0\,, \qquad \partial_j h_{ij} = 0}$$

Outside Source

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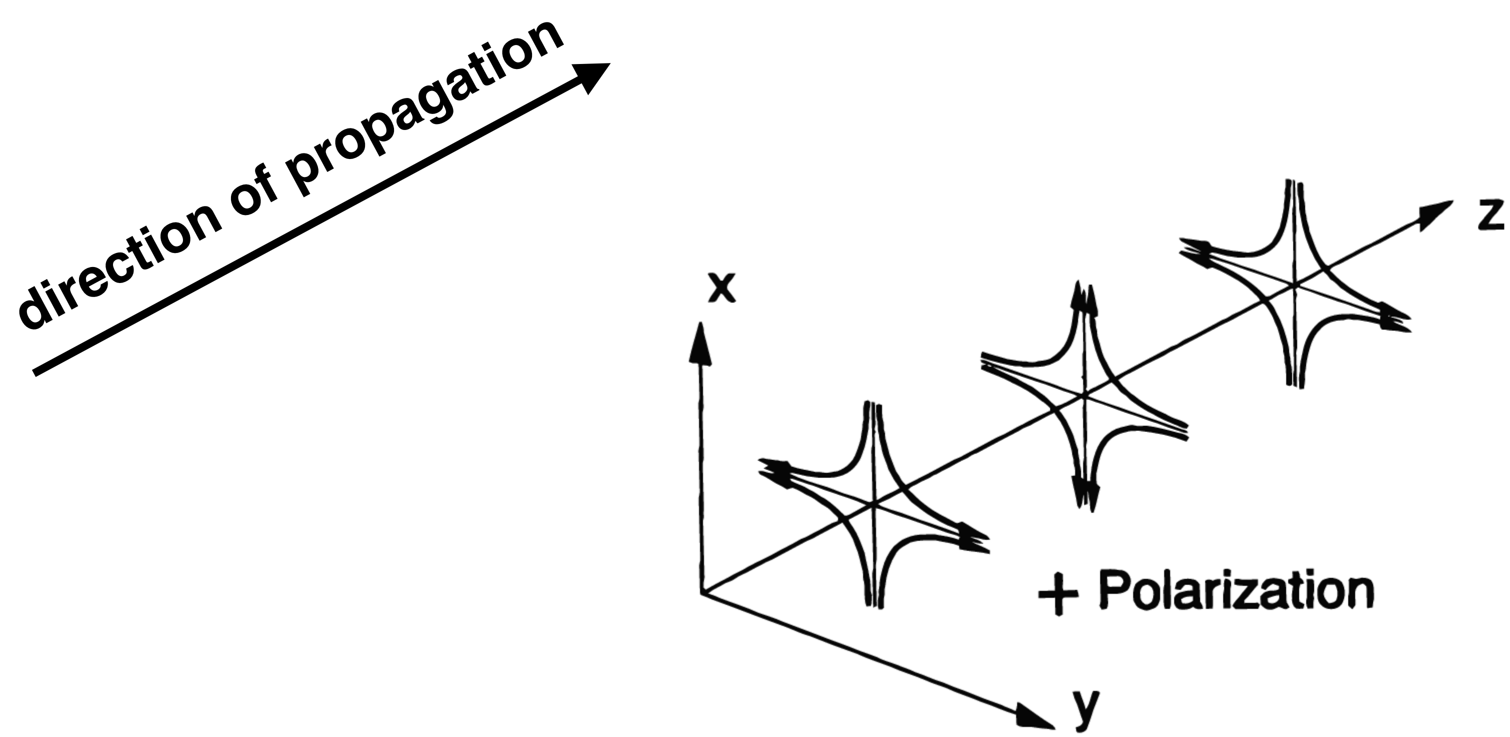

2 *dof* = 2 polarizations

Gravitational Wave Definition

(TT gauge: 6 - 4 = 2 d.o.f.)

1st approach to GWs

$$\boxed{h^{0\mu} = 0\,, \qquad h^i_i = 0\,, \qquad \partial_j h_{ij} = 0}$$

Outside Source

$$\boxed{\partial_\mu \partial^\mu h_{ij} = 0}$$

Wave Eq. → Gravitational Waves !

can GW be 'gauged away' ? No !

---

2 *dof* = 2 polarizations

$$h_{ab}(t, \mathbf{x}) = \int_{-\infty}^{\infty} df \int d\hat{n}\, h_{ab}(f, \hat{n}) e^{-2\pi i f(t - \hat{n}\mathbf{x})}$$

transverse plane

(plane wave)

Gravitational Wave Definition

(TT gauge: 6 - 4 = 2 d.o.f.)

1st approach to GWs $$\boxed{h^{0\mu} = 0\,, \qquad h^i_i = 0\,, \qquad \partial_j h_{ij} = 0}$$

Outside Source

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Wave Eq. ⟶ Gravitational Waves !

can GW be 'gauged away' ? No !

---

2 *dof* = 2 polarizations $$h_{ab}(t, \mathbf{x}) = \int_{-\infty}^{\infty} df \int d\hat{n}\, h_{ab}(f, \hat{n}) e^{-2\pi i f (t - \hat{n}\mathbf{x})}$$

transverse plane

(plane wave)

$$h_{ab}(f, \hat{n}) = \sum_{A=+,\mathrm{x}} h_A(f, \hat{n}) \epsilon^{(A)}_{ab}(\hat{n}) = \begin{pmatrix} h_+ & h_x & 0 \\ h_x & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Transverse-Traceless (2 dof)

Gravitational Wave Definition

(TT gauge: 6 - 4 = 2 d.o.f.)

1st approach to GWs

$$\boxed{h^{0\mu} = 0\,, \quad h^i_i = 0\,, \quad \partial_j h_{ij} = 0}$$

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Wave Eq. ⟶ Gravitational Waves !

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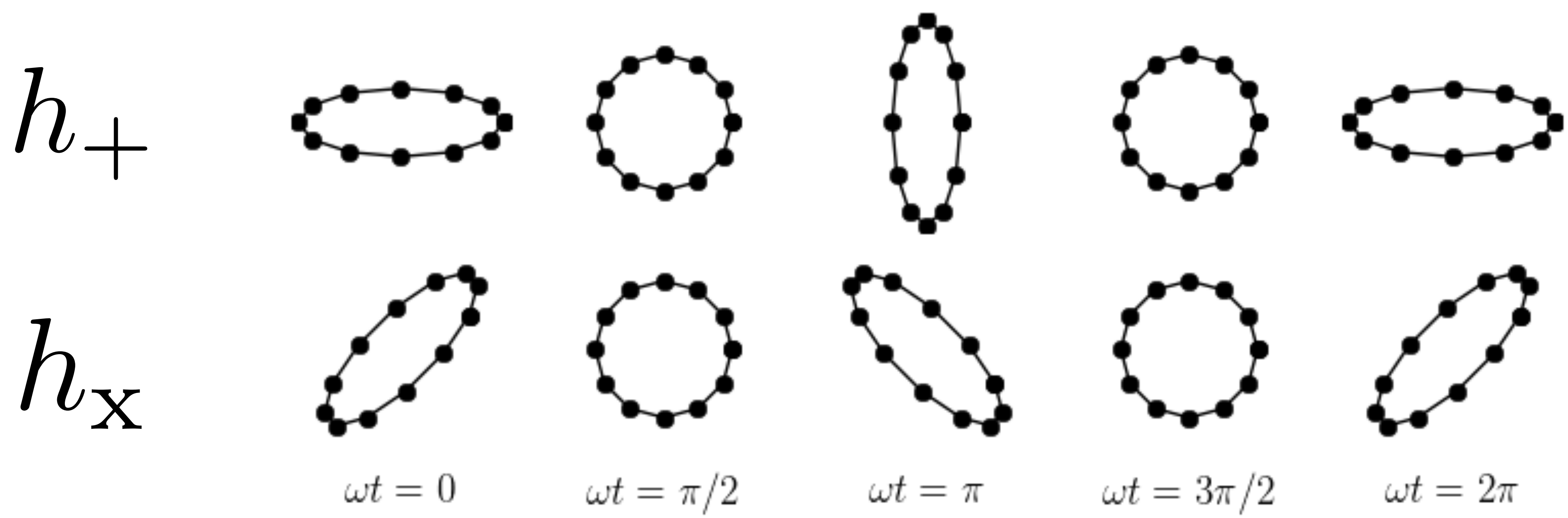

https://www.youtube.com/watch?v=U_hLM1WPDqM

https://www.youtube.com/watch?v=EtL9UyRx_Us

Definition of GWs
2nd approach

Gravitational Wave Definition

2nd approach to GWs

(gauge invariant def.)

Minkowski

$$g_{\mu\nu} = \eta_{\mu\nu} + \delta g_{\mu\nu} \qquad (|\delta g_{\mu\nu}| \ll 1)$$

Gravitational Wave Definition

Minkowski

$$g_{\mu\nu} = \eta_{\mu\nu} + \delta g_{\mu\nu} \qquad (|\delta g_{\mu\nu}| \ll 1)$$

2nd approach to GWs

(gauge invariant def.)

(svt decomposition)

$$\delta g_{00} = -2\phi,$$

$$\delta g_{0i} = \delta g_{i0} = (\partial_i B + S_i)\,,$$

$$\delta g_{ij} = \delta g_{ji} = -2\psi\delta_{ij} + (\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2)E + \partial_i F_j + \partial_j F_i + h_{ij},$$

s: scalar
v: vector
t: tensor

Gravitational Wave Definition

2nd approach to GWs

(gauge invariant def.)

Minkowski

$$g_{\mu\nu} = \eta_{\mu\nu} + \delta g_{\mu\nu} \qquad (|\delta g_{\mu\nu}| \ll 1)$$

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$$T_{00} = \rho,$$

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Gravitational Wave Definition

(svt metric perturbations)

$$\delta g_{00} = -2\phi,$$

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(svt E/p-tensor components)

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	$\delta g_{\mu\nu}$	$T_{\mu\nu}$
Scalar(s) $\in \Re^3$	ϕ, B, ψ, E	ρ, u, p, σ
Vector(s) $\in \Re^3$	S_i, F_i	u_i, v_i
Tensor(s) $\in \Re^3$	h_{ij}	Π_{ij}

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16 degrees of freedom

(svt E/p-tensor components)

$$T_{00} = \rho,$$

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16 degrees of freedom

	$\delta g_{\mu\nu}$	$T_{\mu\nu}$
Scalar(s) $\in \Re^3$	$\phi,\, B,\, \psi,\, E$	$\rho,\, u,\, p,\, \sigma$
Vector(s) $\in \Re^3$	$S_i,\, F_i$	$u_i,\, v_i$
Tensor(s) $\in \Re^3$	h_{ij}	Π_{ij}

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16 degrees of freedom

In order NOT to over-count degrees of freedom

Gravitational Wave Definition

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16 degrees of freedom

In order NOT to over-count degrees of freedom

$$\left.\begin{array}{ll} \partial_i S_i = 0 \text{ (1 constraint)}, & \partial_i F_i = 0 \text{ (1 constraint)}, \\ \partial_i h_{ij} = 0 \text{ (3 constraints)}, & h_{ii} = 0 \text{ (1 constraint)} \end{array}\right\}$$

Metric perturbations

Gravitational Wave Definition

(svt metric perturbations)

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16 degrees of freedom

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16 degrees of freedom

In order NOT to over-count degrees of freedom

$$\left\{\begin{array}{l} \left.\begin{array}{l}\partial_i S_i = 0 \text{ (1 constraint)}, \quad \partial_i F_i = 0 \text{ (1 constraint)}, \\ \partial_i h_{ij} = 0 \text{ (3 constraints)}, \quad h_{ii} = 0 \text{ (1 constraint)}\end{array}\right\} \text{Metric perturbations} \\ \\ \left.\begin{array}{l}\partial_i u_i = 0 \text{ (1 constraint)}, \quad \partial_i v_i = 0 \text{ (1 constraint)}, \\ \partial_i \Pi_{ij} = 0 \text{ (3 constraints)}, \quad \Pi_{ii} = 0 \text{ (1 constraint)},\end{array}\right\} \text{Energy/Momentum tensor} \end{array}\right.$$

Gravitational Wave Definition

(svt metric perturbations)

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6 constraints for metric perturbations

6 constraints for E/p tensor components

Gravitational Wave Definition

(svt metric perturbations)

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~~16~~ 10 degrees of freedom

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~~16~~ 10 degrees of freedom

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6 constraints for metric perturbations

6 constraints for E/p tensor components

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10 degrees of freedom

(svt E/p-tensor components)

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10 degrees of freedom

In order NOT to over-count degrees of freedom

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(svt E/p-tensor components)

$$T_{00} = \rho,$$

$$T_{0i} = T_{i0} = \partial_i u + u_i,$$

$$T_{ij} = T_{ji} = p\,\delta_{ij} + (\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2)\sigma + \partial_i v_j + \partial_j v_i + \Pi_{ij}.$$

10 degrees of freedom

Gravitational Wave Definition

(svt metric perturbations)

$$\delta g_{00} = -2\phi,$$

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Physical Constraints

$$\partial^\mu T_{\mu\nu} = 0$$

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10 degrees of freedom

Physical Constraints

$$\partial^\mu T_{\mu\nu} = 0 \quad \Rightarrow \quad \left\{ \begin{array}{l} \nabla^2 u = \dot{\rho} \;\; (1 \text{ constraint}), \\ \nabla^2 \sigma = \frac{3}{2}(\dot{u} - p) \;\; (1 \text{ constraint}), \\ \nabla^2 v_i = \dot{u}_i \;\; (2 \text{ constraints}). \end{array} \right\}$$

4 constraints (due to E/p conservation)

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Physical Constraints $\quad \partial^\mu G_{\mu\nu} = 0 \;\Rightarrow\; [\ldots]$

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6 degrees of freedom

Physical Symmetry

$$x_\mu \longrightarrow x_\mu + \xi_\mu$$

$$\delta g_{\mu\nu} \longrightarrow \delta g_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$$

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6 degrees of freedom

Physical Symmetry

(4 *d.o.f.* spurious)

$$x_\mu \longrightarrow x_\mu + \xi_\mu$$

$$\delta g_{\mu\nu} \longrightarrow \delta g_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$$

$$\xi_\mu = (\xi_0, \xi_i) \equiv (d_0, \partial_i d + d_i)$$

$$\text{with } \partial_i d_i = 0,$$

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Gauge Invariant !

$$\Phi \equiv -\phi + \dot{B} - \frac{1}{2}\ddot{E},$$

$$\Theta \equiv -2\psi - \frac{1}{3}\nabla^2 E,$$

$$\Sigma_i \equiv S_i - \frac{1}{2}\dot{F}_i,$$

with $\partial_i \Sigma_i = 0$

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$$h_{ij} \equiv h_{ij}\,, \qquad (h_{ii} = \partial_i h_{ij} = 0) \qquad (2)$$

6 gauge invariant degrees of freedom

Gravitational Wave Definition

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6 gauge invariant degrees of freedom

Gauge Invariant Einstein Tensor

$$G_{00} = -\nabla^2\Theta,$$

$$G_{0i} = -\frac{1}{2}\nabla^2\Sigma_i - \partial_i\dot{\Theta},$$

$$G_{ij} = -\frac{1}{2}\Box h_{ij} - \partial_{(i}\dot{\Sigma}_{j)} - \frac{1}{2}\partial_i\partial_j\left(2\Phi + \Theta\right) + \delta_{ij}\left[\frac{1}{2}\nabla^2\left(2\Phi + \Theta\right) - \ddot{\Theta}\right].$$

Gravitational Wave Definition

Gauge Invariant !

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6 gauge invariant degrees of freedom

Gauge Invariant (perturbed) Einstein Eqs.

$$\nabla^2\Theta = -\frac{1}{m_p^2}\rho, \qquad \nabla^2\Phi = \frac{1}{2m_p^2}\left(\rho + 3p - 3\dot{u}\right)$$

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Gravitational Wave Definition

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6 gauge invariant
degrees of freedom

$\Downarrow$

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Gravitational Wave Definition

6 gauge invariant *d.o.f.*

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transverse & traceless (tensor *dof*)

Gravitational Wave Definition

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Only radiative (~ propagating wave Eq.) gauge invariant degrees of freedom !

Gravitational Wave Definition

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Only radiative (~ propagating wave Eq.) gauge invariant degrees of freedom !

Gravitational Waves (GWs) are TT *d.o.f.* metric perturbations, independently of system of reference

Cosmological Backgrounds of Gravitational Waves

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Abstract. Gravitational waves (GWs) have a great potential to probe cosmology. We review early universe sources that can lead to cosmological backgrounds of GWs. We begin by presenting proper definitions of GWs in flat space-time and in a

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)\,, \qquad\qquad |h_{\mu\nu}(x)| \ll 1\,.$$

*Notice that under a Lorentz transformation $x'_\mu = \Lambda_\mu{}^\nu x_\nu$, $g'_{\mu\nu}(x') = \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta g_{\alpha\beta}(x)$, preservation of Eq. (3) requires $|\Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta h_{\alpha\beta}(x)| \ll 1$, so that it remains true that $|h'_{\mu\nu}(x')| \ll 1$. Rotations do not spoil the condition $|h_{\mu\nu}(x)| \ll 1$, but boosts could, and therefore must be restricted to those that do not spoil such condition. As $h_{\mu\nu}(x)$ is invariant under constant displacements $x'^\mu \longrightarrow x^\mu + a^\mu$, linearised gravity Eq. (3) is also invariant under Poincaré transformations.

Definition of GWs
3rd approach

Tbc

Gravitational Wave Definition

3rd approach to GWs

(for a FLRW space-time)

$$g_{\mu\nu}(x) = \underbrace{\bar{g}_{\mu\nu}(x)}_{\text{(FLRW)}} + \delta g_{\mu\nu}(x)\,, \quad |\delta g_{\mu\nu}| \ll 1$$

Gravitational Wave Definition

3rd approach to GWs

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$$g_{\mu\nu}(x) = \underbrace{\bar{g}_{\mu\nu}(x)}_{\text{(FLRW)}} + \delta g_{\mu\nu}(x)\,, \quad |\delta g_{\mu\nu}| \ll 1$$

Flat-FLRW:

$$ds^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu \qquad (t \rightarrow \text{Conformal time})$$

$$= a^2(t)(-dt^2 + d\mathbf{x}\cdot d\mathbf{x})$$

$$= a^2(t)\eta_{\mu\nu} dx^\mu dx^\nu$$

Gravitational Wave Definition

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Flat-FLRW: $ds^2 = a^2(t)\eta_{\mu\nu}dx^\mu dx^\nu \quad (t \rightarrow \text{Conformal time})$

Flat-FLRW + GWs : $ds^2 = a^2(t)[\eta_{\mu\nu} + h_{\mu\nu}]dx^\mu dx^\nu$

where $h_{0\mu} = 0\,, \quad h_{ii} = 0\,, \quad \partial_i h_{ij} = 0$ **Transverse-Traceless (TT) *d.o.f.***

Gravitational Wave Definition

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where $$h_{0\mu} = 0\,, \quad h_{ii} = 0\,, \quad \partial_i h_{ij} = 0$$ **Transverse-Traceless (TT)** ***d.o.f.***

Conformal Transf.: $$ds^2 = \tilde{g}_{\mu\nu}(x)dx^\mu dx^\nu = \Omega^2(x)g_{\mu\nu}(x)dx^\mu dx^\nu$$

Gravitational Wave Definition

3rd approach to GWs
(for a FLRW space-time)

$$g_{\mu\nu}(x) = \underbrace{\bar{g}_{\mu\nu}(x)}_{\text{(FLRW)}} + \delta g_{\mu\nu}(x)\,, \quad |\delta g_{\mu\nu}| \ll 1$$

Flat-FLRW: $ds^2 = a^2(t)\eta_{\mu\nu}dx^\mu dx^\nu \quad (t \to \text{Conformal time})$

Flat-FLRW + GWs : $ds^2 = a^2(t)[\eta_{\mu\nu} + h_{\mu\nu}]dx^\mu dx^\nu$

where $h_{0\mu} = 0\,, \quad h_{ii} = 0\,, \quad \partial_i h_{ij} = 0$ **Transverse-Traceless (TT) *d.o.f.***

Conformal Transf.: $$ds^2 = \tilde{g}_{\mu\nu}(x)dx^\mu dx^\nu = \underbrace{\Omega^2(x)}_{a^2(t)}\underbrace{g_{\mu\nu}(x)}_{[\eta_{\mu\nu} + h_{\mu\nu}]}dx^\mu dx^\nu$$

Gravitational Wave Definition

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Gravitational Wave Definition

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Notation: $$\begin{cases} \partial_{(\mu}\xi_{\nu)} \equiv \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \\ \partial_{[\mu}\xi_{\nu]} \equiv \partial_\mu \xi_\nu - \partial_\nu \xi_\mu \end{cases}$$

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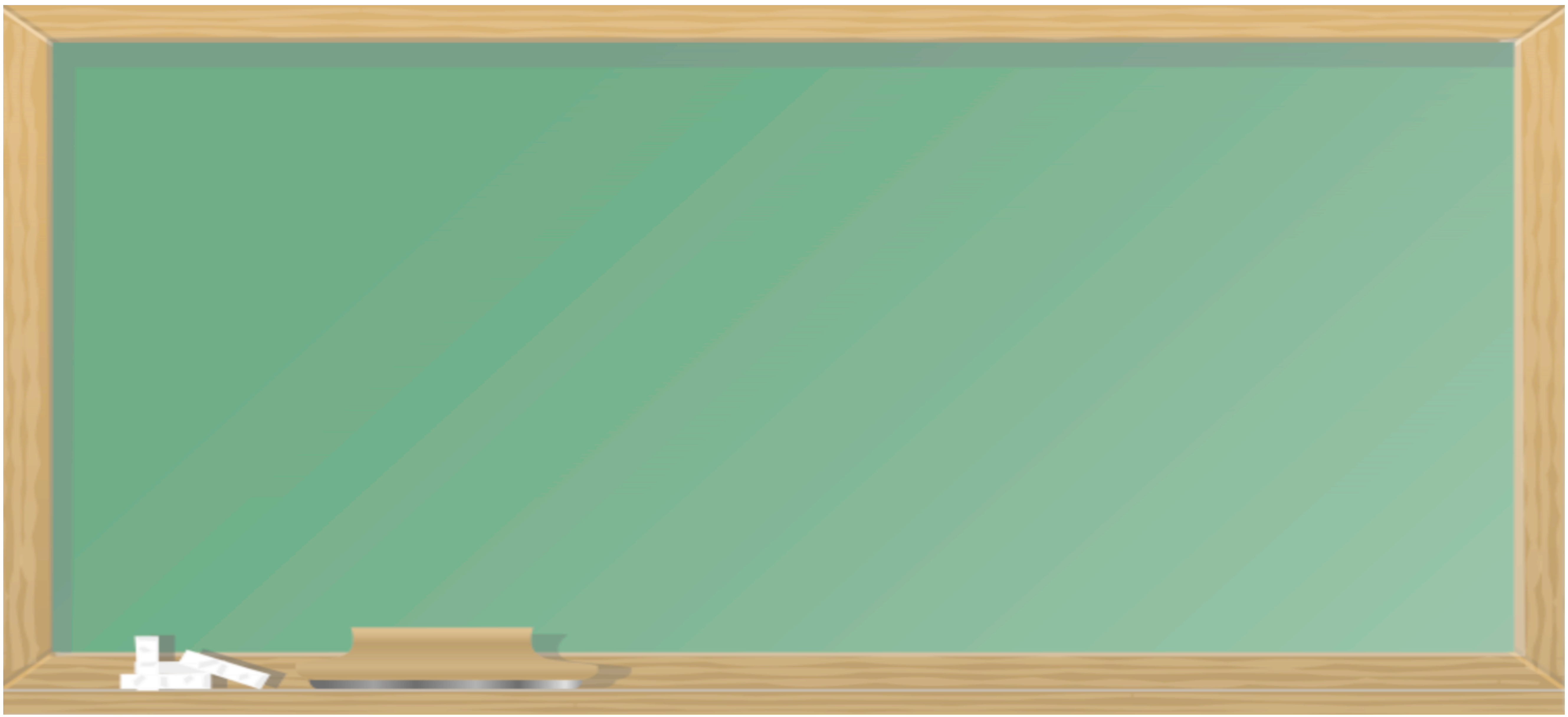

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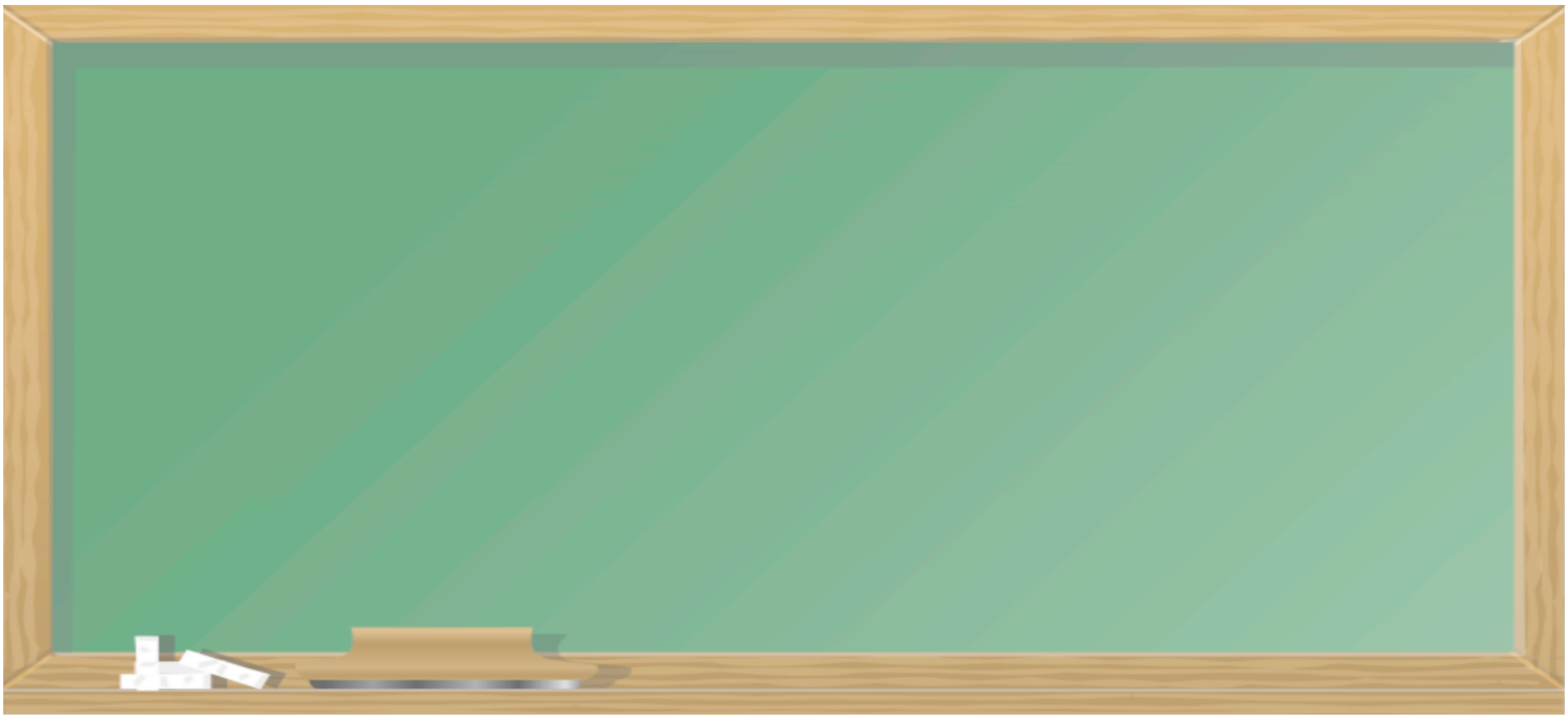

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How does it look $\delta R_{\mu\nu}$ **?**

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$$\boxed{\delta R_{\mu\nu}[g_{**}, \omega] \equiv A\omega_{\mu}\omega_{\mu} + B\omega_{\mu;\nu} + Cg_{\mu\nu}\omega_{\alpha}\omega^{\alpha} + Dg_{\mu\nu}(\omega^{\alpha})_{;\alpha}}$$

(A, B, C, D constants)

It can only take this form !

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After some Calculation… $A = +2\,, B = -2\,, C = -2\,, D = -1$

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$$-2\omega_{\mu;\nu} = 2(\mathscr{H}^2 - a''/a)\delta_{\mu 0}\delta_{\nu 0} + 2\mathscr{H}\Gamma^0_{\mu\nu}[\eta_{**} + h_{**}]$$

$$-2g_{\mu\nu}\omega_\alpha\omega^\alpha = +\,2(\eta_{\mu\nu} + h_{\mu\nu})\mathscr{H}^2$$

$$-g_{\mu\nu}(\omega^\alpha)_{;\alpha} = (\eta_{\mu\nu} + h_{\mu\nu})\big(\mathscr{H}' + \mathscr{H}\Gamma^\alpha_{\alpha 0}[\eta_{**} + h_{**}]\big)$$

$$\Gamma^\alpha_{\mu\nu}[\eta_{**} + h_{**}] \equiv \frac{1}{2}g^{\alpha\beta}\left(\partial_{(\mu}h_{\beta\nu)} - \partial_\beta h_{\mu\nu}\right) \qquad g^{\alpha\beta} \equiv \left(\eta^{\alpha\beta} - h^{\alpha\beta} + h^\alpha{}_\gamma h^{\gamma\beta} + \ldots\right)$$

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu}[\tilde{g}_{**}] \equiv R_{\mu\nu}[\eta_{**} + h_{**}] + \mathscr{D}_{\mu\nu}\omega$$

$$\left[\ \mathscr{D}_{\mu\nu}\omega = 2\omega_{\mu}\omega_{\mu} - 2\omega_{\mu;\nu} - 2g_{\mu\nu}\omega_{\alpha}\omega^{\alpha} - g_{\mu\nu}(\omega^{\alpha})_{;\alpha}\ \right] \quad ; \quad \omega \equiv \log a(t)$$

$$2\omega_{\mu}\omega_{\nu} = 2\mathscr{H}^2\delta_{\mu 0}\delta_{\nu 0}\,, \qquad \mathscr{H} \equiv a'/a$$

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Gravitational Wave Definition

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Gravitational Wave Definition

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$$\Gamma^\alpha_{\mu\nu}[\eta_{**} + h_{**}] \equiv \frac{1}{2}\left(\eta^{\alpha\beta} - h^{\alpha\beta} + h^\alpha{}_\gamma h^{\gamma\beta} + \dots\right)\left(\underbrace{\partial_{(\mu}h_{\beta\nu)} - \partial_\beta h_{\mu\nu}}_{\mathscr{O}(h_{**})}\right) = \overset{(1)}{\Gamma}{}^\alpha_{\mu\nu} + \overset{(2)}{\Gamma}{}^\alpha_{\mu\nu} + \dots$$

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu}[\tilde{g}_{**}] \equiv R_{\mu\nu}[\eta_{**} + h_{**}] + \mathscr{D}_{\mu\nu}\omega$$

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$$\Gamma^\alpha_{\mu\nu}[\eta_{**} + h_{**}] \equiv \overset{(1)}{\Gamma}{}^\alpha_{\mu\nu} + \overset{(2)}{\Gamma}{}^\alpha_{\mu\nu} + \ldots \quad \begin{cases} \overset{(1)}{\Gamma}{}^\alpha_{\mu\nu} \equiv +\dfrac{1}{2}\eta^{\alpha\beta}\left(\partial_{(\mu}h_{\beta\nu)} - \partial_\beta h_{\mu\nu}\right) \\ \overset{(2)}{\Gamma}{}^\alpha_{\mu\nu} \equiv -\dfrac{1}{2}h^{\alpha\beta}\left(\partial_{(\mu}h_{\beta\nu)} - \partial_\beta h_{\mu\nu}\right) \end{cases}$$

Gravitational Wave Definition

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Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv R_{\mu\nu}[\eta_{**} + h_{**}] + (\mathscr{D}_{\mu\nu}\omega)^{(0)} + (\mathscr{D}_{\mu\nu}\omega)^{(1)} + (\mathscr{D}_{\mu\nu}\omega)^{(2)}$$

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$$\left[\ \mathscr{D}_{\mu\nu}\omega = 2\omega_{\mu}\omega_{\mu} - 2\omega_{\mu;\nu} - 2g_{\mu\nu}\omega_{\alpha}\omega^{\alpha} - g_{\mu\nu}(\omega^{\alpha})_{;\alpha}\ \right] \quad ; \quad \omega \equiv \log a(t)$$

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Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \underbrace{R_{\mu\nu}[\eta_{**} + h_{**}]}_{?} + (\mathscr{D}_{\mu\nu}\omega)^{(0)} + (\mathscr{D}_{\mu\nu}\omega)^{(1)} + (\mathscr{D}_{\mu\nu}\omega)^{(2)}$$

$$\Gamma^{\alpha}_{\mu\nu}[\eta_{**} + h_{**}] \equiv \overset{(1)}{\Gamma}{}^{\alpha}_{\mu\nu} + \overset{(2)}{\Gamma}{}^{\alpha}_{\mu\nu} + \ldots \quad \begin{cases} \overset{(1)}{\Gamma}{}^{\alpha}_{\mu\nu} \equiv +\dfrac{1}{2}\eta^{\alpha\beta}\left(\partial_{(\mu}h_{\beta\nu)} - \partial_{\beta}h_{\mu\nu}\right) \\ \overset{(2)}{\Gamma}{}^{\alpha}_{\mu\nu} \equiv -\dfrac{1}{2}h^{\alpha\beta}\left(\partial_{(\mu}h_{\beta\nu)} - \partial_{\beta}h_{\mu\nu}\right) \end{cases}$$

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$$R_{\mu\nu}[\eta_{**} + h_{**}] \equiv \partial_{[\lambda}\Gamma^{\lambda}_{\mu\nu]} + \Gamma^{\alpha}_{[\alpha\lambda}\Gamma^{\lambda}_{\mu\nu]}$$

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \underbrace{R_{\mu\nu}[\eta_{**} + h_{**}]}_{?} + (\mathscr{D}_{\mu\nu}\omega)^{(0)} + (\mathscr{D}_{\mu\nu}\omega)^{(1)} + (\mathscr{D}_{\mu\nu}\omega)^{(2)}$$

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$$R_{\mu\nu}[\eta_{**} + h_{**}] \equiv \partial_{[\lambda}\Gamma^{\lambda}_{\mu\nu]} + \Gamma^{\alpha}_{[\alpha\lambda}\Gamma^{\lambda}_{\mu\nu]}$$

$$\partial_{[\lambda}(\overset{(1)}{\Gamma}{}^{\lambda}_{\mu\nu]} + \overset{(2)}{\Gamma}{}^{\lambda}_{\mu\nu]} + \ldots) + (\overset{(1)}{\Gamma}{}^{\alpha}_{[\alpha\lambda} + \overset{(2)}{\Gamma}{}^{\alpha}_{[\alpha\lambda} + \ldots)(\overset{(1)}{\Gamma}{}^{\lambda}_{\mu\nu]} + \overset{(2)}{\Gamma}{}^{\lambda}_{\mu\nu]} + \ldots)$$

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \underbrace{R_{\mu\nu}[\eta_{**} + h_{**}]}_{?} + (\mathscr{D}_{\mu\nu}\omega)^{(0)} + (\mathscr{D}_{\mu\nu}\omega)^{(1)} + (\mathscr{D}_{\mu\nu}\omega)^{(2)}$$

$$\Gamma^{\alpha}_{\mu\nu}[\eta_{**} + h_{**}] \equiv \overset{(1)}{\Gamma}{}^{\alpha}_{\mu\nu} + \overset{(2)}{\Gamma}{}^{\alpha}_{\mu\nu} + \ldots \quad \begin{cases} \overset{(1)}{\Gamma}{}^{\alpha}_{\mu\nu} \equiv +\dfrac{1}{2}\eta^{\alpha\beta}\left(\partial_{(\mu}h_{\beta\nu)} - \partial_{\beta}h_{\mu\nu}\right) \\ \overset{(2)}{\Gamma}{}^{\alpha}_{\mu\nu} \equiv -\dfrac{1}{2}h^{\alpha\beta}\left(\partial_{(\mu}h_{\beta\nu)} - \partial_{\beta}h_{\mu\nu}\right) \end{cases}$$

$$R_{\mu\nu}[\eta_{**} + h_{**}] \equiv \partial_{[\lambda}\overset{(1)}{\Gamma}{}^{\lambda}_{\mu\nu]} + \partial_{[\lambda}\overset{(2)}{\Gamma}{}^{\lambda}_{\mu\nu]} + \overset{(1)}{\Gamma}{}^{\alpha}_{[\alpha\lambda}\overset{(1)}{\Gamma}{}^{\lambda}_{\mu\nu]} + \ldots$$

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \underbrace{R_{\mu\nu}[\eta_{**} + h_{**}]}_{?} + (\mathscr{D}_{\mu\nu}\omega)^{(0)} + (\mathscr{D}_{\mu\nu}\omega)^{(1)} + (\mathscr{D}_{\mu\nu}\omega)^{(2)}$$

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$$R_{\mu\nu}[\eta_{**} + h_{**}] \equiv \underbrace{\partial_{[\lambda}\overset{(1)}{\Gamma}{}^{\lambda}_{\mu\nu]}}_{\overset{(1)}{\delta R}_{\mu\nu}} + \underbrace{\partial_{[\lambda}\overset{(2)}{\Gamma}{}^{\lambda}_{\mu\nu]} + \overset{(1)}{\Gamma}{}^{\alpha}_{[\alpha\lambda}\overset{(1)}{\Gamma}{}^{\lambda}_{\mu\nu]}}_{\overset{(2)}{\delta R}_{\mu\nu}} + \dots$$

Gravitational Wave Definition

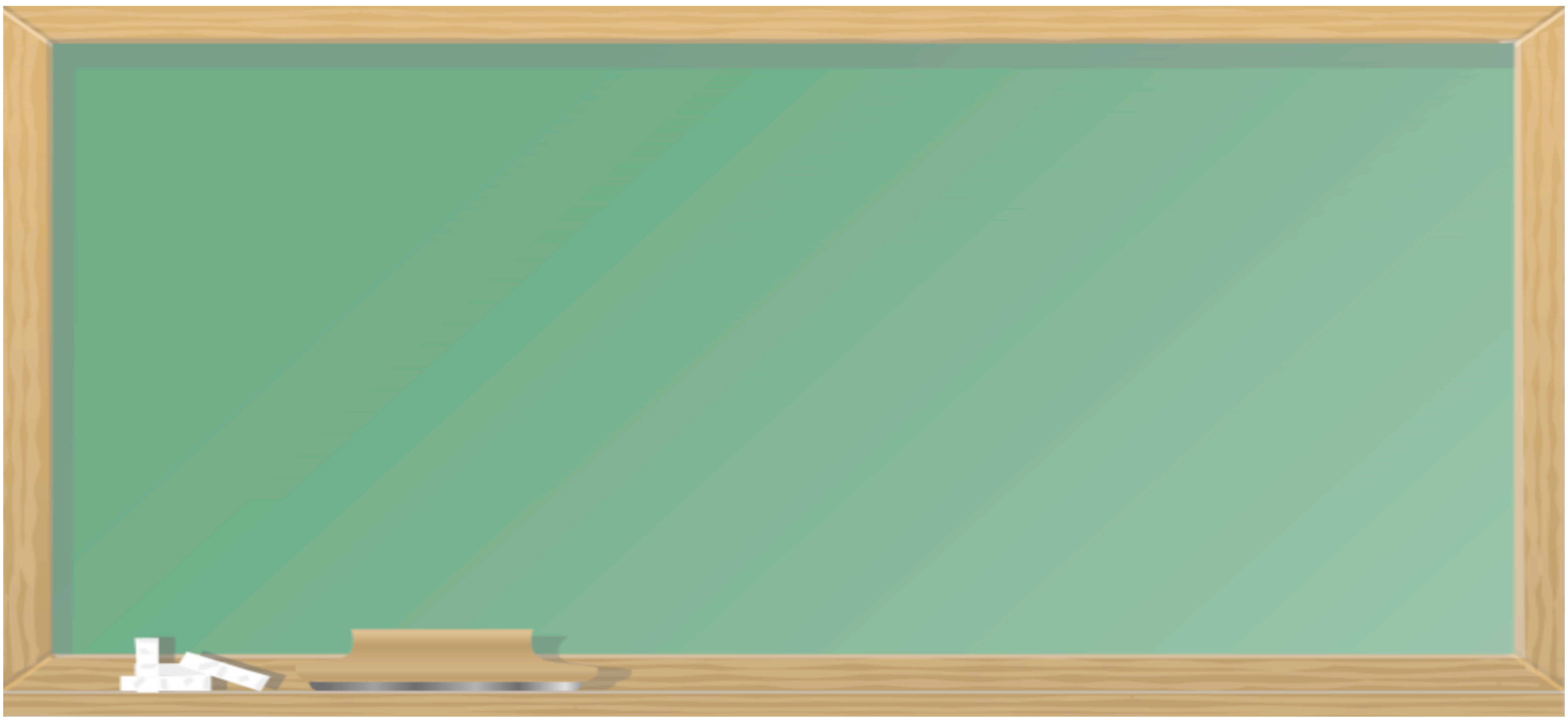
Then: $$\tilde{R}_{\mu\nu} \equiv \underbrace{R_{\mu\nu}[\eta_{**} + h_{**}]}_{0 + \overset{(1)}{\delta R}_{\mu\nu} + \overset{(2)}{\delta R}_{\mu\nu}} + (\mathscr{D}_{\mu\nu}\omega)^{(0)} + (\mathscr{D}_{\mu\nu}\omega)^{(1)} + (\mathscr{D}_{\mu\nu}\omega)^{(2)}$$

$$\Gamma^{\alpha}_{\mu\nu}[\eta_{**} + h_{**}] \equiv \overset{(1)}{\Gamma}{}^{\alpha}_{\mu\nu} + \overset{(2)}{\Gamma}{}^{\alpha}_{\mu\nu} + \ldots \quad \left\{ \begin{array}{l} \overset{(1)}{\Gamma}{}^{\alpha}_{\mu\nu} \equiv +\frac{1}{2}\eta^{\alpha\beta}\left(\partial_{(\mu}h_{\beta\nu)} - \partial_{\beta}h_{\mu\nu}\right) \\ \overset{(2)}{\Gamma}{}^{\alpha}_{\mu\nu} \equiv -\frac{1}{2}h^{\alpha\beta}\left(\partial_{(\mu}h_{\beta\nu)} - \partial_{\beta}h_{\mu\nu}\right) \end{array} \right.$$

$$R_{\mu\nu}[\eta_{**} + h_{**}] \equiv \underbrace{\partial_{[\lambda}\overset{(1)}{\Gamma}{}^{\lambda}_{\mu\nu]}}_{\overset{(1)}{\delta R}_{\mu\nu}} + \underbrace{\partial_{[\lambda}\overset{(2)}{\Gamma}{}^{\lambda}_{\mu\nu]} + \overset{(1)}{\Gamma}{}^{\alpha}_{[\alpha\lambda}\overset{(1)}{\Gamma}{}^{\lambda}_{\mu\nu]}}_{\overset{(2)}{\delta R}_{\mu\nu}} + \ldots$$

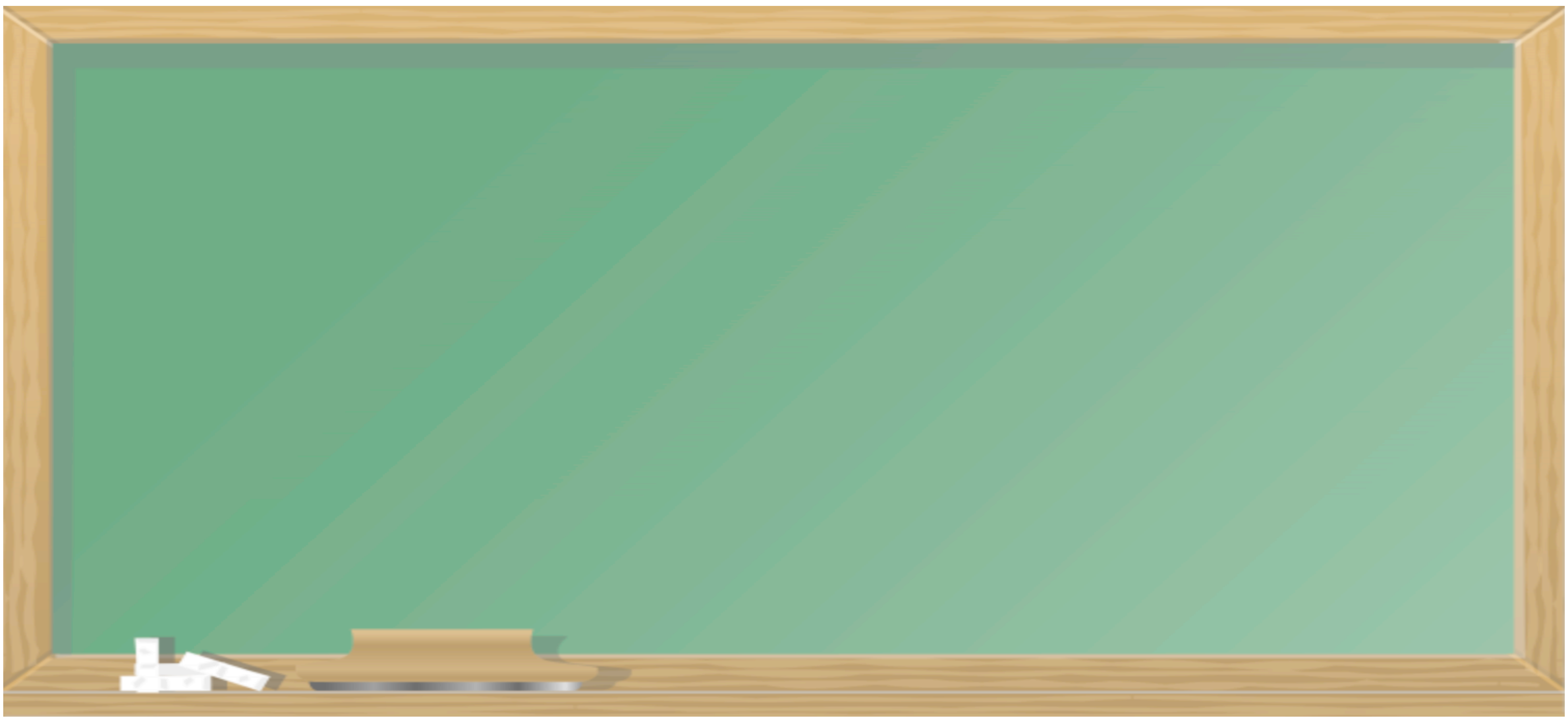
Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \overset{(1)}{\delta R}_{\mu\nu} + \overset{(2)}{\delta R}_{\mu\nu} + (\mathscr{D}_{\mu\nu}\omega)^{(0)} + (\mathscr{D}_{\mu\nu}\omega)^{(1)} + (\mathscr{D}_{\mu\nu}\omega)^{(2)}$$



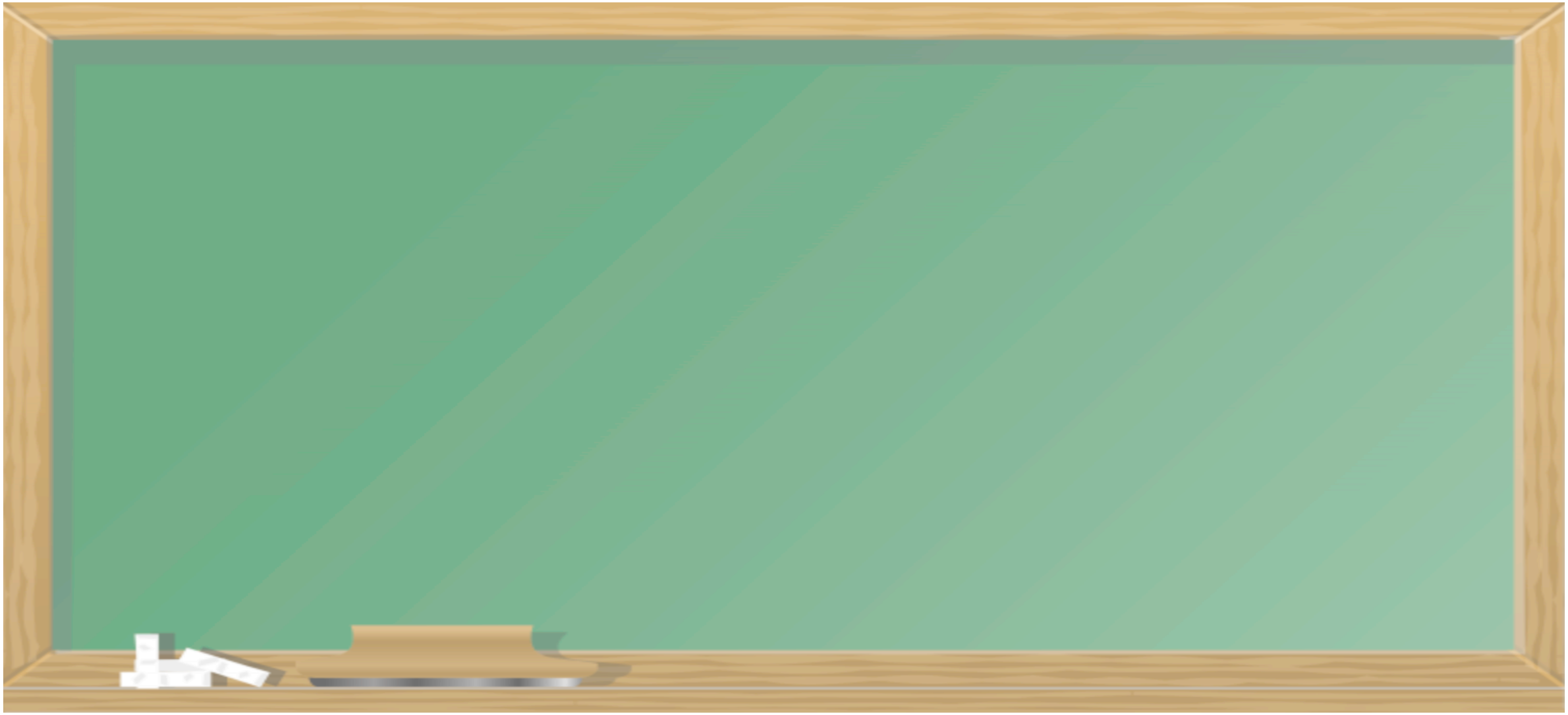
Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv (\mathscr{D}_{\mu\nu}\omega)^{(0)} + \left(\overset{(1)}{\delta R}_{\mu\nu} + (\mathscr{D}_{\mu\nu}\omega)^{(1)} \right) + \left(\overset{(2)}{\delta R}_{\mu\nu} + (\mathscr{D}_{\mu\nu}\omega)^{(2)} \right)$$



Gravitational Wave Definition

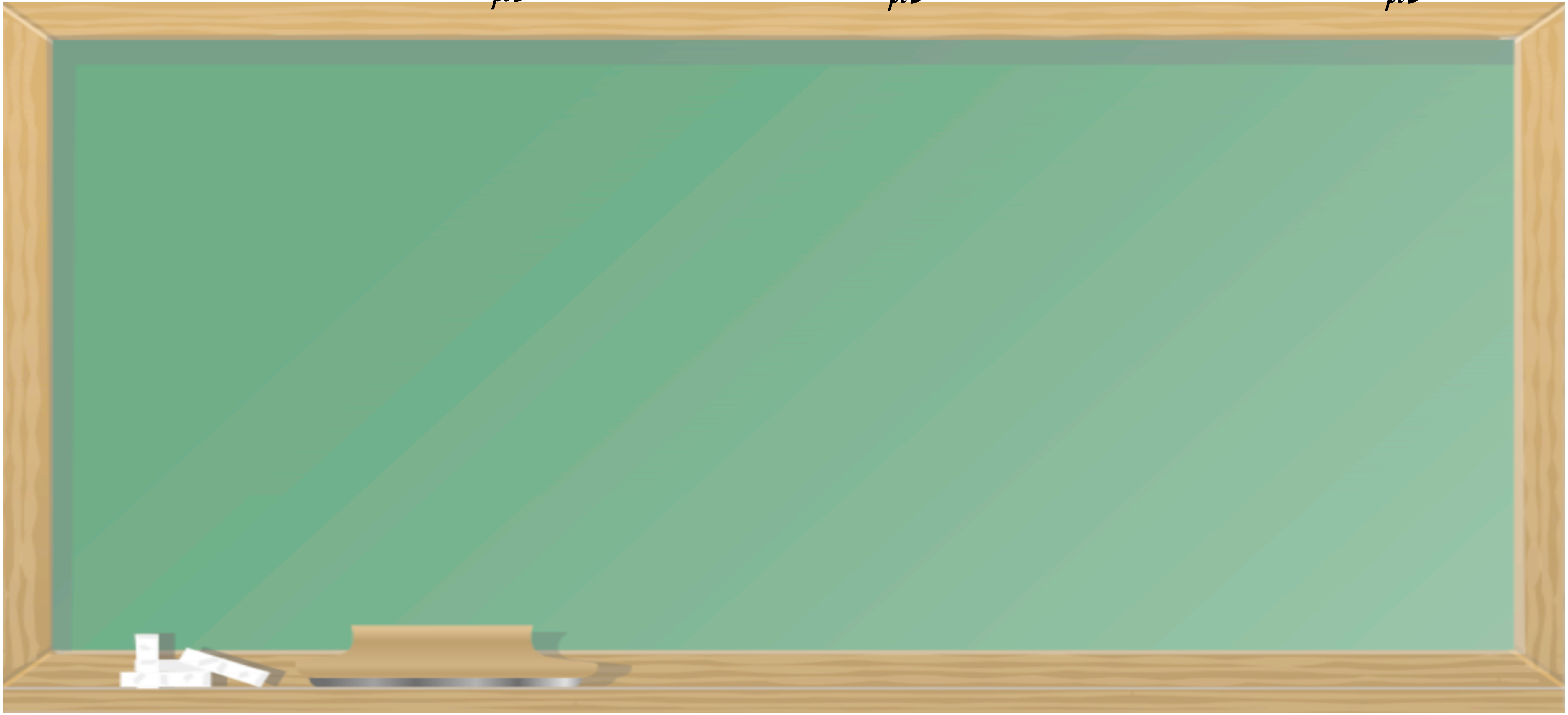
Then: $$\tilde{R}_{\mu\nu} \equiv \underbrace{(\mathscr{D}_{\mu\nu}\omega)^{(0)}}_{\mathcal{O}(h^0_{**})} + \underbrace{\left(\overset{(1)}{\delta R}_{\mu\nu} + (\mathscr{D}_{\mu\nu}\omega)^{(1)}\right)}_{\mathcal{O}(h_{**})} + \underbrace{\left(\overset{(2)}{\delta R}_{\mu\nu} + (\mathscr{D}_{\mu\nu}\omega)^{(2)}\right)}_{\mathcal{O}(h^2_{**})}$$



Gravitational Wave Definition

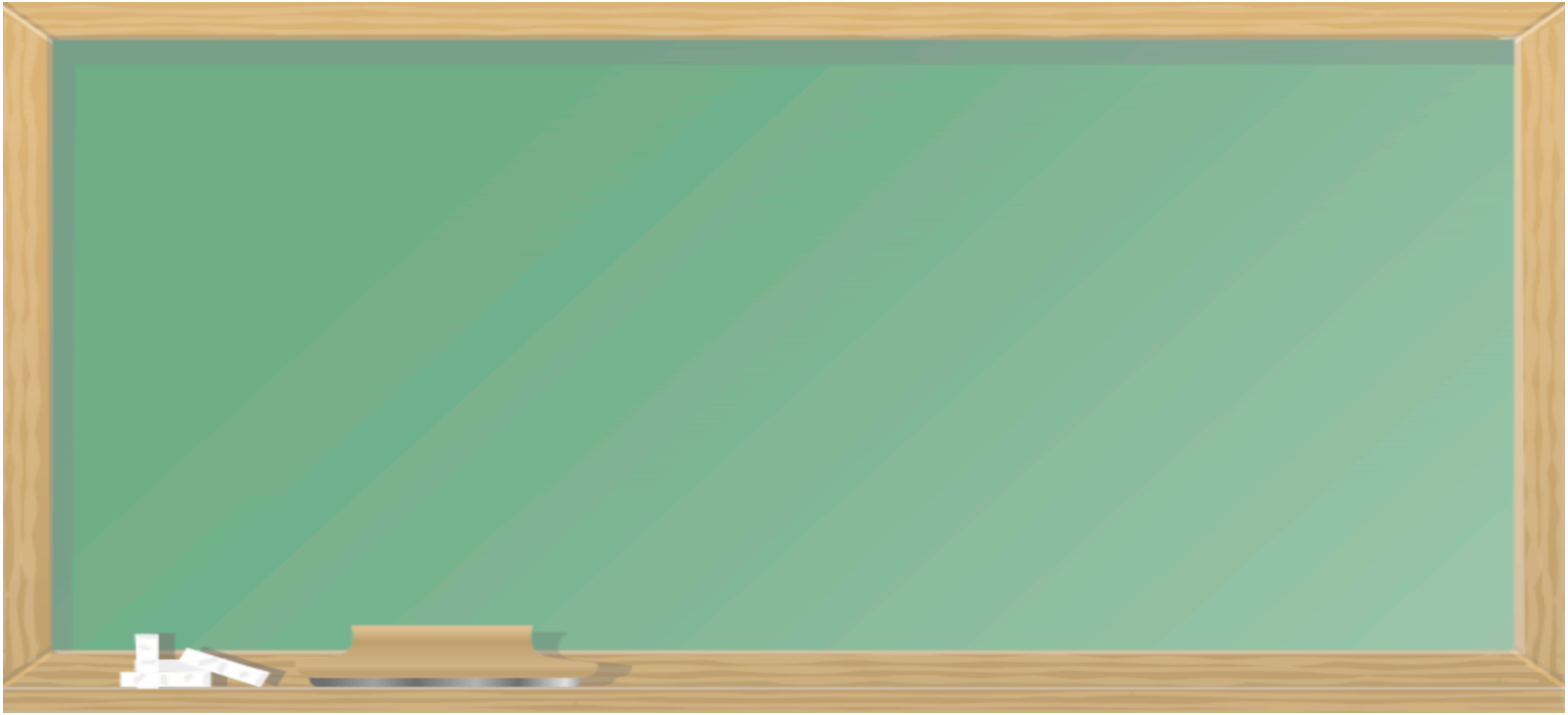
Then:

$$\tilde{R}_{\mu\nu} \equiv \underbrace{(\mathscr{D}_{\mu\nu}\omega)^{(0)}}_{\overset{(0)}{\tilde{R}}_{\mu\nu}} + \underbrace{\left(\overset{(1)}{\delta R}_{\mu\nu} + (\mathscr{D}_{\mu\nu}\omega)^{(1)}\right)}_{\overset{(1)}{\tilde{R}}_{\mu\nu}} + \underbrace{\left(\overset{(2)}{\delta R}_{\mu\nu} + (\mathscr{D}_{\mu\nu}\omega)^{(2)}\right)}_{\overset{(2)}{\tilde{R}}_{\mu\nu}}$$



Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$



Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

$$\overset{(0)}{\tilde{R}}_{\mu\nu} = 2(2\mathscr{H}^2 - a''/a)\delta_{\mu 0}\delta_{\nu 0} + (\mathscr{H}^2 + a''/a)\eta_{\mu\nu}$$ **[Background]**

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

$$\overset{(0)}{\tilde{R}}_{\mu\nu} = 2(2\mathscr{H}^2 - a''/a)\delta_{\mu 0}\delta_{\nu 0} + (\mathscr{H}^2 + a''/a)\eta_{\mu\nu}$$ **[Background]**

$$\overset{(1)}{\tilde{R}}_{\mu\nu} = \delta_{i\mu}\delta_{j\nu}\left(-\frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\partial_\beta h_{ij} + \mathscr{H} h'_{ij} + (\mathscr{H}^2 + a''/a)h_{ij} \right)$$

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

$$\overset{(0)}{\tilde{R}}_{\mu\nu} = 2(2\mathscr{H}^2 - a''/a)\delta_{\mu 0}\delta_{\nu 0} + (\mathscr{H}^2 + a''/a)\eta_{\mu\nu}$$ **[Background]**

$$\overset{(1)}{\tilde{R}}_{\mu\nu} = \delta_{i\mu}\delta_{j\nu}\left(-\frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\partial_\beta h_{ij} + \mathscr{H} h'_{ij} + (\mathscr{H}^2 + a''/a)h_{ij} \right)$$

$$\overset{(2)}{\tilde{R}}_{\mu\nu} = -\frac{1}{2}\mathscr{H}\eta_{\mu\nu}h^{\alpha\beta}h'_{\alpha\beta} + \underbrace{\overset{(2)}{\delta R}_{\mu\nu}}_{\partial_{[\lambda}\overset{(2)}{\Gamma}{}^{\lambda}_{\mu\nu]} + \overset{(1)}{\Gamma}{}^{\alpha}_{[\alpha\lambda}\overset{(1)}{\Gamma}{}^{\lambda}_{\mu\nu]}}$$

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

Let's forget for the moment of second order parts …

$$\overset{(0)}{\tilde{R}}_{\mu\nu} = 2(2\mathscr{H}^2 - a''/a)\delta_{\mu 0}\delta_{\nu 0} + (\mathscr{H}^2 + a''/a)\eta_{\mu\nu}$$ **[Background]**

$$\overset{(1)}{\tilde{R}}_{\mu\nu} = \delta_{i\mu}\delta_{j\nu}\left(-\frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\partial_\beta h_{ij} + \mathscr{H}h'_{ij} + (\mathscr{H}^2 + a''/a)h_{ij}\right)$$

$$\overset{(2)}{\tilde{R}}_{\mu\nu} = -\frac{1}{2}\mathscr{H}\eta_{\mu\nu}h^{\alpha\beta}h'_{\alpha\beta} + \underbrace{\overset{(2)}{\delta R}_{\mu\nu}}_{\partial_{[\lambda}\overset{(2)}{\Gamma}{}^{\lambda}_{\mu\nu]} + \overset{(1)}{\Gamma}{}^{\alpha}_{[\alpha\lambda}\overset{(1)}{\Gamma}{}^{\lambda}_{\mu\nu]}}$$

Gravitational Wave Definition

Then: $\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu}$

Let's forget for the moment of second order parts …

$$\overset{(0)}{\tilde{R}}_{\mu\nu} = 2(2\mathscr{H}^2 - a''/a)\delta_{\mu 0}\delta_{\nu 0} + (\mathscr{H}^2 + a''/a)\eta_{\mu\nu}$$ **[Background]**

$$\overset{(1)}{\tilde{R}}_{\mu\nu} = \delta_{i\mu}\delta_{j\nu}\Big(-\frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\partial_\beta h_{ij} + \mathscr{H} h'_{ij} + (\mathscr{H}^2 + a''/a)h_{ij} \Big)$$

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu}$$

Let's focus on the Einstein Equations

$$\overset{(0)}{\tilde{R}}_{\mu\nu} = 2(2\mathscr{H}^2 - a''/a)\delta_{\mu 0}\delta_{\nu 0} + (\mathscr{H}^2 + a''/a)\eta_{\mu\nu}$$ **[Background]**

$$\overset{(1)}{\tilde{R}}_{\mu\nu} = \delta_{i\mu}\delta_{j\nu}\left(-\frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\partial_\beta h_{ij} + \mathscr{H} h'_{ij} + (\mathscr{H}^2 + a''/a)h_{ij} \right)$$

Gravitational Wave Definition

Then: $\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu}$; $m_p^2 \tilde{R}_{\mu\nu} = S_{\mu\nu}$

$[\ S_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} T_{\alpha\beta}\ ]$

$$\overset{(0)}{\tilde{R}}_{\mu\nu} = 2(2\mathscr{H}^2 - a''/a)\delta_{\mu 0}\delta_{\nu 0} + (\mathscr{H}^2 + a''/a)\eta_{\mu\nu}$$ **[Background]**

$$\overset{(1)}{\tilde{R}}_{\mu\nu} = \delta_{i\mu}\delta_{j\nu}\left(-\frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\partial_\beta h_{ij} + \mathscr{H} h'_{ij} + (\mathscr{H}^2 + a''/a)h_{ij} \right)$$

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} \quad ; \quad m_p^2 \tilde{R}_{\mu\nu} = S_{\mu\nu}$$

$$\overset{(0)}{\tilde{R}}_{\mu\nu} = 2(2\mathscr{H}^2 - a''/a)\delta_{\mu 0}\delta_{\nu 0} + (\mathscr{H}^2 + a''/a)\eta_{\mu\nu}$$ **[Background]**

$$\overset{(1)}{\tilde{R}}_{\mu\nu} = \delta_{i\mu}\delta_{j\nu}\Big(-\frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\partial_\beta h_{ij} + \mathscr{H} h'_{ij} + (\mathscr{H}^2 + a''/a)h_{ij} \Big)$$

Gravitational Wave Definition

Then: $\tilde{R}_{\mu\nu} \equiv \underbrace{\overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu}}$; $\quad m_p^2 \tilde{R}_{\mu\nu} = \underbrace{S_{\mu\nu}}_{= \overset{(0)}{S}_{\mu\nu} + \overset{(1)}{S}_{\mu\nu}}$

$$\overset{(0)}{\tilde{R}}_{\mu\nu} = 2(2\mathscr{H}^2 - a''/a)\delta_{\mu 0}\delta_{\nu 0} + (\mathscr{H}^2 + a''/a)\eta_{\mu\nu}$$ **[Background]**

$$\overset{(1)}{\tilde{R}}_{\mu\nu} = \delta_{i\mu}\delta_{j\nu}\Big(-\frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\partial_\beta h_{ij} + \mathscr{H} h'_{ij} + (\mathscr{H}^2 + a''/a)h_{ij} \Big)$$

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \underbrace{\overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu}} \; ; \qquad m_p^2 \tilde{R}_{\mu\nu} = \underbrace{S_{\mu\nu}}_{= \overset{(0)}{S}_{\mu\nu} + \overset{(1)}{S}_{\mu\nu}} \quad \Rightarrow \quad \begin{cases} m_p^2 \overset{(0)}{\tilde{R}}_{\mu\nu} = \overset{(0)}{S}_{\mu\nu} \\ m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu} \end{cases}$$

$$\overset{(0)}{\tilde{R}}_{\mu\nu} = 2(2\mathscr{H}^2 - a''/a)\delta_{\mu 0}\delta_{\nu 0} + (\mathscr{H}^2 + a''/a)\eta_{\mu\nu}$$ **[Background]**

$$\overset{(1)}{\tilde{R}}_{\mu\nu} = \delta_{i\mu}\delta_{j\nu}\Big(-\frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\partial_\beta h_{ij} + \mathscr{H} h'_{ij} + (\mathscr{H}^2 + a''/a)h_{ij} \Big)$$

Gravitational Wave Definition

Then: $\tilde{R}_{\mu\nu} \equiv \underbrace{\overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu}}$; $m_p^2 \tilde{R}_{\mu\nu} = \underbrace{S_{\mu\nu}}_{= \overset{(0)}{S}_{\mu\nu} + \overset{(1)}{S}_{\mu\nu}}$ $\Rightarrow$ $\begin{cases} m_p^2 \overset{(0)}{\tilde{R}}_{\mu\nu} = \overset{(0)}{S}_{\mu\nu} \\ m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu} \end{cases}$

$$S_{\mu\nu} = \underbrace{(\rho + p)u_\mu u_\nu + \frac{1}{2}(\rho - p)\tilde{g}_{\mu\nu}}_{\textbf{Perfect fluid}} + \underbrace{\Pi_{ij}}_{\textbf{Anisotropic Stress}} \quad ; \quad u_\mu \equiv (a,0,0,0)$$

Gravitational Wave Definition

Then: $\tilde{R}_{\mu\nu} \equiv \underbrace{\overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu}}$; $m_p^2 \tilde{R}_{\mu\nu} = \underbrace{S_{\mu\nu}}_{= \overset{(0)}{S}_{\mu\nu} + \overset{(1)}{S}_{\mu\nu}}$ $\Rightarrow$ $\begin{cases} m_p^2 \overset{(0)}{\tilde{R}}_{\mu\nu} = \overset{(0)}{S}_{\mu\nu} \\ m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu} \end{cases}$

$$
\begin{aligned}
S_{\mu\nu} &= (\rho + p) u_\mu u_\nu + \frac{1}{2}(\rho - p)\tilde{g}_{\mu\nu} + \Pi_{ij} \qquad ; \quad u_\mu \equiv (a,0,0,0) \\
&= \underbrace{(\rho + p) a^2 \delta_{\mu 0}\delta_{\mu 0} + \frac{1}{2}(\rho - p) a^2 \eta_{\mu\nu}}_{\overset{(0)}{S}_{\mu\nu}} + \underbrace{\frac{1}{2}(\rho - p) a^2 h_{\mu\nu} + \Pi_{ij}}_{\overset{(1)}{S}_{\mu\nu}} \\
&= \overset{(0)}{S}_{\mu\nu} + \overset{(1)}{S}_{\mu\nu}
\end{aligned}
$$

Gravitational Wave Definition

Then:

$$\tilde{R}_{\mu\nu} \equiv \underbrace{\overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu}} \quad ; \quad m_p^2 \tilde{R}_{\mu\nu} = \underbrace{S_{\mu\nu}}_{= \overset{(0)}{S}_{\mu\nu} + \overset{(1)}{S}_{\mu\nu}} \quad \Rightarrow \quad \begin{cases} m_p^2 \overset{(0)}{\tilde{R}}_{\mu\nu} = \overset{(0)}{S}_{\mu\nu} \\ m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu} \end{cases}$$

$$S_{\mu\nu} = (\rho + p) u_\mu u_\nu + \frac{1}{2}(\rho - p)\tilde{g}_{\mu\nu} + \Pi_{ij} \qquad ; \quad u_\mu \equiv (a,0,0,0)$$

$$= \underbrace{(\rho + p) a^2 \delta_{\mu 0}\delta_{\mu 0} + \frac{1}{2}(\rho - p) a^2 \eta_{\mu\nu}}_{\overset{(0)}{S}_{\mu\nu}} + \underbrace{\frac{1}{2}(\rho - p) a^2 h_{\mu\nu} + \Pi_{ij}}_{\overset{(1)}{S}_{\mu\nu}}$$

$$= \overset{(0)}{S}_{\mu\nu} + \overset{(1)}{S}_{\mu\nu}$$

$$[\ \Pi_{ij} = \Pi_{ij}^{(S)} + \Pi_{ij}^{(V)} + \Pi_{ij}^{(T)}\ ]$$

Gravitational Wave Definition

Then:

$$\tilde{R}_{\mu\nu} \equiv \underbrace{\overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu}} \; ; \qquad m_p^2 \tilde{R}_{\mu\nu} = \underbrace{S_{\mu\nu}}_{= \overset{(0)}{S}_{\mu\nu} + \overset{(1)}{S}_{\mu\nu}} \quad \Rightarrow \quad \begin{cases} m_p^2 \overset{(0)}{\tilde{R}}_{\mu\nu} = \overset{(0)}{S}_{\mu\nu} \\ m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu} \end{cases}$$

$$m_p^2 \left(\overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} \right) = \underbrace{(\rho + p) a^2 \delta_{\mu 0} \delta_{\mu 0} + \frac{1}{2} (\rho - p) a^2 \eta_{\mu\nu}}_{\overset{(0)}{S}_{\mu\nu}} + \underbrace{\frac{1}{2} (\rho - p) a^2 h_{\mu\nu} + \Pi_{ij}^{(\mathrm{T})}}_{\overset{(1)}{S}_{\mu\nu}}$$

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \underbrace{\overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu}} \; ; \qquad m_p^2 \tilde{R}_{\mu\nu} = \underbrace{S_{\mu\nu}}_{= \overset{(0)}{S}_{\mu\nu} + \overset{(1)}{S}_{\mu\nu}} \quad \Rightarrow \quad \begin{cases} m_p^2 \overset{(0)}{\tilde{R}}_{\mu\nu} = \overset{(0)}{S}_{\mu\nu} \\ m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu} \end{cases}$$

Background: $$m_p^2 \overset{(0)}{\tilde{R}}_{\mu\nu} = \overset{(0)}{S}_{\mu\nu}$$

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \underbrace{\overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu}} \; ; \quad m_p^2 \tilde{R}_{\mu\nu} = \underbrace{S_{\mu\nu}}_{= \overset{(0)}{S}_{\mu\nu} + \overset{(1)}{S}_{\mu\nu}} \;\Rightarrow\; \begin{cases} m_p^2 \overset{(0)}{\tilde{R}}_{\mu\nu} = \overset{(0)}{S}_{\mu\nu} \\ m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu} \end{cases}$$

Background: $m_p^2 \overset{(0)}{\tilde{R}}_{\mu\nu} = \overset{(0)}{S}_{\mu\nu}$

$$\begin{cases} (\mu,\nu) = (0,0) : \; (\mathscr{H}^2 - a''/a) = \dfrac{a^2}{6m_p^2}(\rho + 3p) & \text{(I)} \\[2ex] (\mu,\nu) = (i,i) : \; (\mathscr{H}^2 + a''/a) = \dfrac{a^2}{2m_p^2}(\rho - p) & \text{(II)} \end{cases}$$

Gravitational Wave Definition

Then: $\tilde{R}_{\mu\nu} \equiv \underbrace{\overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu}}$; $m_p^2 \tilde{R}_{\mu\nu} = \underbrace{S_{\mu\nu}}_{= \overset{(0)}{S}_{\mu\nu} + \overset{(1)}{S}_{\mu\nu}}$ $\Rightarrow$ $\begin{cases} m_p^2 \overset{(0)}{\tilde{R}}_{\mu\nu} = \overset{(0)}{S}_{\mu\nu} \\ m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu} \end{cases}$

Background: $m_p^2 \overset{(0)}{\tilde{R}}_{\mu\nu} = \overset{(0)}{S}_{\mu\nu}$

$$\begin{cases} \text{(I) + (II) :} \qquad \mathscr{H}^2 = \dfrac{a^2}{3m_p^2}\rho \\[2ex] \text{(II) - (I) :} \qquad \dfrac{a''}{a} = \dfrac{a^2}{6m_p^2}(\rho - 3p) \end{cases}$$

Friedmann Equations !

Gravitational Wave Definition

Then:

$$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} \;;\qquad m_p^2 \tilde{R}_{\mu\nu} = \underbrace{S_{\mu\nu}}_{= \overset{(0)}{S}_{\mu\nu} + \overset{(1)}{S}_{\mu\nu}} \;\Rightarrow\; \begin{cases} m_p^2 \overset{(0)}{\tilde{R}}_{\mu\nu} = \overset{(0)}{S}_{\mu\nu} \\ m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu} \end{cases}$$

First Order: $m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu}$

Gravitational Wave Definition

Then: $\tilde{R}_{\mu\nu} \equiv \underbrace{\overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu}}$; $\quad m_p^2 \tilde{R}_{\mu\nu} = \underbrace{S_{\mu\nu}}_{= \overset{(0)}{S}_{\mu\nu} + \overset{(1)}{S}_{\mu\nu}}$ $\Rightarrow$ $\begin{cases} m_p^2 \overset{(0)}{\tilde{R}}_{\mu\nu} = \overset{(0)}{S}_{\mu\nu} \\ m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu} \end{cases}$

First Order: $m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu}$

$$h''_{ij} - \nabla^2 h_{ij} + 2\mathscr{H} h'_{ij} + 2(\mathscr{H}^2 + a''/a) h_{ij} = \frac{2}{m_p^2} \Pi^{(\mathrm{T})}_{ij} + \frac{a^2(\rho - p)}{m_p^2} h_{ij}$$

Gravitational Wave Definition

Then: $\tilde{R}_{\mu\nu} \equiv \underbrace{\overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu}}$; $m_p^2 \tilde{R}_{\mu\nu} = \underbrace{S_{\mu\nu}}_{= \overset{(0)}{S}_{\mu\nu} + \overset{(1)}{S}_{\mu\nu}}$ $\Rightarrow$ $\begin{cases} m_p^2 \overset{(0)}{\tilde{R}}_{\mu\nu} = \overset{(0)}{S}_{\mu\nu} \\ m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu} \end{cases}$

First Order: $m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu}$

$$\underbrace{h_{ij}'' - \nabla^2 h_{ij} + 2\mathscr{H} h_{ij}'}_{\text{wave operator}} + \underbrace{2(\mathscr{H}^2 + a''/a) h_{ij}}_{\text{mass term?}} = \frac{2}{m_p^2} \Pi_{ij}^{(\mathrm{T})} + \frac{a^2(\rho - p)}{m_p^2} h_{ij}$$

Gravitational Wave Definition

Then: $\tilde{R}_{\mu\nu} \equiv \underbrace{\overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu}}$; $m_p^2 \tilde{R}_{\mu\nu} = \underbrace{S_{\mu\nu}}_{= \overset{(0)}{S}_{\mu\nu} + \overset{(1)}{S}_{\mu\nu}}$ $\Rightarrow$ $\begin{cases} m_p^2 \overset{(0)}{\tilde{R}}_{\mu\nu} = \overset{(0)}{S}_{\mu\nu} \\ m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu} \end{cases}$

First Order: $m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu}$

$$\underbrace{h_{ij}'' - \nabla^2 h_{ij} + 2\mathscr{H} h_{ij}'}_{\text{wave operator}} + \underbrace{\cancel{2(\mathscr{H}^2 + a''/a)h_{ij}}}_{\text{mass term?}} = \frac{2}{m_p^2}\Pi_{ij}^{(\mathrm{T})} + \cancel{\frac{a^2(\rho - p)}{m_p^2} h_{ij}}$$

Gravitational Wave Definition

Then: $\tilde{R}_{\mu\nu} \equiv \underbrace{\overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu}}$; $m_p^2 \tilde{R}_{\mu\nu} = \underbrace{S_{\mu\nu}}_{= \overset{(0)}{S}_{\mu\nu} + \overset{(1)}{S}_{\mu\nu}}$ $\Rightarrow$ $\begin{cases} m_p^2 \overset{(0)}{\tilde{R}}_{\mu\nu} = \overset{(0)}{S}_{\mu\nu} \\ m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu} \end{cases}$

First Order: $m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu}$

$$\boxed{h''_{ij} - \nabla^2 h_{ij} + 2\mathscr{H} h'_{ij} = \frac{2}{m_p^2}\Pi^{(T)}_{ij}}$$

Grav. Wave
Eq. of motion

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \underbrace{\overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu}} \quad ; \quad m_p^2 \tilde{R}_{\mu\nu} = \underbrace{S_{\mu\nu}}_{= \overset{(0)}{S}_{\mu\nu} + \overset{(1)}{S}_{\mu\nu}} \Rightarrow \begin{cases} m_p^2 \overset{(0)}{\tilde{R}}_{\mu\nu} = \overset{(0)}{S}_{\mu\nu} \\ m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu} \end{cases}$$

First Order: $$m_p^2 \overset{(1)}{\tilde{R}}_{\mu\nu} = \overset{(1)}{S}_{\mu\nu}$$

$$\boxed{h''_{ij} - \nabla^2 h_{ij} + 2\mathscr{H} h'_{ij} = \frac{2}{m_p^2}\Pi_{ij}^{(\mathrm{T})}}$$

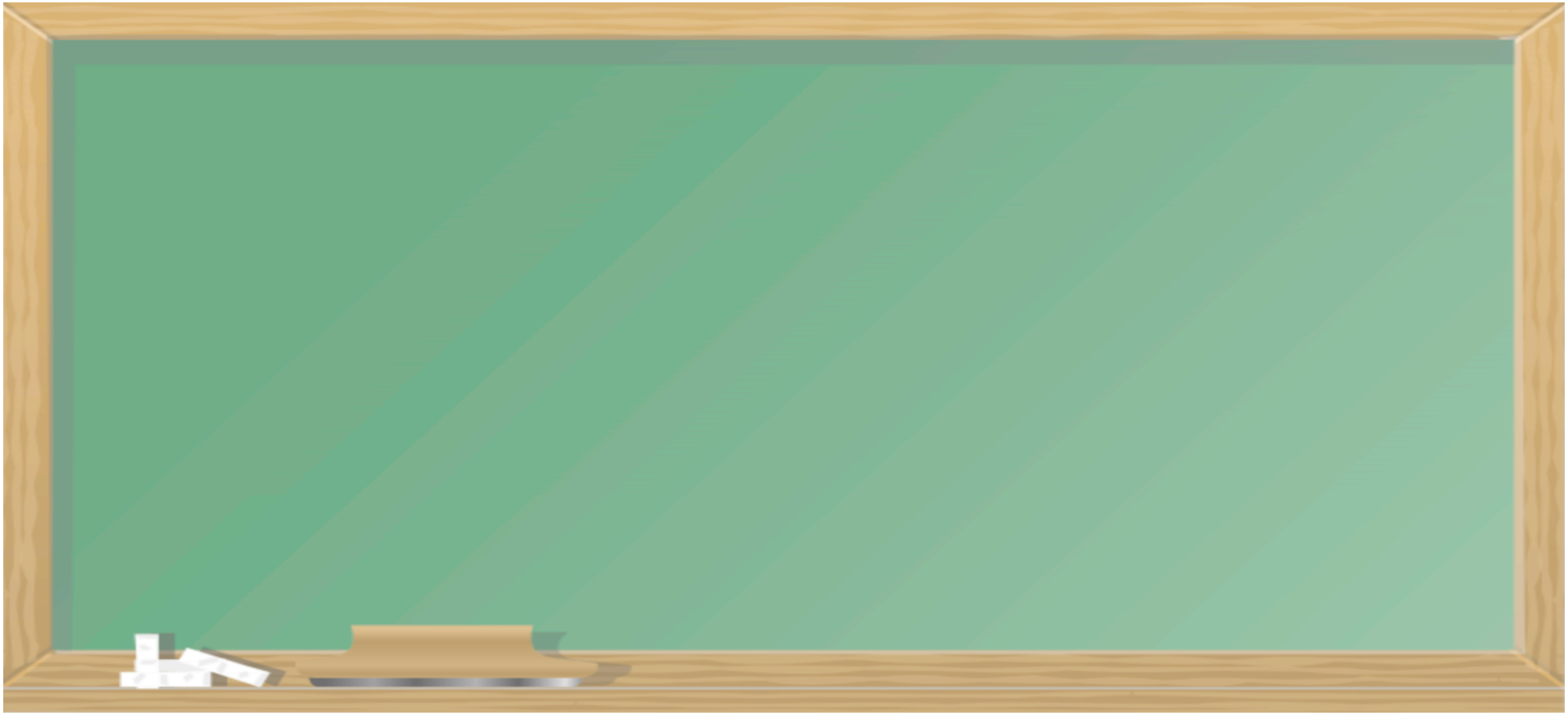
Grav. Wave
Eq. of motion

Friction

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu}$$

[Friedmann Equations] [GW Eq. motion]



Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

[Friedmann Equations] **[GW Eq. motion]** **?**

$$\overset{(2)}{\tilde{R}}_{\mu\nu} = -\frac{1}{2}\mathscr{H}\eta_{\mu\nu}h^{\alpha\beta}h'_{\alpha\beta} + \overset{(2)}{\delta R}_{\mu\nu} \quad ; \quad \overset{(2)}{\delta R}_{\mu\nu} \equiv \partial_{[\lambda}\overset{(2)}{\Gamma^{\lambda}_{\mu\nu]}} + \overset{(1)}{\Gamma^{\alpha}_{[\alpha\lambda}}\overset{(1)}{\Gamma^{\lambda}_{\mu\nu]}}$$

$$\begin{cases} \overset{(1)}{\Gamma^{\alpha}_{\mu\nu}} \equiv +\frac{1}{2}\eta^{\alpha\beta}\left(\partial_{(\mu}h_{\beta\nu)} - \partial_{\beta}h_{\mu\nu}\right) \\ \overset{(2)}{\Gamma^{\alpha}_{\mu\nu}} \equiv -\frac{1}{2}h^{\alpha\beta}\left(\partial_{(\mu}h_{\beta\nu)} - \partial_{\beta}h_{\mu\nu}\right) \end{cases}$$

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

[Friedmann Equations] [GW Eq. motion] ?

$$S_{\mathrm{HE}} = \frac{m_p^2}{2}\int d^4x\sqrt{-\tilde{g}}\,\tilde{R}$$

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

[Friedmann Equations] [GW Eq. motion] ?

$$S_{\mathrm{HE}} = \frac{m_p^2}{2}\int d^4x\sqrt{-\tilde{g}}\,\tilde{R}$$

$$= \frac{m_p^2}{2}\int d^4x\sqrt{-\tilde{g}}\;\tilde{g}^{\mu\nu}\tilde{R}_{\mu\nu}$$

$$= \frac{m_p^2}{2}\int d^4x\;\tilde{f}^{\mu\nu}\tilde{R}_{\mu\nu}$$

Gravitational Wave Definition

Then: $\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$

[Friedmann Equations] **[GW Eq. motion]** **?**

$$S_{\text{HE}} = \frac{m_p^2}{2} \int d^4x \, \tilde{f}^{\mu\nu} \tilde{R}_{\mu\nu}$$

Gravitational Wave Definition

Then:

$$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

[Friedmann Equations] | [GW Eq. motion] | ?

$$S_{\mathrm{HE}} = \frac{m_p^2}{2}\int d^4x\, \tilde{f}^{\mu\nu}\tilde{R}_{\mu\nu}$$

$$\tilde{f}^{\mu\nu} \equiv \sqrt{-\tilde{g}}\;\tilde{g}^{\mu\nu} = a(t)^4\left(1 - \frac{1}{4}h_{\mu\nu}h^{\mu\nu}\right)a^{-2}\left(\eta^{\mu\nu} - h^{\mu\nu} + h^{\mu}{}_{\alpha}h^{\alpha\nu}\right)$$

Gravitational Wave Definition

Then: $\tilde{R}_{\mu\nu} \equiv \underset{\text{[Friedmann Equations]}}{\overset{(0)}{\tilde{R}}_{\mu\nu}} + \underset{\text{[GW Eq. motion]}}{\overset{(1)}{\tilde{R}}_{\mu\nu}} + \underset{\textbf{?}}{\overset{(2)}{\tilde{R}}_{\mu\nu}}$

$$S_{\mathrm{HE}} = \frac{m_p^2}{2}\int d^4x\, \tilde{f}^{\mu\nu}\tilde{R}_{\mu\nu}$$

$$\tilde{f}^{\mu\nu} \equiv \sqrt{-\tilde{g}}\;\tilde{g}^{\mu\nu} = a(t)^4\left(1 - \frac{1}{4}h_{\mu\nu}h^{\mu\nu}\right)a^{-2}\left(\eta^{\mu\nu} - h^{\mu\nu} + h^{\mu}{}_{\alpha}h^{\alpha\nu}\right)$$

$$= \underbrace{a(t)^2\eta^{\mu\nu}}_{\overset{(0)}{\tilde{f}}{}^{\mu\nu}} \underbrace{- a(t)^2h^{\mu\nu}}_{+\overset{(1)}{\tilde{f}}{}^{\mu\nu}} \underbrace{+ h^{\mu}{}_{\alpha}h^{\alpha\nu} - \frac{1}{4}\eta^{\mu\nu}h_{\alpha\beta}h^{\alpha\beta}}_{+\overset{(2)}{\tilde{f}}{}^{\mu\nu}}$$

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

[Friedmann Equations] [GW Eq. motion] ?

$$S_{\mathrm{HE}} = \frac{m_p^2}{2}\int d^4x\, \tilde{f}^{\mu\nu}\tilde{R}_{\mu\nu}$$

$$= \frac{m_p^2}{2}\int d^4x \left(\overset{(0)}{\tilde{f}^{\mu\nu}} + \overset{(1)}{\tilde{f}^{\mu\nu}} + \overset{(2)}{\tilde{f}^{\mu\nu}}\right)\left(\overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}\right)$$

$$= \overset{(0)}{S}_{\mathrm{HE}} + \overset{(1)}{S}_{\mathrm{HE}} + \overset{(2)}{S}_{\mathrm{HE}}$$

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

[Friedmann Equations] [GW Eq. motion] ?

$$S_{\text{HE}} = \overset{(0)}{S}_{\text{HE}} + \overset{(1)}{S}_{\text{HE}} + \overset{(2)}{S}_{\text{HE}}$$

$$\overset{(0)}{S}_{\text{HE}} \equiv \frac{m_p^2}{2} \int d^4x \, \overset{(0)}{\tilde{f}^{\mu\nu}} \overset{(0)}{\tilde{R}}_{\mu\nu}$$

$$\overset{(1)}{S}_{\text{HE}} \equiv \frac{m_p^2}{2} \int d^4x \left(\overset{(0)}{\tilde{f}^{\mu\nu}} \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{f}^{\mu\nu}} \overset{(0)}{\tilde{R}}_{\mu\nu} \right)$$

$$\overset{(2)}{S}_{\text{HE}} \equiv \frac{m_p^2}{2} \int d^4x \left(\overset{(0)}{\tilde{f}^{\mu\nu}} \overset{(2)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{f}^{\mu\nu}} \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{f}^{\mu\nu}} \overset{(0)}{\tilde{R}}_{\mu\nu} \right)$$

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

[Friedmann Equations] [GW Eq. motion] ?

$$S_{\mathrm{HE}} = \overset{(0)}{S}_{\mathrm{HE}} + \overset{(1)}{S}_{\mathrm{HE}} + \overset{(2)}{S}_{\mathrm{HE}}$$

$$\overset{(0)}{S}_{\mathrm{HE}} = 3m_p^2 \int d^4x \; a(t)a''(t)$$

$$\overset{(1)}{S}_{\mathrm{HE}} = 0$$

$$\overset{(2)}{S}_{\mathrm{HE}} \equiv -\frac{m_p^2}{4}\int d^4x\, a^2(t)\left(\frac{1}{2}\eta^{\mu\nu}\partial_\mu h_{ij}\partial_\nu h_{ij} + 2\mathscr{H} h_{ij}h'_{ij} + 3\frac{a''}{a}h_{ij}h_{ij}\right)$$

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

[Friedmann Equations] **[GW Eq. motion]** **?**

$$\overset{(2)}{S}_{\mathrm{HE}} \equiv -\frac{m_p^2}{4}\int d^4x\, a^2(t)\left(\frac{1}{2}\eta^{\mu\nu}\partial_\mu h_{ij}\partial_\nu h_{ij} + 2\mathscr{H} h_{ij}h'_{ij} + 3\frac{a''}{a}h_{ij}h_{ij}\right)$$

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

[Friedmann Equations] [GW Eq. motion] ?

$$\overset{(2)}{S}_{\mathrm{HE}} \equiv -\frac{m_p^2}{4}\int d^4x\, a^2(t)\left(\frac{1}{2}\eta^{\mu\nu}\partial_\mu h_{ij}\partial_\nu h_{ij} + 2\mathscr{H} h_{ij}h'_{ij} + 3\frac{a''}{a}h_{ij}h_{ij}\right)$$

Consistency check: Find Eq.'s of motion of h_{ij}

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

[Friedmann Equations] [GW Eq. motion] ?

$$\overset{(2)}{S}_{\text{HE}} \equiv -\frac{m_p^2}{4}\int d^4x\, a^2(t)\left(\frac{1}{2}\eta^{\mu\nu}\partial_\mu h_{ij}\partial_\nu h_{ij} + 2\mathscr{H} h_{ij}h'_{ij} + 3\frac{a''}{a}h_{ij}h_{ij}\right)$$

$$\delta\overset{(2)}{S}_{\text{HE}} \equiv -\frac{m_p^2}{4}\int d^4x\, a^2\Big(\underbrace{h''_{ij} + 2\mathscr{H} h'_{ij} - \nabla^2 h_{ij}}_{\text{wave operator}} + \underbrace{2(\mathscr{H}' + a''/a)}_{-\frac{2a^2p}{m_p^2}}h_{ij}\Big)\delta h_{ij}$$

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

[Friedmann Equations] [GW Eq. motion] ?

$$\delta \overset{(2)}{S}_{\rm HE} \equiv -\frac{m_p^2}{4}\int d^4x\, a^2\left(h''_{ij} + 2\mathscr{H}h'_{ij} - \nabla^2 h_{ij}\right)\delta h_{ij} + \frac{1}{2}\int d^4x\, a^4\, p\, h_{ij}\delta h_{ij}$$

Gravitational Wave Definition

Then: $\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$

[Friedmann Equations] **[GW Eq. motion]** **?**

$$\delta \overset{(2)}{S}_{\mathrm{HE}} \equiv -\frac{m_p^2}{4}\int d^4x\, a^2\left(h''_{ij} + 2\mathscr{H} h'_{ij} - \nabla^2 h_{ij}\right)\delta h_{ij} + \frac{1}{2}\int d^4x\, a^4\, p\, h_{ij}\delta h_{ij}$$

$$S_{\mathrm{m}} \equiv \int d^4x \sqrt{-\tilde{g}}\, \mathscr{L}_{\mathrm{m}}$$ (**matter sector**)

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

[Friedmann Equations] [GW Eq. motion] ?

$$\delta \overset{(2)}{S}_{\mathrm{HE}} \equiv -\frac{m_p^2}{4}\int d^4x\, a^2\left(h''_{ij} + 2\mathcal{H}h'_{ij} - \nabla^2 h_{ij}\right)\delta h_{ij} + \frac{1}{2}\int d^4x\, a^4\, p\, h_{ij}\delta h_{ij}$$

$$S_{\mathrm{m}} \equiv \int d^4x\sqrt{-\tilde{g}}\,\mathscr{L}_{\mathrm{m}} = \overset{(0)}{S}_{\mathrm{m}} + \overset{(1)}{S}_{\mathrm{m}} + \overset{(2)}{S}_{\mathrm{m}} + \ldots$$

Gravitational Wave Definition

Then: $\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$

[Friedmann Equations] — [GW Eq. motion] — ?

$$\delta \overset{(2)}{S}_{\rm HE} \equiv -\frac{m_p^2}{4}\int d^4x\, a^2\left(h''_{ij} + 2\mathscr{H} h'_{ij} - \nabla^2 h_{ij}\right)\delta h_{ij} + \frac{1}{2}\int d^4x\, a^4\, p\, h_{ij}\delta h_{ij}$$

$$S_{\rm m} \equiv \int d^4x \sqrt{-\tilde{g}}\,\mathscr{L}_{\rm m} = \overset{(0)}{S}_{\rm m} + \overset{(1)}{S}_{\rm m} + \overset{(2)}{S}_{\rm m} + \ldots$$

$$\overset{(2)}{S}_{\rm m} \equiv -\frac{1}{2}\int d^4x \sqrt{-\tilde{g}}\, T_{\mu\nu}\,\delta\tilde{g}^{\mu\nu} - \frac{1}{4}\int d^4x \sqrt{-\tilde{g}}\left(\frac{1}{\sqrt{-\tilde{g}}}\frac{\delta(\sqrt{-\tilde{g}}\,T_{\mu\nu})}{\delta\tilde{g}^{\alpha\beta}}\right)\delta\tilde{g}^{\mu\nu}\delta\tilde{g}^{\alpha\beta}$$

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

[Friedmann Equations] [GW Eq. motion] ?

$$\delta \overset{(2)}{S}_{\mathrm{HE}} \equiv -\frac{m_p^2}{4}\int d^4x\, a^2\left(h_{ij}'' + 2\mathscr{H} h_{ij}' - \nabla^2 h_{ij}\right)\delta h_{ij} + \frac{1}{2}\int d^4x\, a^4\, p\, h_{ij}\delta h_{ij}$$

$$S_{\mathrm{m}} \equiv \int d^4x \sqrt{-\tilde{g}}\, \mathscr{L}_{\mathrm{m}} = \overset{(0)}{S}_{\mathrm{m}} + \overset{(1)}{S}_{\mathrm{m}} + \overset{(2)}{S}_{\mathrm{m}} + \ldots$$

$$\overset{(2)}{S}_{\mathrm{m}} \equiv \underbrace{-\frac{1}{2}\int d^4x \sqrt{-\tilde{g}}\, T_{\mu\nu}\,\delta\tilde{g}^{\mu\nu}}_{\frac{1}{2}\int d^4x\, a^2(t)\, \Pi^{(\mathrm{T})}_{ij}\, h_{ij}} \underbrace{-\frac{1}{4}\int d^4x \sqrt{-\tilde{g}}\left(\frac{1}{\sqrt{-\tilde{g}}}\frac{\delta(\sqrt{-\tilde{g}}\, T_{\mu\nu})}{\delta\tilde{g}^{\alpha\beta}}\right)\delta\tilde{g}^{\mu\nu}\delta\tilde{g}^{\alpha\beta}}_{-\frac{1}{4}\int d^4x\, a^4(t)\, p\, h_{ij}h_{ij}}$$

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

[Friedmann Equations] [GW Eq. motion] ?

$$\delta \overset{(2)}{S}_{\mathrm{HE}} \equiv -\frac{m_p^2}{4}\int d^4x\, a^2\left(h''_{ij} + 2\mathcal{H} h'_{ij} - \nabla^2 h_{ij}\right)\delta h_{ij} + \frac{1}{2}\int d^4x\, a^4\, p\, h_{ij}\delta h_{ij}$$

$$\overset{(2)}{S}_{\mathrm{m}} \equiv \frac{1}{2}\int d^4x\, a^2(t)\,\Pi^{(\mathrm{T})}_{ij}\, h_{ij} - \frac{1}{4}\int d^4x\, a^4(t)\, p\, h_{ij}h_{ij}$$

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

[Friedmann Equations] **[GW Eq. motion]** **?**

$$\delta \overset{(2)}{S}_{\mathrm{HE}} \equiv -\frac{m_p^2}{4}\int d^4x\, a^2\left(h_{ij}'' + 2\mathscr{H} h_{ij}' - \nabla^2 h_{ij}\right)\delta h_{ij} + \frac{1}{2}\int d^4x\, a^4\, p\, h_{ij}\delta h_{ij}$$

$$\delta \overset{(2)}{S}_{\mathrm{m}} = \frac{1}{2}\int d^4x\, a^2(t)\, \Pi_{ij}^{(\mathrm{T})}\, \delta h_{ij} - \frac{1}{2}\int d^4x\, a^4(t)\, p\, h_{ij}\, \delta h_{ij}$$

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

[Friedmann Equations] **[GW Eq. motion]** **?**

$$\overset{(2)}{\delta S}_{\mathrm{HE}} \equiv -\frac{m_p^2}{4}\int d^4x\, a^2\left(h''_{ij} + 2\mathscr{H} h'_{ij} - \nabla^2 h_{ij}\right)\delta h_{ij} + \frac{1}{2}\cancel{\int d^4x\, a^4\, p\, h_{ij}\delta h_{ij}}$$

$$\overset{(2)}{\delta S}_{\mathrm{m}} = \frac{1}{2}\int d^4x\, a^2(t)\, \Pi^{(\mathrm{T})}_{ij}\, \delta h_{ij} - \frac{1}{2}\cancel{\int d^4x\, a^4(t)\, p\, h_{ij}\, \delta h_{ij}}$$

$$\overset{(2)}{\delta S}_{\mathrm{m}} + \overset{(2)}{\delta S}_{\mathrm{HE}} = 0$$

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

[Friedmann Equations] [GW Eq. motion] ?

$$\overset{(2)}{\delta S}_{\mathrm{HE}} \equiv -\frac{m_p^2}{4}\int d^4x\, a^2\left(h''_{ij} + 2\mathscr{H} h'_{ij} - \nabla^2 h_{ij}\right)\delta h_{ij} + \frac{1}{2}\cancel{\int d^4x\, a^4\, p\, h_{ij}\delta h_{ij}}$$

$$\overset{(2)}{\delta S}_{\mathrm{m}} = \frac{1}{2}\int d^4x\, a^2(t)\, \Pi^{(\mathrm{T})}_{ij}\, \delta h_{ij} - \frac{1}{2}\cancel{\int d^4x\, a^4(t)\, p\, h_{ij}\, \delta h_{ij}}$$

$$\overset{(2)}{\delta S}_{\mathrm{m}} + \overset{(2)}{\delta S}_{\mathrm{HE}} = 0 = \int d^4x\, a^2\left[-\frac{m_p^2}{4}\left(h''_{ij} + 2\mathscr{H} h'_{ij} - \nabla^2 h_{ij}\right) + \frac{1}{2}\Pi^{(\mathrm{T})}_{ij}\right]\delta h_{ij}$$

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

[Friedmann Equations] **[GW Eq. motion]** **?**

$$\delta \overset{(2)}{S}_{\rm HE} \equiv -\frac{m_p^2}{4}\int d^4x\, a^2\left(h_{ij}'' + 2\mathscr{H} h_{ij}' - \nabla^2 h_{ij}\right)\delta h_{ij} + \frac{1}{2}\cancel{\int d^4x\, a^4\, p\, h_{ij}\delta h_{ij}}$$

$$\delta \overset{(2)}{S}_{\rm m} = \frac{1}{2}\int d^4x\, a^2(t)\, \Pi_{ij}^{(\rm T)}\, \delta h_{ij} - \frac{1}{2}\cancel{\int d^4x\, a^4(t)\, p\, h_{ij}\, \delta h_{ij}}$$

$$\delta \overset{(2)}{S}_{\rm m} + \delta \overset{(2)}{S}_{\rm HE} = -\frac{m_p^2}{4}\int d^4x\, a^2\Big(\underbrace{h_{ij}'' + 2\mathscr{H} h_{ij}' - \nabla^2 h_{ij}}_{\text{wave operator}} - \underbrace{\frac{2}{m_p^2}\Pi_{ij}^{(\rm T)}}_{\text{Source}}\Big)\delta h_{ij} = 0$$

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

[Friedmann Equations] [GW Eq. motion] ?

$$\overset{(2)}{\delta S}_{\mathrm{HE}} \equiv -\frac{m_p^2}{4}\int d^4x\, a^2\left(h''_{ij} + 2\mathscr{H} h'_{ij} - \nabla^2 h_{ij}\right)\delta h_{ij} + \frac{1}{2}\cancel{\int d^4x\, a^4\, p\, h_{ij}\delta h_{ij}}$$

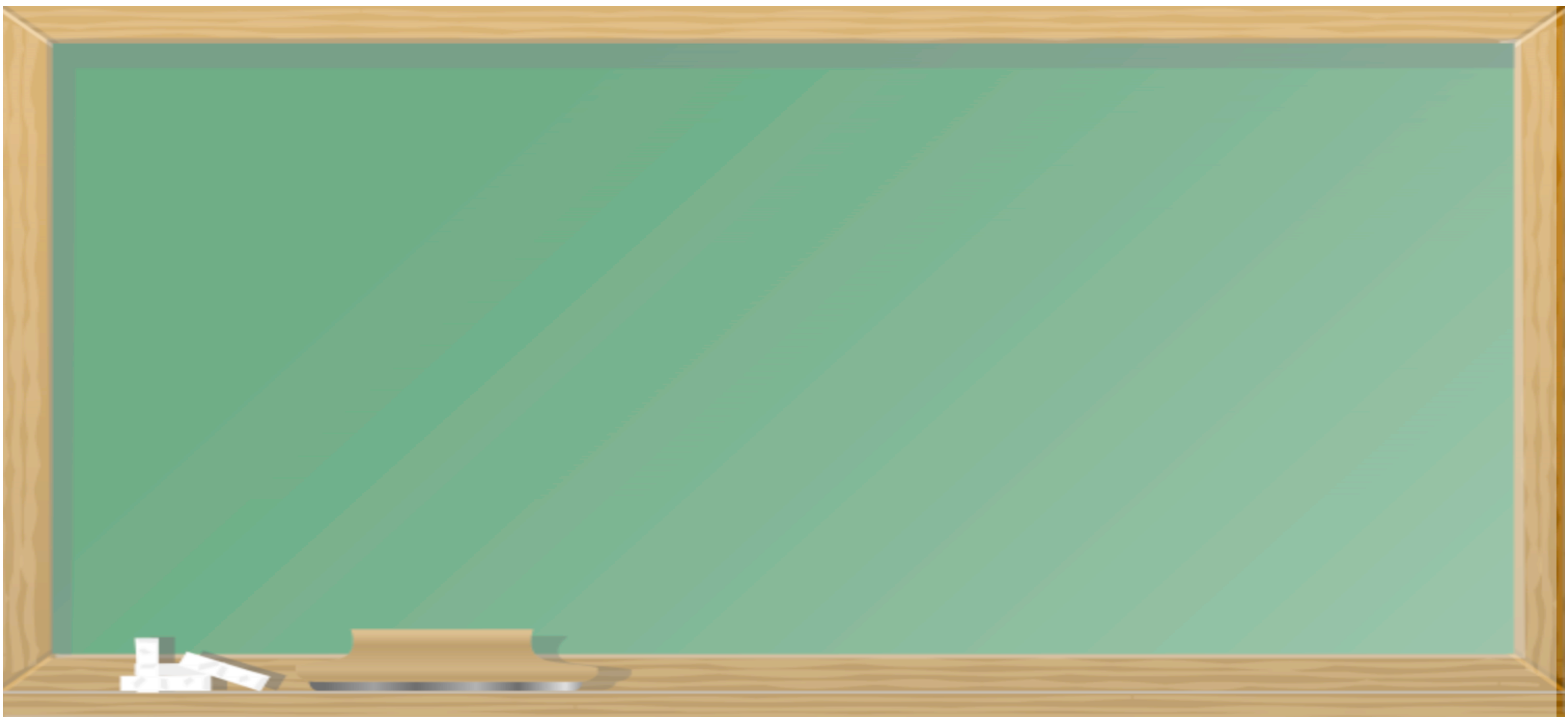
$$\overset{(2)}{\delta S}_{\mathrm{m}} = \frac{1}{2}\int d^4x\, a^2(t)\, \Pi^{(\mathrm{T})}_{ij}\, \delta h_{ij} - \cancel{\frac{1}{2}\int d^4x\, a^4(t)\, p\, h_{ij}\, \delta h_{ij}}$$

$$\overset{(2)}{\delta S}_{\mathrm{m}} + \overset{(2)}{\delta S}_{\mathrm{HE}} = -\frac{m_p^2}{4}\int d^4x\, a^2\underbrace{\left(h''_{ij} + 2\mathscr{H} h'_{ij} - \nabla^2 h_{ij} - \frac{2}{m_p^2}\Pi^{(\mathrm{T})}_{ij}\right)}_{\text{Correct Eq. of motion !}}\delta h_{ij} = 0$$

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

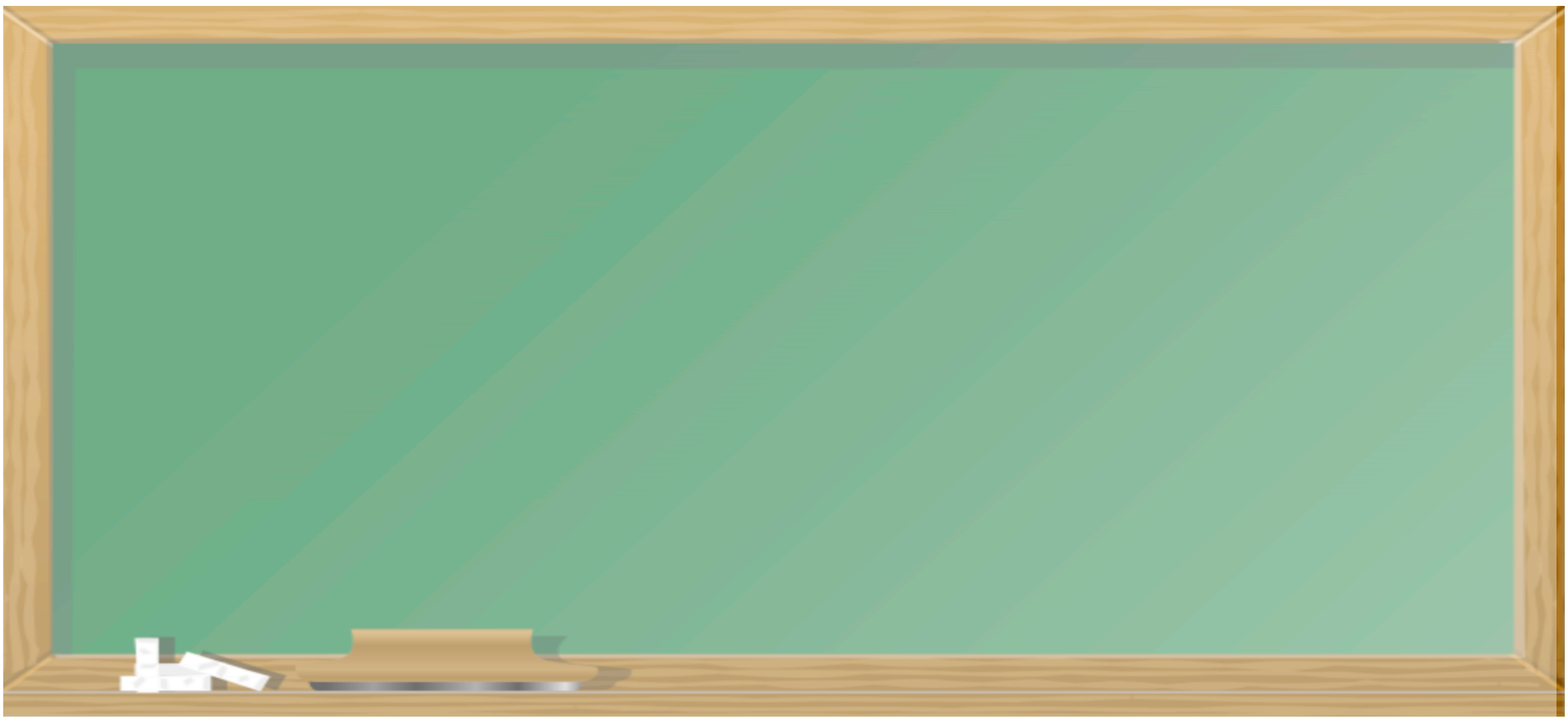
[Friedmann Equations]	[GW Eq. motion]	**?**



Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

[Friedmann Equations] **[GW Eq. motion]** $\mathcal{O}(h_{**}^2) \longrightarrow$ **GW's Energy-momentum ?**



Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

[Friedmann Equations] [GW Eq. motion] $\mathcal{O}(h_{**}^2) \longrightarrow$ **GW's Energy-momentum ?**

$$\overset{(2)}{S}_{\rm tot} \equiv \overset{(2)}{S}_{\rm m} + \overset{(2)}{S}_{\rm HE}$$

Gravitational Wave Definition

Then: $\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$

[Friedmann Equations] **[GW Eq. motion]** $\mathcal{O}(h_{**}^2) \longrightarrow$ **GW's Energy-momentum ?**

$$\overset{(2)}{S}_{\rm tot} \equiv \overset{(2)}{S}_{\rm m} + \overset{(2)}{S}_{\rm HE}$$

$$= -\frac{m_p^2}{4}\int d^4x\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu h_{ij}\partial_\nu h_{ij} + 4\mathscr{H}h_{ij}g^{0\mu}\partial_\mu h_{ij} + \frac{1}{a^2}\left(\mathscr{H}^2 + \frac{a''}{a}\right)h_{ij}h_{ij} - \frac{2}{a^2m_p^2}h_{ij}\Pi_{ij}^{(\rm T)}\right]$$

Gravitational Wave Definition

Then: $\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$

[Friedmann Equations] **[GW Eq. motion]** $\mathcal{O}(h_{**}^2) \longrightarrow$ **GW's Energy-momentum ?**

$$\overset{(2)}{S}_{\text{tot}} \equiv \int d^4x \sqrt{-g}\, \mathscr{L}(h_{ij}, \partial_\mu h_{ij}) \quad ; \quad g_{\mu\nu} \equiv a^2(t)\eta_{\mu\nu} \quad \text{[FLRW]}$$

Gravitational Wave Definition

Then: $\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$

[Friedmann Equations] [GW Eq. motion] $\mathcal{O}(h^2_{**}) \longrightarrow$ **GW's Energy-momentum ?**

$$\overset{(2)}{S}_{\text{tot}} \equiv \int d^4x \sqrt{-g}\, \mathscr{L}(h_{ij}, \partial_\mu h_{ij}) \quad ; \quad g_{\mu\nu} \equiv a^2(t)\eta_{\mu\nu} \quad \text{[FLRW]}$$

Noether's Theorem: $$T_{\mu\nu} \equiv -\frac{\partial \mathscr{L}}{\partial(\partial^\mu h_{ij})}\partial_\nu h_{ij} + g_{\mu\nu}\mathscr{L}$$

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

[Friedmann Equations] **[GW Eq. motion]** $\mathcal{O}(h_{**}^2) \longrightarrow$ **GW's Energy-momentum ?**

$$\overset{(2)}{S}_{\rm tot} \equiv \int d^4x \sqrt{-g}\, \mathcal{L}(h_{ij}, \partial_\mu h_{ij}) \quad ; \quad g_{\mu\nu} \equiv a^2(t)\eta_{\mu\nu} \quad \text{[FLRW]}$$

Noether's Theorem: $$T_{\mu\nu} \equiv -\frac{\partial \mathcal{L}}{\partial(\partial^\mu h_{ij})}\partial_\nu h_{ij} + g_{\mu\nu}\mathcal{L} + \partial_\lambda f^{\lambda\mu\nu}$$

Gravitational Wave Definition

Then: $\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$

[Friedmann Equations] [GW Eq. motion] $\mathcal{O}(h_{**}^2) \longrightarrow$ **GW's Energy-momentum ?**

$$\overset{(2)}{S}_{\text{tot}} \equiv \int d^4x \sqrt{-g}\, \mathscr{L}(h_{ij}, \partial_\mu h_{ij}) \quad ; \quad g_{\mu\nu} \equiv a^2(t)\eta_{\mu\nu} \quad \text{[FLRW]}$$

Noether's Theorem: $T_{\mu\nu} \equiv \left\langle -\dfrac{\partial \mathscr{L}}{\partial(\partial^\mu h_{ij})}\partial_\nu h_{ij} + g_{\mu\nu}\mathscr{L} + \partial_\lambda f^{\lambda\mu\nu} \right\rangle$

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

[Friedmann Equations] **[GW Eq. motion]** $\mathcal{O}(h_{**}^2) \longrightarrow$ **GW's Energy-momentum ?**

$$\overset{(2)}{S}_{\rm tot} \equiv \int d^4x \sqrt{-g}\, \mathscr{L}(h_{ij}, \partial_\mu h_{ij}) \quad ; \quad g_{\mu\nu} \equiv a^2(t)\eta_{\mu\nu} \quad \text{[FLRW]}$$

Noether's Theorem: $$T_{\mu\nu} \equiv \left\langle - \frac{\partial \mathscr{L}}{\partial(\partial^\mu h_{ij})}\partial_\nu h_{ij} + g_{\mu\nu}\mathscr{L} + \cancel{\partial_\lambda f^{\lambda\mu\nu}} \right\rangle$$

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

[Friedmann Equations] [GW Eq. motion] $\mathcal{O}(h_{**}^2) \longrightarrow$ **GW's Energy-momentum ?**

$$\overset{(2)}{S}_{\text{tot}} \equiv \int d^4x \sqrt{-g}\, \mathscr{L}(h_{ij}, \partial_\mu h_{ij}) \quad ; \quad g_{\mu\nu} \equiv a^2(t)\eta_{\mu\nu} \quad \text{[FLRW]}$$

Noether's Theorem: $$\bar{T}_{\mu\nu} \equiv \left\langle -\frac{\partial \mathscr{L}}{\partial(\partial^\mu h_{ij})}\partial_\nu h_{ij} + g_{\mu\nu}\mathscr{L} \right\rangle$$

[Volume averaging over $V \gg \lambda^3$]

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

[Friedmann Equations] [GW Eq. motion] $\mathcal{O}(h_{**}^2) \longrightarrow$ **GW's Energy-momentum ?**

$$\overset{(2)}{S}_{\text{tot}} \equiv \int d^4x \sqrt{-g}\, \mathscr{L}(h_{ij}, \partial_\mu h_{ij}) \quad ; \quad g_{\mu\nu} \equiv a^2(t)\eta_{\mu\nu} \quad \text{[FLRW]}$$

Noether's Theorem: $$\bar{T}_{\mu\nu} \equiv \left\langle -\frac{\partial \mathscr{L}}{\partial(\partial^\mu h_{ij})}\partial_\nu h_{ij} + g_{\mu\nu}\mathscr{L} \right\rangle$$

$$\rho_{\text{GW}} = a^{-2}\bar{T}_{00} \equiv \left\langle \frac{m_p^2}{4}\left(\frac{1}{2a^2}(h'_{ij})^2 + \frac{1}{2a^2}(\nabla h_{ij})^2 + \left(\mathscr{H}^2 + \frac{a''}{a}\right)h_{ij}^2\right) - \frac{1}{2a^2}\Pi_{ij}^{(\mathrm{T})}h_{ij} \right\rangle$$

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

[Friedmann Equations] **[GW Eq. motion]** $\mathcal{O}(h_{**}^2) \longrightarrow$ **GW's Energy-momentum ?**

$$\overset{(2)}{S}_{\text{tot}} \equiv \int d^4x \sqrt{-g}\; \mathcal{L}(h_{ij}, \partial_\mu h_{ij}) \qquad ; \quad g_{\mu\nu} \equiv a^2(t)\eta_{\mu\nu} \quad \text{[FLRW]}$$

Noether's Theorem: $$\bar{T}_{\mu\nu} \equiv \left\langle - \frac{\partial \mathcal{L}}{\partial(\partial^\mu h_{ij})}\partial_\nu h_{ij} + g_{\mu\nu}\mathcal{L} \right\rangle$$

$$\rho_{\text{GW}} = a^{-2}\bar{T}_{00} \equiv \left\langle \frac{m_p^2}{4}\left(\frac{1}{2a^2}(h'_{ij})^2 + \frac{1}{2a^2}(\nabla h_{ij})^2 + \left(\mathcal{H}^2 + \frac{a''}{a}\right)h_{ij}^2\right) - \frac{1}{2a^2}\Pi^{(\mathrm{T})}_{ij}h_{ij} \right\rangle$$

Kinetic **Gradient** **Interaction**

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

[Friedmann Equations] [GW Eq. motion] $\mathcal{O}(h_{**}^2) \longrightarrow$ **GW's Energy-momentum ?**

$$\overset{(2)}{S}_{\rm tot} \equiv \int d^4x \sqrt{-g}\, \mathcal{L}(h_{ij}, \partial_\mu h_{ij}) \quad ; \quad g_{\mu\nu} \equiv a^2(t)\eta_{\mu\nu} \quad \text{[FLRW]}$$

Noether's Theorem: $$\bar{T}_{\mu\nu} \equiv \left\langle -\frac{\partial \mathcal{L}}{\partial(\partial^\mu h_{ij})}\partial_\nu h_{ij} + g_{\mu\nu}\mathcal{L} \right\rangle$$

$$\rho_{\rm GW} = a^{-2}\bar{T}_{00} \equiv \left\langle \frac{m_p^2}{4}\left(\underbrace{\frac{1}{2a^2}(h_{ij}')^2}_{\text{Kinetic}} + \underbrace{\frac{1}{2a^2}(\nabla h_{ij})^2}_{\text{Gradient}} + \underbrace{\left(\mathcal{H}^2 + \frac{a''}{a}\right)h_{ij}^2}_{\text{Caution! differs in the literature}} \right) - \underbrace{\frac{1}{2a^2}\Pi_{ij}^{(\rm T)} h_{ij}}_{\text{Interaction}} \right\rangle$$

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

[Friedmann Equations] **[GW Eq. motion]** $\mathcal{O}(h_{**}^2) \longrightarrow$ **GW's Energy-momentum ?**

$$\rho_{\rm GW} = \left\langle \frac{m_p^2}{4}\left(\frac{1}{2a^2}(h'_{ij})^2 + \frac{1}{2a^2}(\nabla h_{ij})^2 + \left(\mathcal{H}^2 + \frac{a''}{a}\right)h_{ij}^2\right) - \frac{1}{2a^2}\Pi_{ij}^{(\mathrm{T})} h_{ij} \right\rangle$$

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

[Friedmann Equations] **[GW Eq. motion]** $\mathcal{O}(h_{**}^2) \longrightarrow$ **GW's Energy-momentum ?**

$$\rho_{\rm GW} = \left\langle \frac{m_p^2}{4}\left(\frac{1}{2a^2}(h_{ij}')^2 + \frac{1}{2a^2}(\nabla h_{ij})^2 + \left(\mathcal{H}^2 + \frac{a''}{a}\right)h_{ij}^2\right) - \frac{1}{2a^2}\Pi_{ij}^{(\rm T)}h_{ij} \right\rangle$$

Sub-horizon : $(k \gg \mathcal{H})$ $\qquad \sim k^2h^2 \gg \sim \mathcal{H}^2h^2$

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

[Friedmann Equations]

[GW Eq. motion]

$\mathcal{O}(h_{**}^2) \longrightarrow$ **GW's Energy-momentum ?**

$$\rho_{\rm GW} = \left\langle \frac{m_p^2}{4}\left(\frac{1}{2a^2}(h_{ij}')^2 + \frac{1}{2a^2}(\nabla h_{ij})^2 + \cancel{\left(\mathcal{H}^2 + \frac{a''}{a}\right)h_{ij}^2}\right) - \frac{1}{2a^2}\Pi_{ij}^{(\rm T)}h_{ij} \right\rangle$$

Sub-horizon : $\sim k^2h^2 \gg \sim \mathcal{H}^2h^2$

$(k \gg \mathcal{H})$

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

[Friedmann Equations] **[GW Eq. motion]** $\mathcal{O}(h_{**}^2) \longrightarrow$ **GW's Energy-momentum ?**

$$\rho_{\rm GW} = \left\langle \frac{m_p^2}{4}\left(\frac{1}{2a^2}(h'_{ij})^2 + \frac{1}{2a^2}(\nabla h_{ij})^2 + \cancel{\left(\mathcal{H}^2 + \frac{a''}{a}\right)h_{ij}^2}\right) - \cancel{\frac{1}{2a^2}\Pi_{ij}^{(T)}}h_{ij} \right\rangle$$

Sub-horizon : $(k \gg \mathcal{H})$ $\quad \sim k^2h^2 \gg \sim \mathcal{H}^2h^2$

Free fields : (after emission) $\quad \Pi_{ij} \to 0$

Gravitational Wave Definition

Then: $\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$

[Friedmann Equations] [GW Eq. motion] $\mathcal{O}(h_{**}^2) \longrightarrow$ **GW's Energy-momentum ?**

$$\rho_{\rm GW} = \left\langle \frac{m_p^2}{4}\left(\frac{1}{2a^2}(h'_{ij})^2 + \frac{1}{2a^2}(\nabla h_{ij})^2 + \cancel{\left(\mathcal{H}^2 + \frac{a''}{a}\right) h_{ij}^2} \right) - \cancel{\frac{1}{2a^2}\Pi^{(T)}_{ij}} h_{ij} \right\rangle$$

Sub-horizon : $(k \gg \mathcal{H})$ $\quad \sim k^2 h^2 \gg \sim \mathcal{H}^2 h^2$

Free fields : (after emission) $\quad \frac{1}{2a^2}(h'_{ij})^2 = \frac{1}{2a^2}(\nabla h_{ij})^2$

$\Pi_{ij} \to 0$

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

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Energy density carried by Grav. Waves

Sub-horizon $(k \gg \mathcal{H})$ **&** **Free fields** (after emission)

[Volume averaging over $V \gg \lambda^3$ **]**

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[Friedmann Equations]   [GW Eq. motion] $\mathcal{O}(h_{**}^2) \longrightarrow$ **GW's Energy-momentum ?**

$$\boxed{\rho_{\rm GW} = \frac{m_p^2}{4a^2}\left\langle (h'_{ij})^2 \right\rangle}$$

Energy density carried by Grav. Waves

Sub-horizon $(k \gg \mathcal{H})$ **&** **Free fields** (after emission)

[Volume averaging over $V \gg \lambda^3$]

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \overset{(2)}{\tilde{R}}_{\mu\nu}$$

[Friedmann Equations] **[GW Eq. motion]** $\mathcal{O}(h_{**}^2) \longrightarrow$ **GW's Energy-momentum ?**

Energy density carried by Gravitational Waves

$$\boxed{\rho_{\rm GW} = \frac{m_p^2}{4a^2}\Big\langle\, h'_{ij}\, h'_{ij}\,\Big\rangle_{V\gg\lambda^3}}$$

Sub-horizon $(k \gg \mathcal{H})$ **&** **Free fields** (after emission)

Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \left\langle \overset{(2)}{\tilde{R}}_{\mu\nu} \right\rangle$$

[Friedmann Equations]

[GW Eq. motion]

GW energy-momentum over background !

→ How gravity gravitates !

Energy density carried by Gravitational Waves

$$\boxed{\rho_{\rm GW} = \frac{m_p^2}{4a^2} \left\langle h'_{ij} h'_{ij} \right\rangle_{V \gg \lambda^3}}$$

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Gravitational Wave Definition

Then: $$\tilde{R}_{\mu\nu} \equiv \overset{(0)}{\tilde{R}}_{\mu\nu} + \overset{(1)}{\tilde{R}}_{\mu\nu} + \left\langle \overset{(2)}{\tilde{R}}_{\mu\nu} \right\rangle$$

[Friedmann Equations] **[GW Eq. motion]** **GW energy-momentum over background !** → **How gravity gravitates !**

Energy density carried by Gravitational Waves

$$\boxed{\rho_{\rm GW} = \frac{1}{32\pi G}\left\langle \dot{h}_{ij}\dot{h}_{ij} \right\rangle_{V\gg\lambda^3}}$$

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Gravitational Wave Definition

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GW energy-momentum over background !

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Sub-horizon $(k \gg \mathscr{H})$ **&** **Free fields** (after emission)

$$\rho_{\rm GW} = \int d\log f \left(\frac{\partial\rho_{\rm GW}}{d\log f} \right)$$

→ **Energy density Spectrum of Gravitational Waves**

Definition of GWs
4th approach

Gravitational Wave Definition

4th approach to GWs

(for a curved space-time)

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + \delta g_{\mu\nu}(x)\,, \quad |\delta g_{\mu\nu}| \ll 1$$

(separation not well defined)

Gravitational Wave Definition

4th approach to GWs

(for a curved space-time)

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + \delta g_{\mu\nu}(x)\,, \quad |\delta g_{\mu\nu}| \ll 1$$

(separation not well defined)

See e.g. Maggiore's 1st Book on GWs

More subtle problem! Solution: Separation of scales !

Gravitational Wave Definition

4th approach to GWs

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$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + \delta g_{\mu\nu}(x)\,, \quad |\delta g_{\mu\nu}| \ll 1$$

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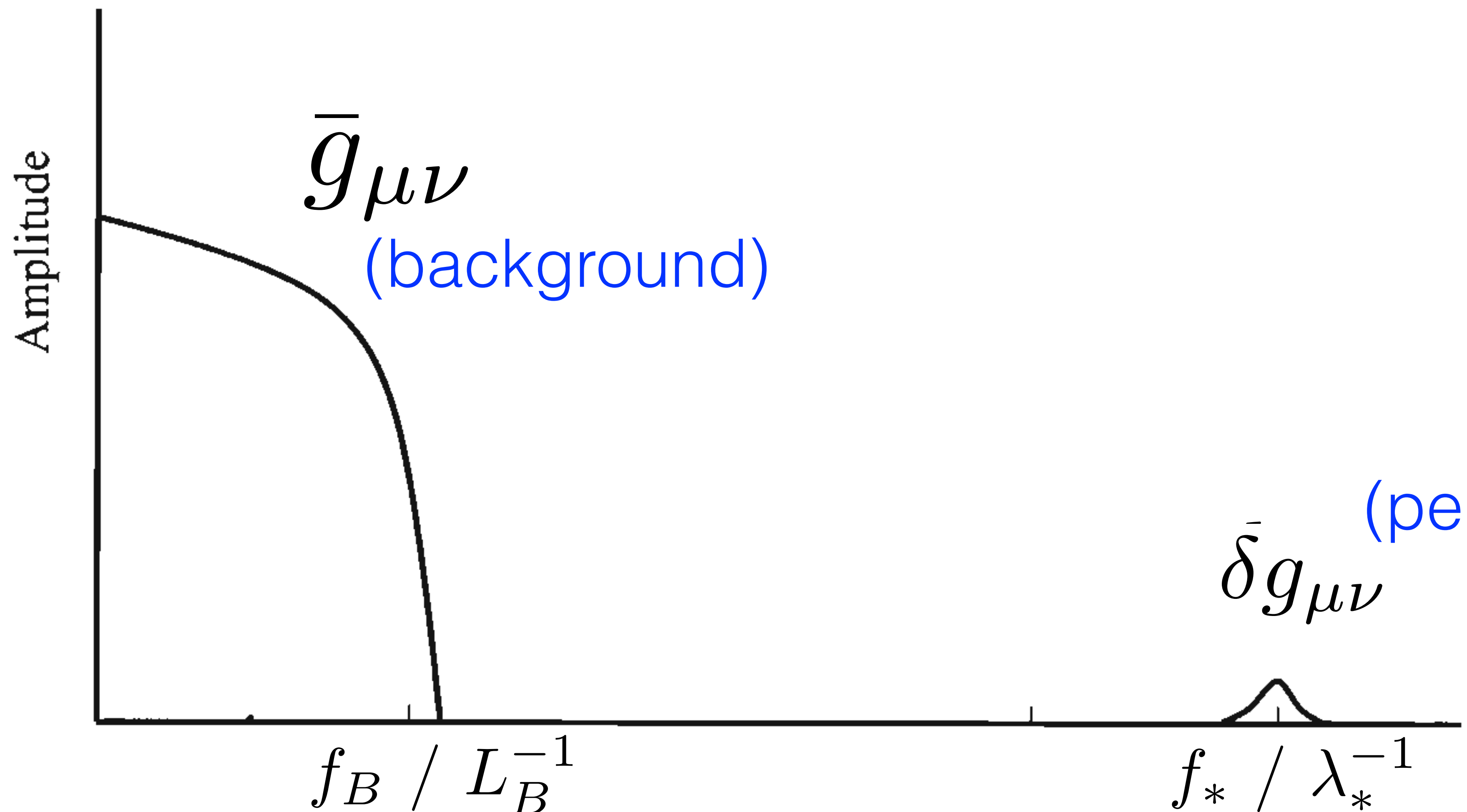

Gravitational Wave Definition

4th approach to GWs

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$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + \delta g_{\mu\nu}(x)\,, \quad |h_{\mu\nu}| \ll 1$$

(separation not well defined)

More subtle problem! Solution: Separation of scales !

$$R_{\mu\nu} = \frac{1}{m_p^2}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)$$

Gravitational Wave Definition

4th approach to GWs

(for a curved space-time)

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + \delta g_{\mu\nu}(x)\,, \quad |h_{\mu\nu}| \ll 1$$

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More subtle problem! Solution: Separation of scales !

$$R_{\mu\nu} = \frac{1}{m_p^2}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) \quad \Rightarrow \quad R_{\mu\nu} = \underbrace{\bar{R}_{\mu\nu}}_{\text{(background)}} + \underbrace{R^{(1)}_{\mu\nu}}_{\mathcal{O}(\delta g)} + \underbrace{R^{(2)}_{\mu\nu}}_{\mathcal{O}(\delta g^2)} + \ldots\,,$$

Gravitational Wave Definition

4th approach to GWs

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$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + \delta g_{\mu\nu}(x)\,, \quad |h_{\mu\nu}| \ll 1$$

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Low Freq. / Long Scale: $\bar{R}_{\mu\nu} = -[R^{(2)}_{\mu\nu}]^{\rm Low} + \frac{1}{m_p^2}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{\rm Low}$

Gravitational Wave Definition

4th approach to GWs

(for a curved space-time)

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + \delta g_{\mu\nu}(x)\,, \quad |h_{\mu\nu}| \ll 1$$

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High Freq. / Short Scale: $$R^{(1)}_{\mu\nu} = -[R^{(2)}_{\mu\nu}]^{\rm High} + \frac{1}{m_p^2}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{\rm High}$$

Gravitational Wave Definition

4th approach to GWs

(for a curved space-time)

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More subtle problem! Solution: Separation of scales !

$$R_{\mu\nu} = \frac{1}{m_p^2}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) \rightarrow R_{\mu\nu} = \underbrace{\bar{R}_{\mu\nu}}_{\text{(background)}} + \underbrace{R^{(1)}_{\mu\nu}}_{\mathcal{O}(h)} + \underbrace{R^{(2)}_{\mu\nu}}_{\mathcal{O}(h^2)} + \ldots\,,$$

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Gravitational Wave Definition

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Gravitational Wave Definition

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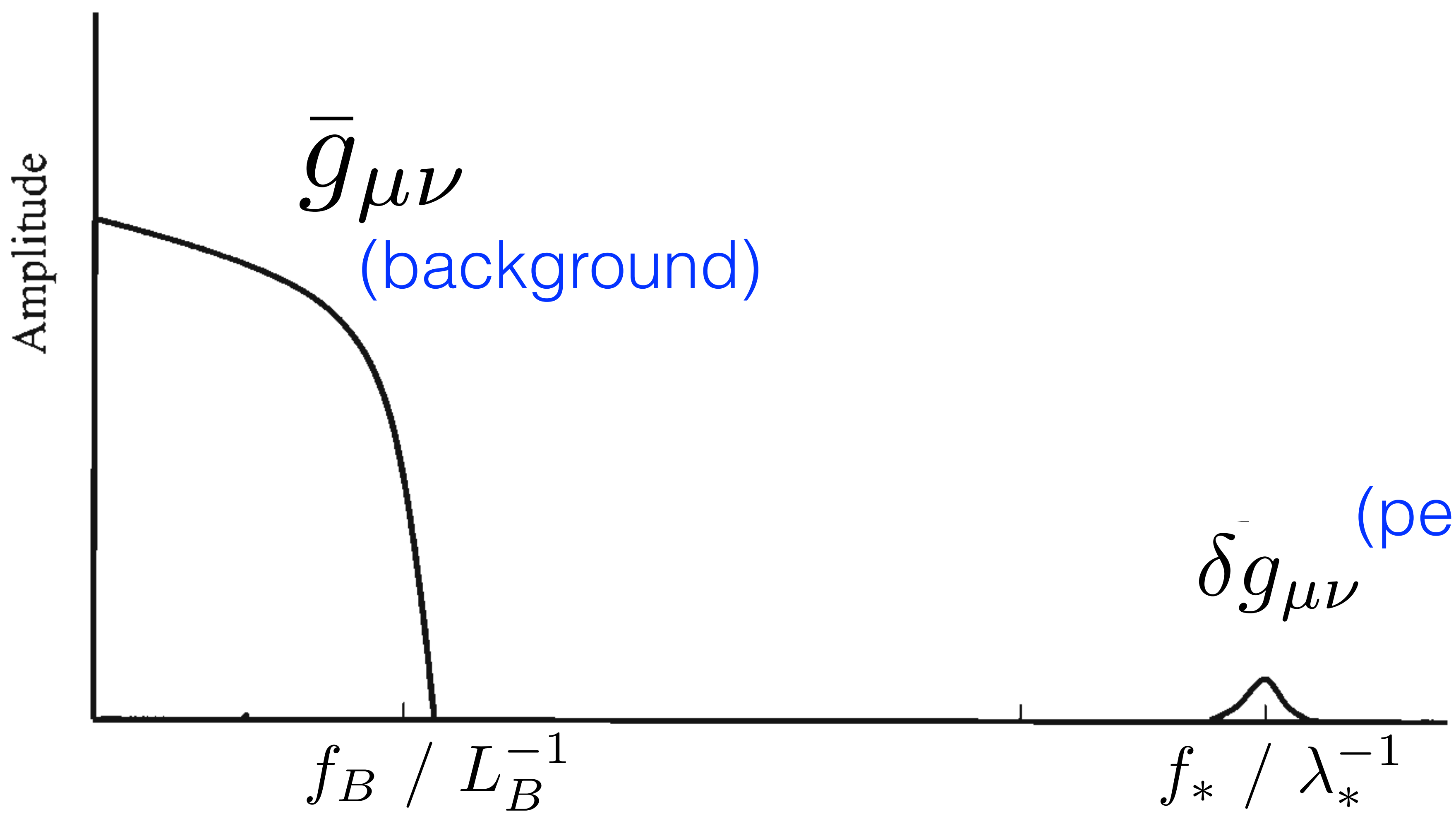
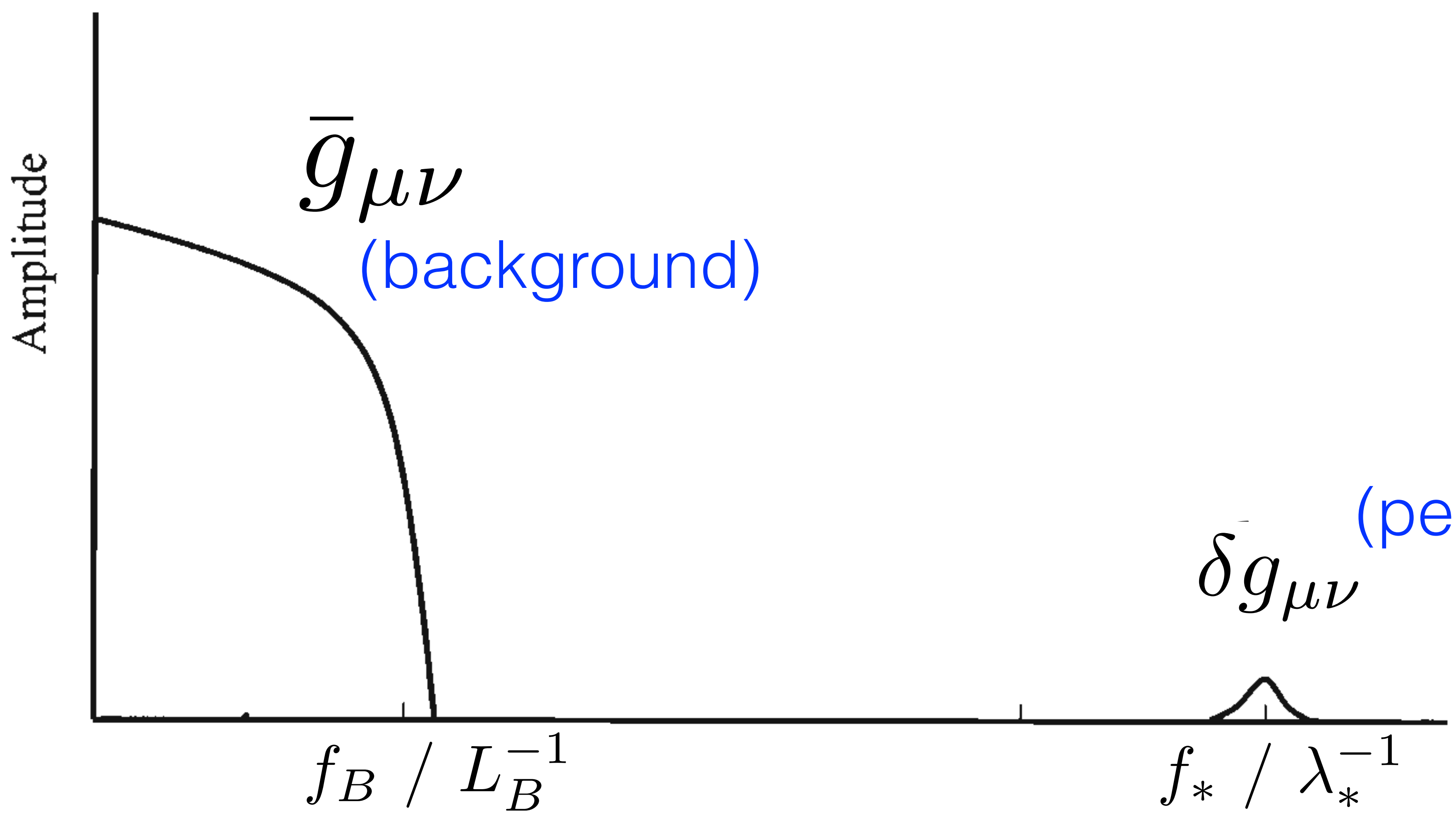


Gravitational Wave Definition

Low Freq. / Long Scale: $$\bar{R}_{\mu\nu} = -\langle R^{(2)}_{\mu\nu} \rangle + \frac{1}{m_p^2} \langle T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \rangle$$

(space/time average)



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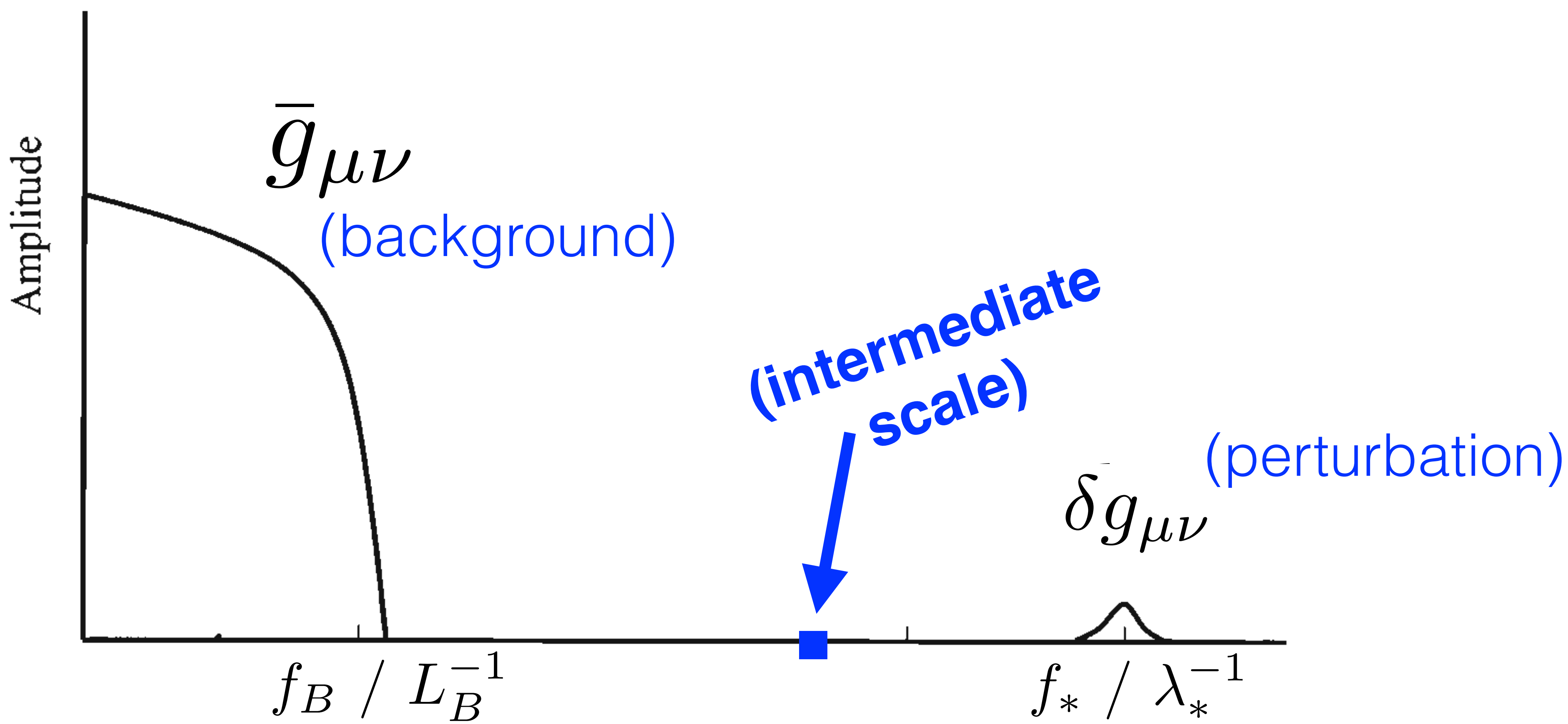
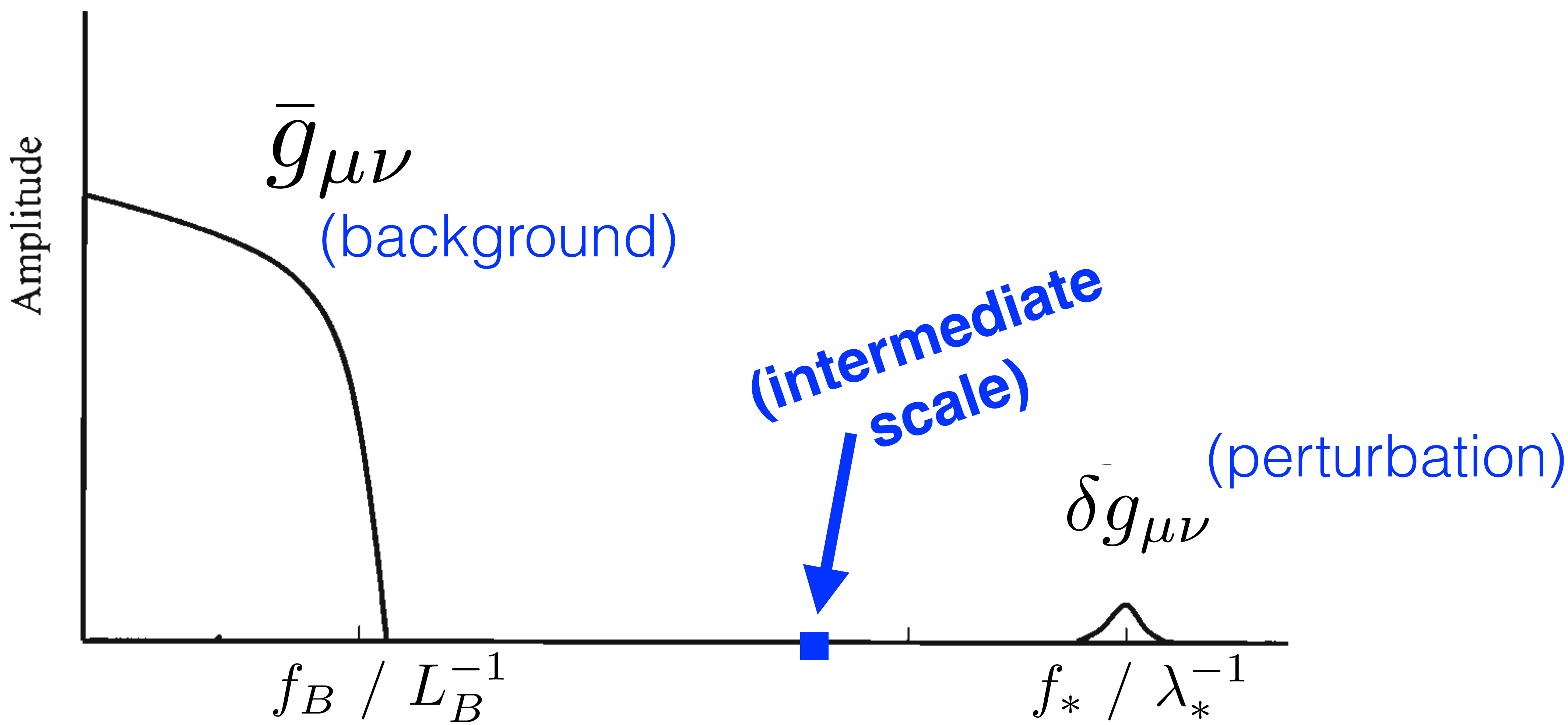


Gravitational Wave Definition

Low Freq. / Long Scale: $$\bar{R}_{\mu\nu} = -\langle R^{(2)}_{\mu\nu} \rangle + \frac{1}{m_p^2} \langle T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \rangle$$ **(space/time average)**

$$t_{\mu\nu} = -\frac{1}{m_p^2} \langle R^{(2)}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} R^{(2)} \rangle$$

$$\langle T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \rangle = \bar{T}^{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{T}$$



Gravitational Wave Definition

Low Freq. / Long Scale:

$$\boxed{\bar{R}_{\mu\nu} = \frac{1}{m_p^2}\left(t_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}t\right) + \frac{1}{m_p^2}\left(\bar{T}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{T}\right)}$$

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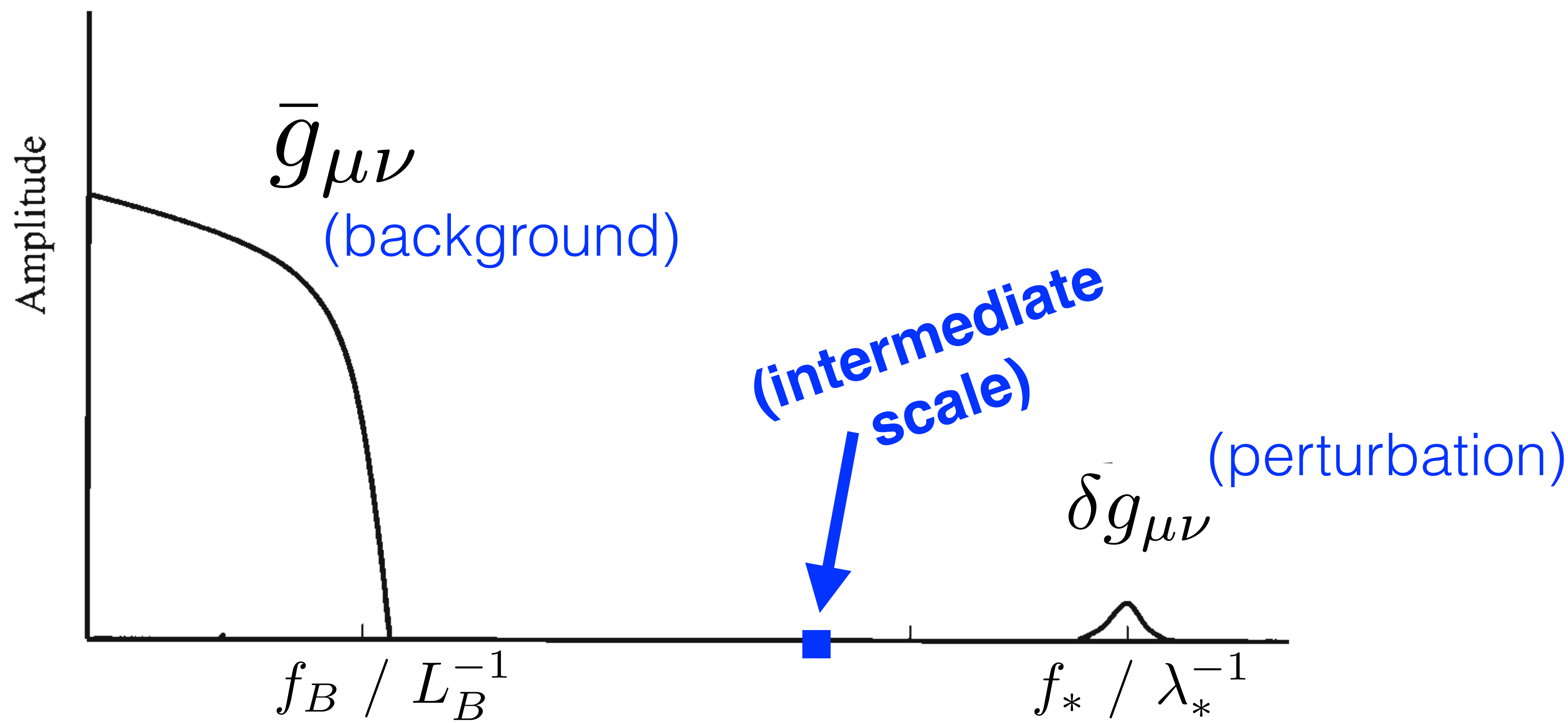

Gravitational Wave Definition

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$$\langle R^{(2)}_{\mu\nu}\rangle = -\frac{1}{4}\langle\partial_\mu\delta g_{\alpha\beta}\,\partial_\nu\delta g^{\alpha\beta}\rangle \Rightarrow t_{\mu\nu} = \frac{m_p^2}{4}\langle\partial_\mu\delta g_{\alpha\beta}\,\partial_\nu\delta g^{\alpha\beta}\rangle$$

Gravitational Wave Definition

Low Freq. / Long Scale:

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It can be shown that only TT *dof* contribute to < … >

Gravitational Wave Definition

Low Freq. / Long Scale:

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$$\boxed{t_{\mu\nu} = \frac{m_p^2}{4}\langle\partial_\mu\delta g^{\rm TT}_{ij}\,\partial_\nu\delta g^{\rm TT}_{ij}\rangle}$$

GW energy-momentum tensor

Gravitational Wave Definition

Low Freq. / Long Scale:

$$\boxed{\bar{R}_{\mu\nu} = \frac{1}{m_p^2}\left(t_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}t\right) + \frac{1}{m_p^2}\left(\bar{T}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{T}\right)}$$

$$t_{\mu\nu} = -\frac{1}{m_p^2}\langle R^{(2)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^{(2)}\rangle \qquad \langle T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\rangle = \bar{T}^{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{T}$$

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It can be shown that only TT *dof* contribute to < … >

$$\boxed{t_{\mu\nu} = \frac{m_p^2}{4}\langle\partial_\mu\delta g_{ij}^{\rm TT}\,\partial_\nu\delta g_{ij}^{\rm TT}\rangle} \;\xrightarrow[(\delta g_{ij}\equiv h_{ij})]{}\; \boxed{\frac{dE}{dAdt} = \frac{m_p^2}{4}\langle\dot{h}_{ij}^{\rm TT}\,\dot{h}_{ij}^{\rm TT}\rangle}$$

GW energy-momentum tensor

GW power/area radiated

Gravitational Wave Definition

Low Freq. / Long Scale: $$\boxed{\bar{R}_{\mu\nu} = \frac{1}{m_p^2}\left(t_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}t\right) + \frac{1}{m_p^2}\left(\bar{T}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{T}\right)}$$

$$t_{\mu\nu} = -\frac{1}{m_p^2}\langle R^{(2)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^{(2)}\rangle \qquad \langle T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\rangle = \bar{T}^{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{T}$$

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It can be shown that only TT *dof* contribute to < … >

$$\boxed{t_{\mu\nu} = \frac{m_p^2}{4}\langle\partial_\mu\delta g^{\rm TT}_{ij}\,\partial_\nu\delta g^{\rm TT}_{ij}\rangle} \;\xrightarrow[(\delta g_{ij}\equiv h_{ij})]{}\; \boxed{\rho_{\rm GW} \equiv \frac{m_p^2}{4}\langle\dot{h}^{\rm TT}_{ij}\,\dot{h}^{\rm TT}_{ij}\rangle}$$

GW energy-momentum tensor

GW energy density

Gravitational Wave Propagation

What about the
High Freq. / Short Scale?

$$R^{(1)}_{\mu\nu} = -[R^{(2)}_{\mu\nu}]^{\mathrm{High}} + \frac{1}{m_p^2}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{\mathrm{High}}$$

Gravitational Wave Propagation

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$$\frac{|R^{(2)}_{\mu}|^{\text{High}}}{|R^{(1)}_{\mu}|} \sim \mathcal{O}\left(\frac{\lambda_*}{L_B}\right) \longrightarrow |R^{(2)}_{\mu}|^{\text{High}}$$ negligible

Gravitational Wave Propagation

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$$R^{(1)}_{\mu\nu} = \bar{g}^{\alpha\beta}\left(D_\alpha D_{(\mu}\delta g_{\nu)\beta} - D_\mu D_\nu \delta g_{\alpha\beta} - D_\alpha D_\beta \delta g_{\mu\nu}\right)$$

$$D_\mu \overline{\delta g}_{\mu\nu} = 0 \quad \left(\overline{\delta g}_{\mu\nu} \equiv \delta g_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}\delta g_{\alpha\beta} \right)$$

Lorentz gauge

Gravitational Wave Propagation

What about the
High Freq. / Short Scale?

$$R^{(1)}_{\mu\nu} = -[R^{(2)}_{\mu\nu}]^{\rm High} + \frac{1}{m_p^2}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{\rm High}$$

$$\frac{|R^{(2)}_{\mu}|^{\rm High}}{|R^{(1)}_{\mu}|} \sim \mathcal{O}\left(\frac{\lambda_*}{L_B}\right) \longrightarrow |R^{(2)}_{\mu}|^{\rm High}$$ negligible

$$R^{(1)}_{\mu\nu} = \bar{g}^{\alpha\beta}\left(D_\alpha D_{(\mu}\delta g_{\nu)\beta} - D_\mu D_\nu \delta g_{\alpha\beta} - D_\alpha D_\beta \delta g_{\mu\nu}\right)$$

$$D_\mu \overline{\delta g}_{\mu\nu} = 0 \quad \left(\overline{\delta g}_{\mu\nu} \equiv \delta g_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}\delta g_{\alpha\beta}\right)$$

Lorentz gauge

vacuum

$$\longrightarrow \boxed{D_\alpha D^\alpha \overline{\delta g}_{\mu\nu} = 0}$$

Propagation of GWs
in curved space-time

Gravitational Wave Propagation

What about the
High Freq. / Short Scale?

$$R^{(1)}_{\mu\nu} = -[R^{(2)}_{\mu\nu}]^{\rm High} + \frac{1}{m_p^2}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{\rm High}$$

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$$D_\mu \overline{\delta g}_{\mu\nu} = 0 \quad \left(\ \overline{\delta g}_{\mu\nu} \equiv \delta g_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}\delta g_{\alpha\beta}\ \right)$$

Lorentz
gauge

$\longrightarrow$

vacuum

$$\boxed{D_\alpha D^\alpha \delta g^{\rm TT}_{ij} = 0}$$

Propagation of GWs
in curved space-time

$\left(\ D_i \delta g^{\rm TT}_{ij} = \bar{g}^{ij}\delta g^{\rm TT}_{ij} = 0\ \right)$

Gravitational Wave Propagation

What about the
High Freq. / Short Scale?

$$R^{(1)}_{\mu\nu} = -[R^{(2)}_{\mu\nu}]^{\rm High} + \frac{1}{m_p^2}\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)^{\rm High}$$

$$\frac{|R^{(2)}_{\mu}|^{\rm High}}{|R^{(1)}_{\mu}|} \sim \mathcal{O}\left(\frac{\lambda_*}{L_B}\right) \longrightarrow |R^{(2)}_{\mu}|^{\rm High}$$ negligible

$$R^{(1)}_{\mu\nu} = \bar{g}^{\alpha\beta}\left(D_\alpha D_{(\mu}\delta g_{\nu)\beta} - D_\mu D_\nu \delta g_{\alpha\beta} - D_\alpha D_\beta \delta g_{\mu\nu}\right)$$

$$D_\mu \overline{\delta g}_{\mu\nu} = 0 \quad \left(\overline{\delta g}_{\mu\nu} \equiv \delta g_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}\delta g_{\alpha\beta}\right)$$

Lorentz gauge

matter

$$\boxed{D_\alpha D^\alpha \overline{\delta g}_{\mu\nu} = \Pi_{\mu\nu}}$$

Creation of GWs
in curved space-time

Gravitational Wave Propagation

What about the
High Freq. / Short Scale?

$$R^{(1)}_{\mu\nu} = -[R^{(2)}_{\mu\nu}]^{\mathrm{High}} + \frac{1}{m_p^2}\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)^{\mathrm{High}}$$

$$\frac{|R^{(2)}_{\mu}|^{\mathrm{High}}}{|R^{(1)}_{\mu}|} \sim \mathcal{O}\left(\frac{\lambda_*}{L_B}\right) \longrightarrow |R^{(2)}_{\mu}|^{\mathrm{High}}$$ negligible

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matter

$$\boxed{D_\alpha D^\alpha \overline{\delta g}^{\mathrm{TT}}_{\mu\nu} = \Pi^{\mathrm{TT}}_{\mu\nu}}$$

Creation of GWs
in curved space-time
TT dof = truly radiative !
[no gauge choice]

GW Propagation/Creation in Cosmology

FLRW: $ds^2 = a^2(-d\eta^2 + (\delta_{ij} + h_{ij})dx^i dx^j),\qquad \mathrm{TT}: \begin{cases} h_{ii} = 0 \\ h_{ij},_j = 0 \end{cases}$

GW Propagation/Creation in Cosmology

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Creation of GWs in curved space-time

Source: Anisotropic Stress

Eom: $\boxed{h''_{ij} + 2\mathcal{H}h'_{ij} - \nabla^2 h_{ij} = 16\pi G\Pi^{\mathrm{TT}}_{ij},}\qquad \boxed{\Pi_{ij} = T_{ij} - \langle T_{ij}\rangle_{\mathrm{FRW}}}$

GW Propagation/Creation in Cosmology

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Creation of GWs in curved space-time

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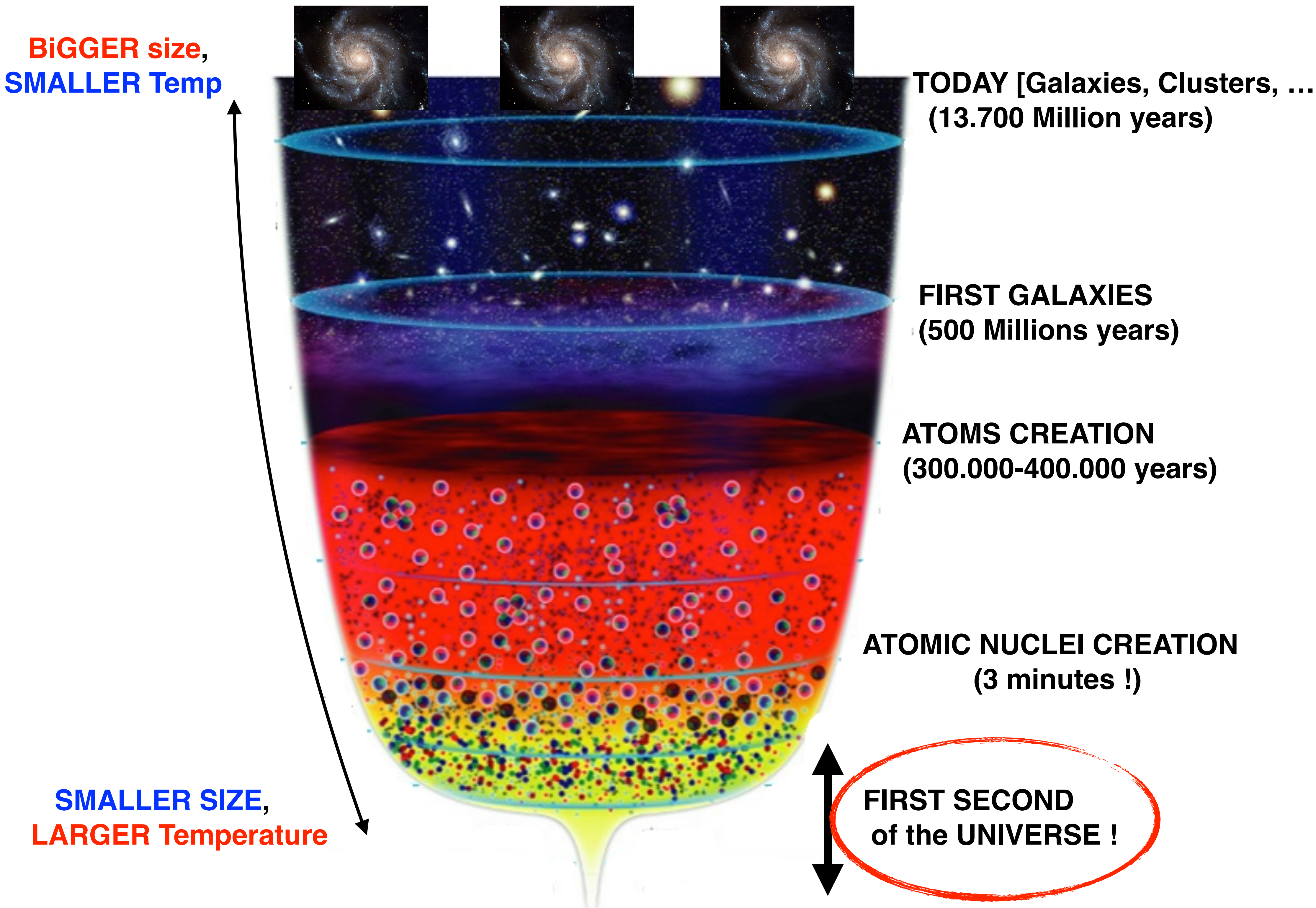
Source: Anisotropic Stress

$\boxed{\Pi_{ij} = T_{ij} - \langle T_{ij}\rangle_{\mathrm{FRW}}}$

GW Source(s): (SCALARS , VECTOR , FERMIONS)

$\Pi^{TT}_{ij} \quad\propto\quad \{\partial_i\chi^a\partial_j\chi^a\}^{TT},\ \ \{E_iE_j + B_iB_j\}^{TT},\ \ \{\bar\psi\gamma_i D_j\psi\}^{TT}$

Cosmic History



GWs: probe of the early Universe

1. **WEAKNESS of GRAVITY**:

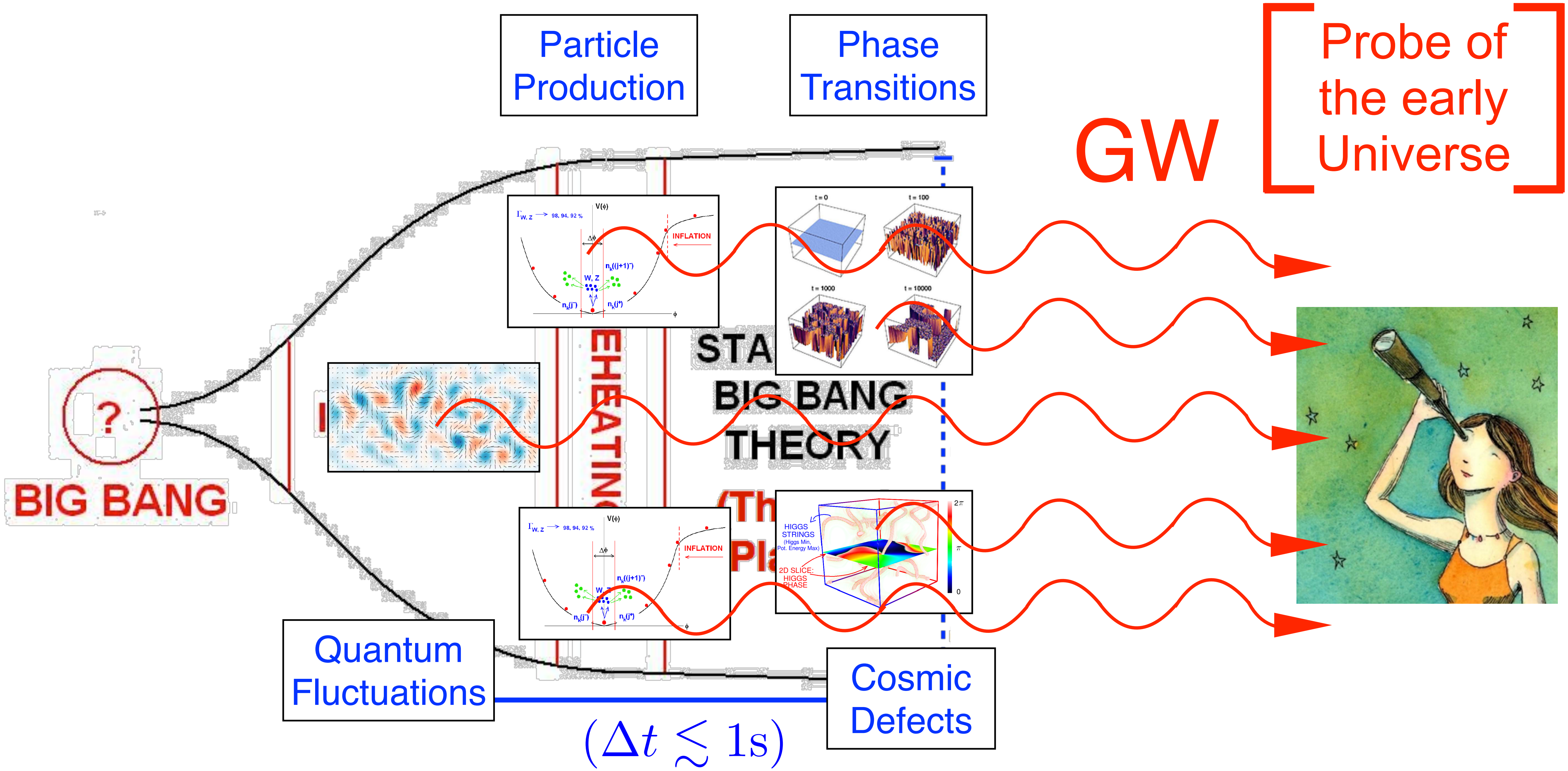
   ADVANTAGE: GW DECOUPLE upon Production
   DISADVANTAGE: DIFFICULT DETECTION

2. **ADVANTAGE**: GW $\rightarrow$ Probe for Early Universe

$$\rightarrow \begin{cases} \textbf{Decouple} \rightarrow \text{Spectral Form Retained} \\ \textbf{Specific HEP} \;\; \Leftrightarrow \;\; \text{Specific GW} \end{cases}$$

3. **Physical Processes**: $\begin{cases} \text{Inflation} \\ \text{Reheating} \\ \text{Phase Transitions} \\ \text{Cosmic Defects} \end{cases}$

The Early Universe



$$\Pi_{ij}^{\rm TT}[\phi, A_\mu, \psi, ...] \xrightarrow{\textbf{GWs}} \frac{d\rho_{\rm GW}}{d\log f}$$

GWs: probe of the early Universe

OUTLINE

1) Cosmology + GWs ✓

Early Universe

2) GWs from Inflation

3) GWs from Preheating

4) GWs from Phase Transitions

5) GWs from Cosmic Defects

GWs: probe of the early Universe

OUTLINE

1) Cosmology + GWs ✓

Early Universe

2) GWs from I...

4) ...itions

5) GWs from Cosmic Defects

To Be … Continued