Quantum Chromo Dynamics

LECTURE PLAN

① COLLIDER PHENOMENOLOGY

- COLLIDING PARTICLES
- PARTON-MODEL FOR THE DRELL-YAN PROCESS
- BREIT-WIGNER & NWA

② QUANTUM CHROMO DYNAMICS

- A REVIEW OF QCD
- RADIATIVE CORRECTIONS to DY
- DGLAP EVOLUTION

③ RESUMMATION

- SOFT-COLLINEAR FACTORISATION, IRC SAFETY
- THE TRANSVERSE MOMENTUM OF THE Z BOSON

④ JET PHYSICS

- WHY JETS?
- JET DEFINITIONS

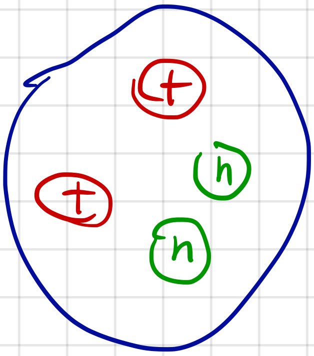
⑤ JET SUBSTRUCTURE

- GROOMING and TAGGING
- MACHINE LEARNING (briefly)

ENTER QCD!

- So far we've only mentioned QCD!

A bit of history: the quest for understanding strong interactions started in 1930s:



nuclei are NOT elementary particles.
What binds together protons and neutrons?

- very short-ranged force (1 fm)
⇝ maybe a massive force carrier?

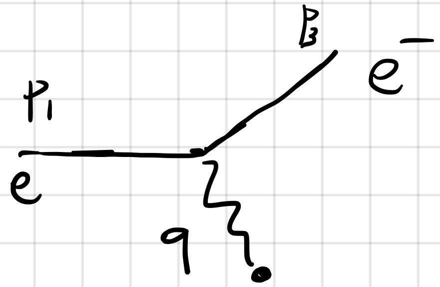
YUKAWA MODEL: (1934) $\begin{pmatrix} p \\ n \end{pmatrix}$ isospin doublet interacting through pions π.

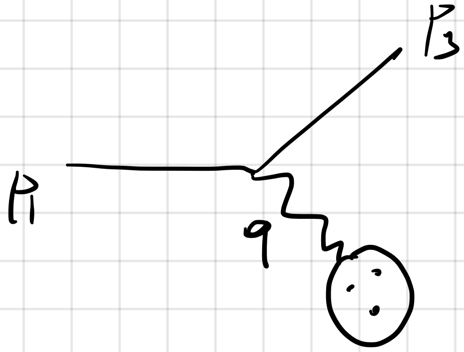
- Better experiments lead to a revolution:

(a) more and more hadronic states created leading to the so-called PARTICLE ZOO:
kaons, Λ baryons, ρ meson, Δ ...
Is there an organizing principle?

(b) very high energy ep scattering allowed us to probe the structure of the proton

$$Q^2 = -(p_1 - p_3)^2 \Rightarrow \lambda_\gamma \sim 1/\sqrt{Q^2}$$



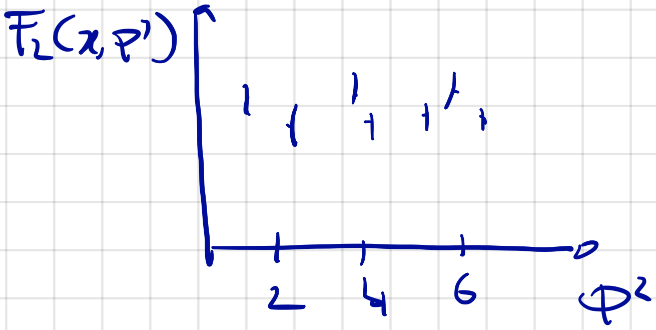


$$x = \frac{Q^2}{2p_1 \cdot q}$$

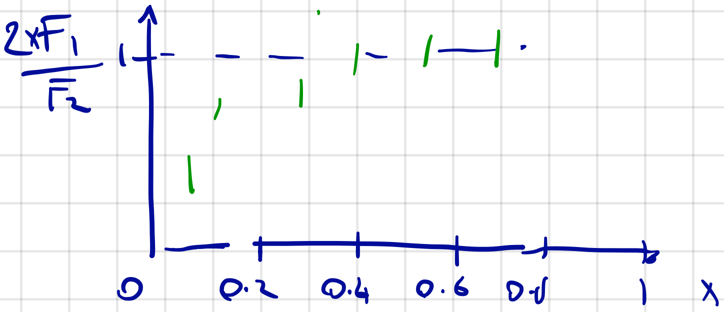
$\lambda \gtrsim r_p$
the proton appears point-like

$\lambda \ll r_p$
we can explore the proton structure

- DEEP INELASTIC SCATTERING experiments (1970s) show peculiar properities of the proton structure functions:



Bjorken scaling



Callan - Gross relations

ⓐ ⇒ THE QUARK MODEL (Gell-Mann) allow us to classify mesons and baryons in terms of representantion of the flavour symmetry group. It leads to the introduction of a new quantum number: COLOUR.

$SU(3)_L$ $\binom{u}{d}$; $SU(4)$ $(u\ d\ s\ c)$ . . .

(b) => THE PARTON MODEL (Bjorken/Feynman)

describes the $\gamma^* p$ interactions as elastic collisions between γ^* and a free$^{(*)}$ spin $1/2^{(**)}$ parton carry a fraction z of the proton momentum:

$$d\sigma_{e^-p} = d\sigma_{e^-q}(zP) \cdot f_{q/p}(z)$$

(*) this explains Bjorken scaling

(**) this explains Callan-Gross.

STRONG INTERACTIONS

→ HIGH energy: hadrons behave as made of "free" partons

→ LOW ENERGY: hadrons are bound states with symmetry properties of quark "constituents"

- Is there a theory that can account for both models?
  - people understood QUANTUM FIELD THEORY & GAUGE SYMMETRY (eg QED).

- Can we build a theory of strong interactions using the gauge principle?

QED : $U(1)$ gauge symmetry $\psi'(x) = e^{i\alpha(x)}\psi(x)$

QCD : $\underset{\overset{\uparrow}{N_c=3}}{SU(N_c)}$ non-Abelian gauge symmetry $\psi_i'(x) = V_{ij}(x)\psi_j(x)$

Classical QCD lagrangian

$i,j = 1,\dots,N_c$
$a = 1\dots,N_c^2-1$

$$\mathcal{L} = -\frac{1}{4} G_{\mu\nu}^a G^{a\mu\nu} +$$

$$\sum_{\substack{f=u,d,\\ s,c\dots}} \bar{\psi}_i^f (i\not{D}_{ij} - m_f \delta_{ij})\psi_j^f$$

$$D_{ij}^\mu = \partial_\mu \delta_{ij} - i g_s A_\mu^a t_{ij}^a$$

$$G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c$$

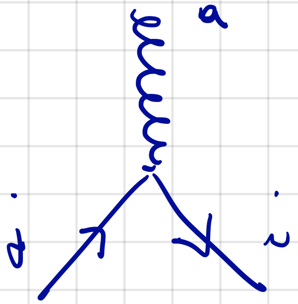
$$[t^a, t^b] = i f^{abc} t^c$$

- Quantisation of non-Abelian gauge theories is far from trivial but well-understood (Faddeev-Popov ghost, BRST symmetry etc).

- So we have a consistent QFT, characterized by $SU(3)$ gauge symmetry:
  - quarks transform in the fundamental rep.: 3 (anti-quarks in $\bar{3}$): we often say that quarks have 3 possible colours.
  - gluons transform in the adjoint rep.: 8 (N_c^2-1): we often say that gluons have 8 possible colours.

$\Rightarrow$ QUALITATIVE DIFFERENCE wrt QED: the gauge vectors carry charge!

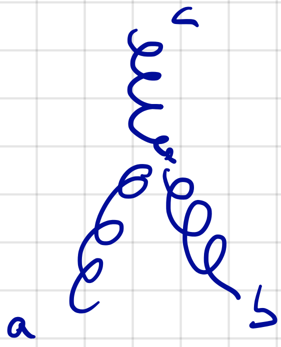
- We can work out Feynman rules and start computing scattering amplitudes involving quarks & gluons (whether this has anything to do with Nature is at this point an act of faith).

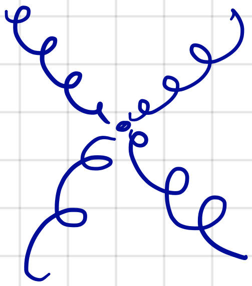


$$-ig_s\, t^a_{ij}\, \gamma^\mu$$

from the covariant derivative

but also





from the gluon field strength

- There are also vertices coupling the gluon field to the Fadeev-Popov ghosts.

- We can compute scattering amplitudes at tree-level: only novelty are colour factor (more later)

- If we go at 1-loop we (unsurprisingly) find UV divergences.

- QCD is renormalisable so all UV divergencies can be cured using a finite number of renormalisation constants: $Z_1, Z_2, Z_3, (Z_m, Z_5)$

- NOTE: in QED Ward-Takahashi identities ensure $Z_1 = Z_2$; in QCD Slavnov-Taylor identies do not allow us to conclude so.

- FACT: renormalisation of g_s leads to a scale-dependent coupling $\alpha_s = \frac{g_s^2}{4\pi}$

$$\boxed{\mu^2 \frac{d}{d\mu^2} \alpha_s = - \beta_0 \alpha_s^2 .}$$

$$\beta_0 = \frac{11 C_A - 2 n_f}{12 \pi}$$

(Gross, Politzer, Wilczek NOBEL PRIZE 2004)

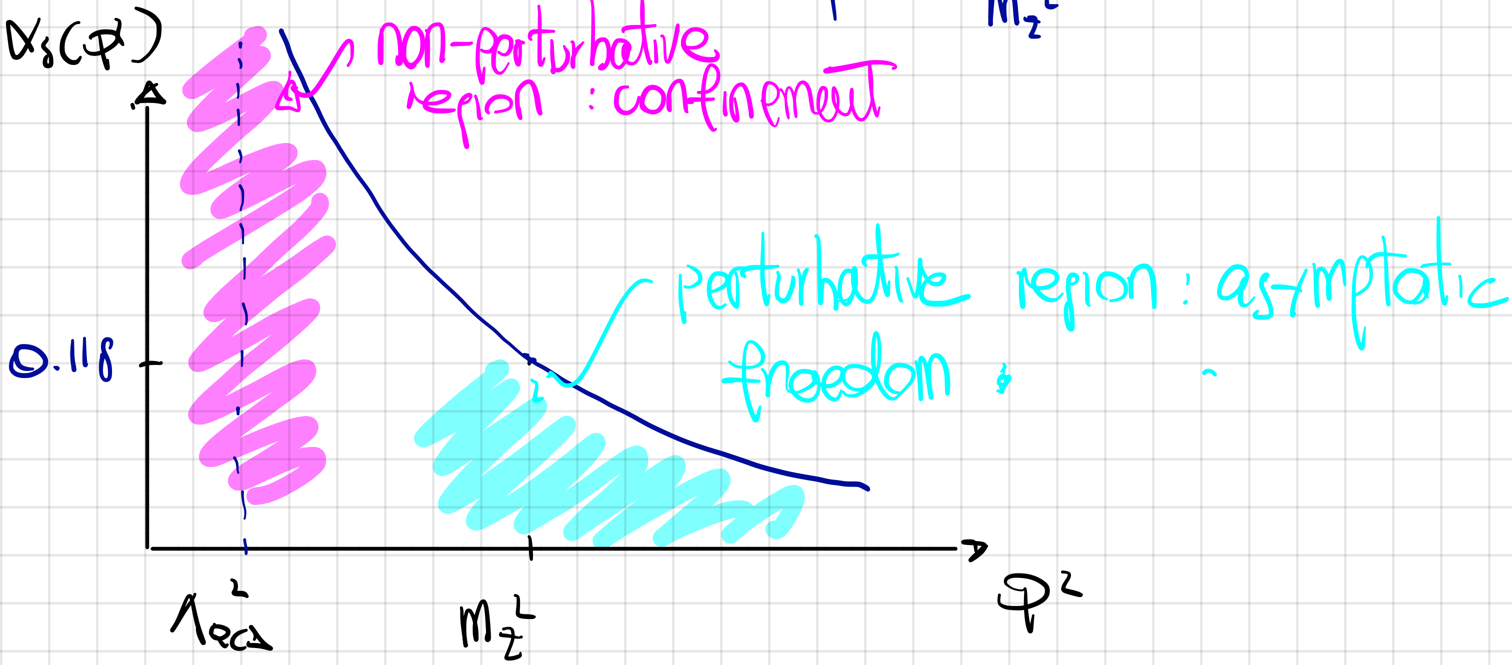
$$\frac{d\alpha_s}{\alpha_s^2} = -\beta_0 \frac{d\mu^2}{\mu^2}$$

$$\int_{\alpha_s(m_Z^2)}^{\alpha_s(Q^2)} \frac{d\alpha}{\alpha^2} = -\beta_0 \int_{m_Z^2}^{Q^2} \frac{d\mu^3}{\mu^2}$$

→ reference-scale is arbitrary. Nowadays we often choose m_Z because of LEP

$$\frac{1}{\alpha_s(m_Z^2)} - \frac{1}{\alpha_s(Q^2)} = -\beta_0 \ln \frac{Q^2}{m_Z^2}$$

$$\alpha_s(Q^2) = \frac{\alpha_s(m_Z^2)}{1 + \alpha_s \beta_0 \ln \frac{Q^2}{m_Z^2}}$$



↓

$$1 + \alpha_s \beta_0 \ln \frac{\Lambda_{QCD}^2}{m_Z^2} = 0$$

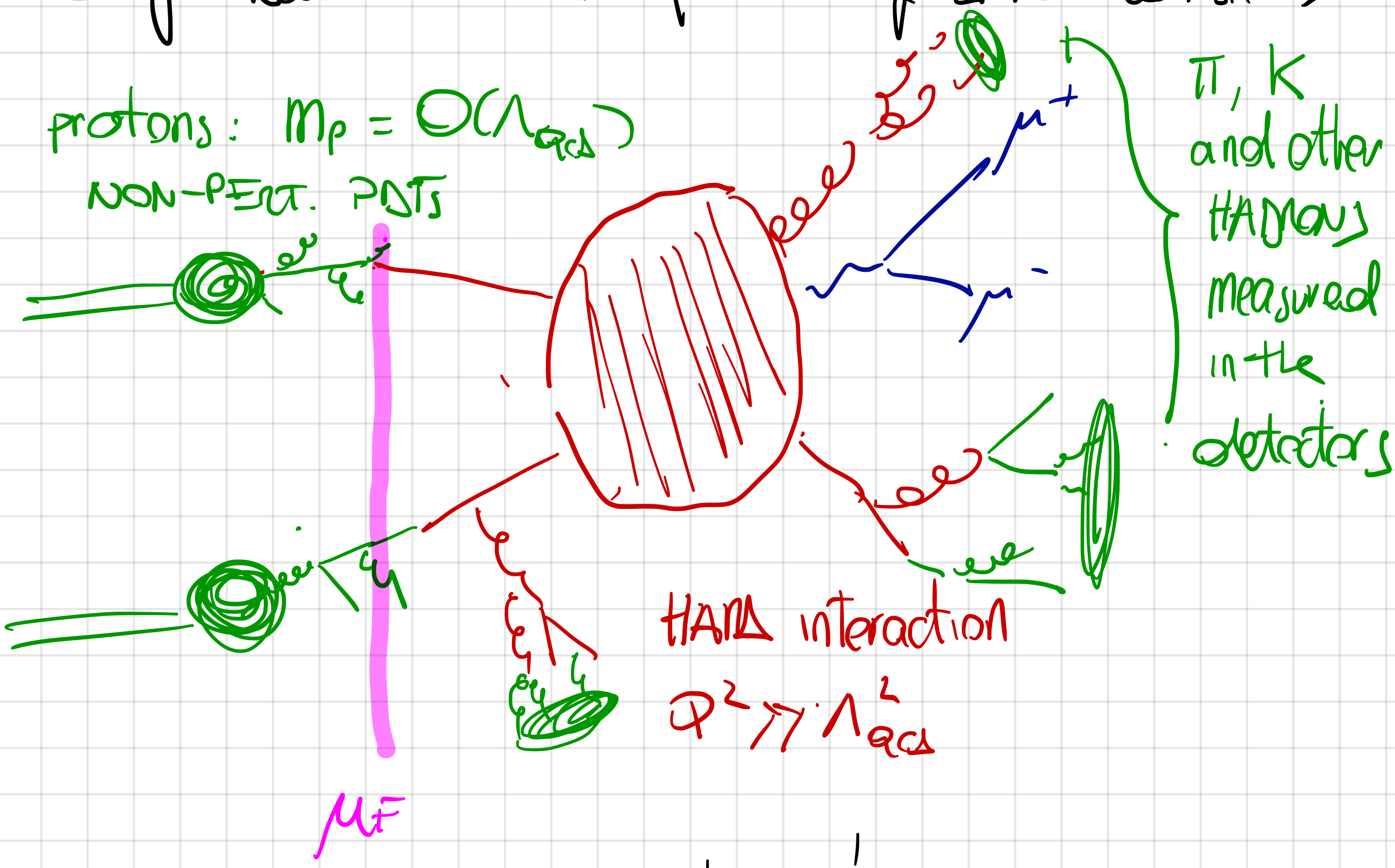
$$\Lambda_{QCD} = m_Z \exp\left(-\frac{1}{2\alpha_s\beta_0}\right)$$

for $Q^2 \gg \Lambda_{QCD}^2$ we can talk about quarks and gluons

for $Q^2 \sim \Lambda_{QCD}^2$ we have to talk about hadrons.
(models, LATTICE QCD, χ-PT...)

- Thus QCD is able (in principle) to generalise both the quark model and the parton model.

- Let's go back to our picture of LHC collisions



$$\sigma_{pp\to\mu^+\mu^-}(\tau, Q^2) = \sum_{\substack{i=q,g\\ j=q,g}} \int_0^1 dz_1 \int_0^1 dz_2 \, f_i(z_1, \mu_F^2)\, f_j(z_2, \mu_F^2)$$

$$\hat{\sigma}_{ij\to\mu^+\mu^-+X}\left(\frac{\tau}{z_1 z_2}, Q^2, \frac{Q^2}{\mu_F^2}, \frac{Q^2}{\mu_R^2}\right)$$

$$+\ O\left(\frac{\Lambda^2_{QCD}}{Q^2}\right)$$

any final state hadronic state.

μ_R : renormalisation scale

μ_F : factorisation scale

- This is the essence of the renowed "FACTORISATION" THEOREM by Collins, Soper, Sterman.

- In this very inclusive form is well-established and the result is analogous to DIS (where can also be proved using the OPE: see eg. SCHWARTZ 32.4.3)

- Things become more difficult if we want to ask more differential questions, ie we want to probe QCD final states.

RADIATIVE CORRECTIONS : $\mathcal{O}(\alpha_s)$

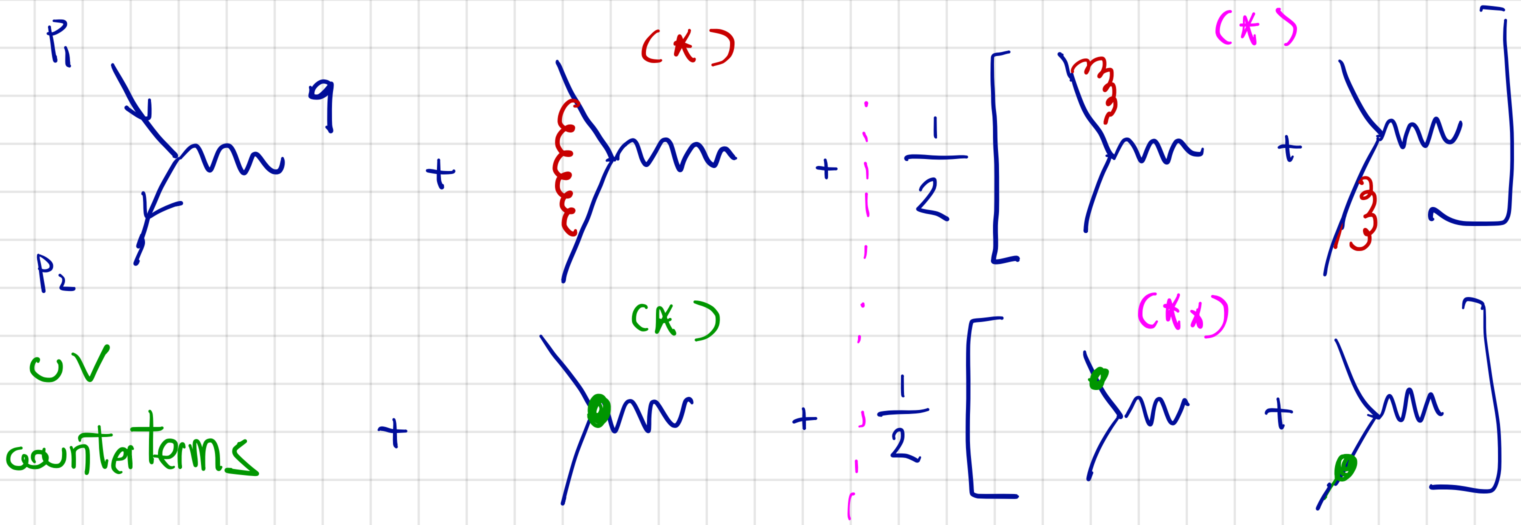
- Let's go back to our main example: DY
- We want to include $\mathcal{O}(\alpha_s)$ QCD corrections
- Following an S-matrix approach we begin by considering $\mathcal{O}(\alpha_s)$ correction to the process

$$q\bar{q} \to Z/\gamma^* \to \mu^+\mu^- ;$$

i.e. so-called loop or virtual corrections

- For simplicity we work close to the Z-pole and use NWA :

$$q\bar{q} \to Z$$



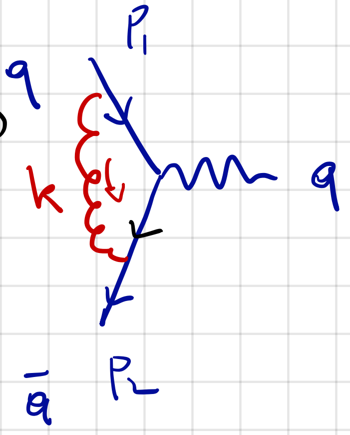
- these contributions would deserve a long discussion.
- LSZ formula dictates to amputate external legs. However, in schemes like $\overline{MS}$ $\sqrt{Z_2}$ contributions enter LSZ

- We work in $d = 4 - \varepsilon$ dimensions and use $\overline{MS}$ scheme.
  (if you are not familiar with DIMREG, see Sct B.3)
- In $\overline{MS}$ with only massless quarks, we have:
  - $(*)$ are zero: scaleless integrals
  - $(*)$ and $(**)$ contain only UV poles, which cancel when added together

So, we only have to compute $(*)$

PROBLEM Show that the integral $\int \frac{d^4 \ell_E}{\ell_E^4}$ (with ℓ_E an Euclidean vector) is both UV & IR divergent, but it's "zero" in DIMREG: $\int \frac{d^d \ell_E}{\ell_E^4} = 0$.

$i\mathcal{M}_d^{(1\text{-loop})} =$



$$= \int \frac{d^d k}{(2\pi)^d} \bar{v}(p_2) (-i\bar{g}_s \gamma^\nu t^a) \frac{i(-\not{p}_2 - \not{k})}{(p_2+k)^2 + i\varepsilon} \frac{(-ig_w)}{2\cos\vartheta_w} \gamma^\mu (V_q - A_q \gamma_5) \varepsilon_\mu^*(q) \frac{(\not{p}_1 - \not{k}) i}{(p_1 - k)^2 + i\varepsilon} (-i\bar{g}_s \gamma^\sigma t^b) \frac{-i g_{\rho\sigma} \delta^{ab}}{k^2 + i\varepsilon} u(p_1)$$

$$\bar{g}_s = g_s \mu^{(4-d)/2}$$

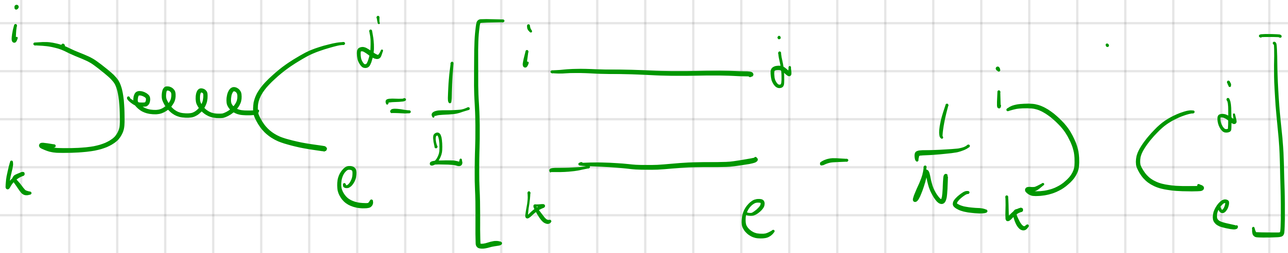
so

$$i\mathcal{M}_d^{(1-\text{loop})} = i\,\frac{g_W}{2\cos\theta_W}\, g_s^2\, \mu^{4-d}\, \bar{v}_i(p_2)\, t^a_{ik} t^a_{kj} \Bigg[\int \frac{d^dk}{(2\pi)^d} \frac{N^\mu(k,p_1,p_2)\; \varepsilon^*_\mu(q)}{[(p_2+k)^2+i\varepsilon][(p_1-k)^2+i\varepsilon][k^2+i\varepsilon]}$$

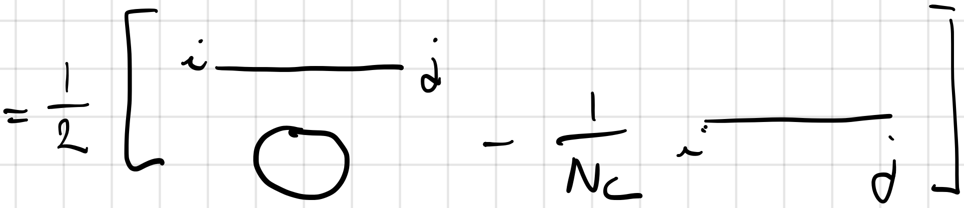
$$u_j(p_1)$$

① colour factor $t^a_{ik} t^a_{kj}$

Fierz identity



we want $k = e$



$$= \frac{1}{2}\left[N_c - \frac{1}{N_c} \right] \delta_{ij} = C_F\, \delta_{ij}$$

see book on Birdtracks by Citanovic

- We now have to evaluate the loop integral
- This can be done in a standard way by combining the denominators with Feynman parameters, shifting the loop momentum and Wick-rotate it.
- the resulting integral is UV divergent (no surprise)
  UV divergences are dealt with renormalisation

· However we note something troublesome: because

$$\sim \frac{1}{[k^2+i\varepsilon][-2p_1\cdot k+k^2+i\varepsilon][2p_2\cdot k+k^2+i\varepsilon]}$$

when the loop momentum is in the infrared region, i.e. it's either SOFT $(k^\mu \ll p^\mu_{1,2})$ or COLLINEAR $(k^\mu \sim a_i p_i^\mu)$ to massless legs NEW SINGULARITIES APPEAR.

e.g. $$\int \frac{d^d \ell_E}{\ell_E^4} \sim \int \frac{\ell_E^{d-1}\, d\ell_E}{\ell_E^4} \overset{d=4-\varepsilon}{=} \int \frac{d\ell_E}{\ell_E^{1+\varepsilon}}$$

where DIMREG regularize the integral if $\varepsilon<0$.

· A dirty trick is usually employed: compute everything in $d=4-\varepsilon$ dimensions and formally interpret $\varepsilon \equiv \varepsilon_{UV}>0$ to regulate the UV and $\varepsilon \equiv \varepsilon_{IR}<0$ to regulate soft and collinear divergencies.

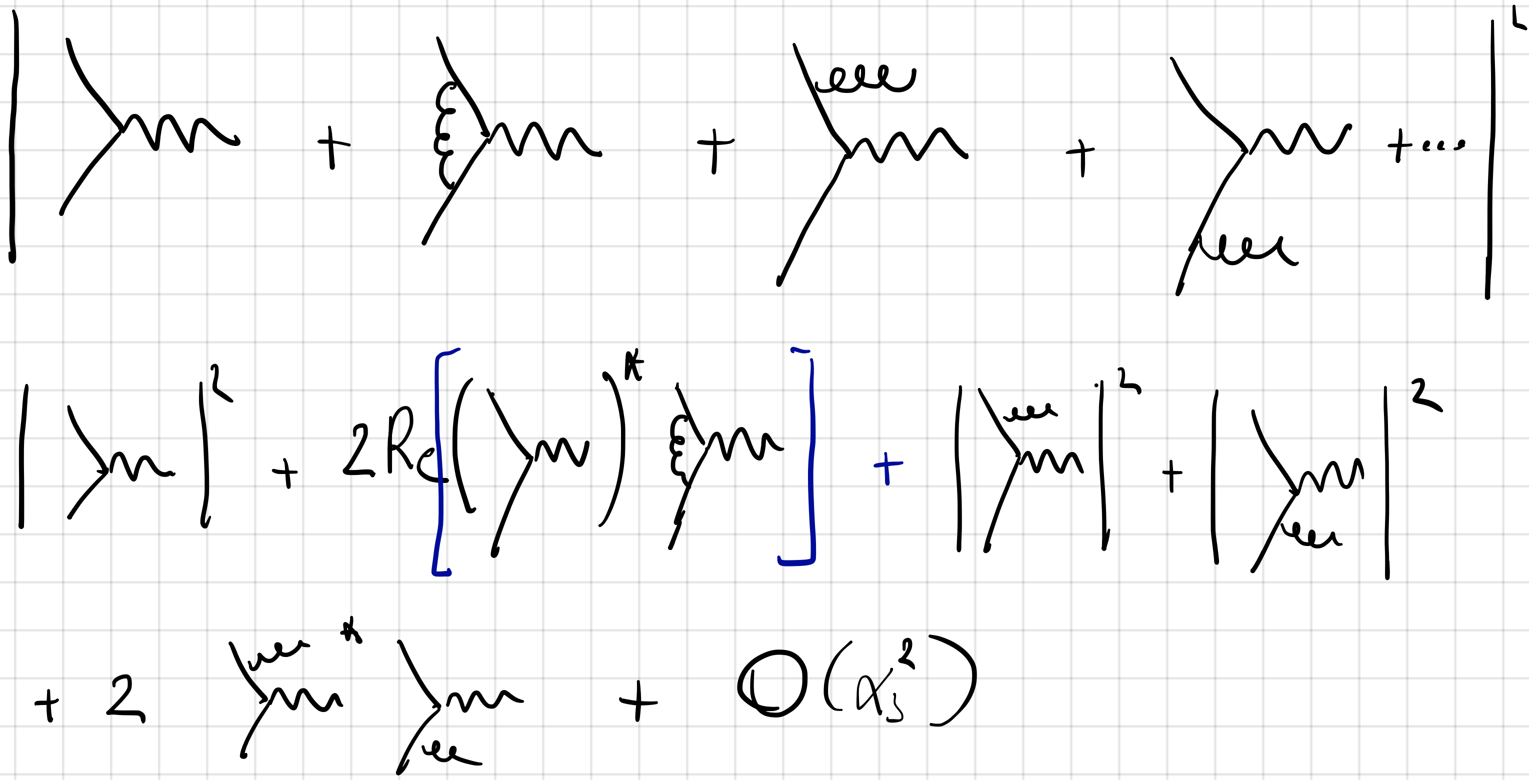
· How do we deal with IR singularities?

they arise because of long range interactions of massless gauge bosons (QED & QCD). In QCD they are eventually masked by hadronisation.

· Two approaches:

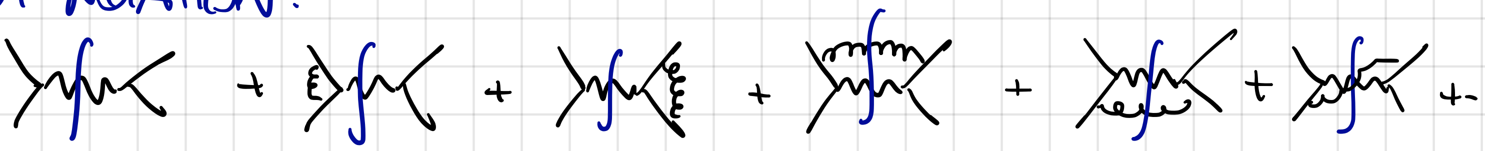
- "THEORY"-inspired: define a finite S-matrix which accounts for non-trivial IN/OUT states [Fadeev-Kulish ...]

- "PHENO"-INSPIRED: note that at the end of the day we are interested in x-sec non amplitudes:



- the real-radiation diagrams describe DIFFERENT final states, however in the soft/collinear limits we are worried about, their kinematics resemble the LO and virtual one. There is HOPE!

OUT NOTATION:



- In this x-section approach, we have:

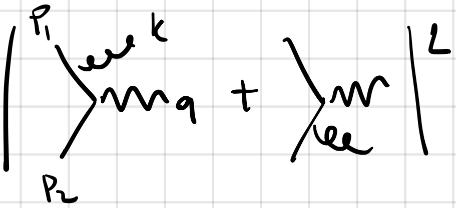
$$\lim_{\varepsilon\to 0} 2\,\mathrm{Re}\left[\mathcal{M}^{(1-\text{loop})}_{4-\varepsilon}\mathcal{M}^{(0)}_{4-\varepsilon}\right] = -2\,|\mathcal{M}^{(0)}|^2\, 2\frac{\alpha_s}{\pi} C_F \left(\frac{\mu^2}{q^2}\right)^{\varepsilon/2}\frac{4\pi^{\varepsilon/2}}{\Gamma(1-\frac{\varepsilon}{2})}\left[\frac{1}{\varepsilon^2}+\frac{3}{4}\frac{1}{\varepsilon}+1-\frac{\pi^2}{8}\right].$$

- the $1/\varepsilon^k$ poles left are purely IR
- Note that the kinematics is the same of LO:

$$\hat{s} = Q^2 \,, \text{ ie } \quad \hat{\tau} = \frac{Q^2}{\hat{s}} = 1 \,,$$

ie we can write the whole contribution as proportional to $\delta(1-\hat{\tau})$.

- We now have to consider:



and integrate it over the real gluon $k^2=0$ phase space:

$$\int \frac{d^{d-1}\vec{k}}{2E_k(2\pi)^{d-1}}$$

- the kinematics is now less trivial:

$$p_1+p_2 = k+q \quad \text{so} \quad \hat{s} = (k+q)^2 = Q^2+2q\cdot k$$

$$\hat{\tau} = 1 - \frac{2q\cdot k}{\hat{s}} < 1$$

we write the real-emission result as a function of $\hat{\tau}$

$$\sim |\mathcal{M}^{(0)}|^2 \frac{4\alpha_s}{\pi} C_F \left(\frac{\mu^2}{Q^2}\right)^{\varepsilon/2} \frac{4\pi^{\varepsilon/2}}{\Gamma(1-\frac{\varepsilon}{2})} \Bigg\{ \frac{1}{\varepsilon^2}\delta(1-\hat{\tau}) + \frac{3}{4}\frac{1}{\varepsilon}\delta(1-\hat{\tau}) - \underbrace{\frac{1}{\varepsilon}\left[\frac{1+\hat{\tau}^2}{(1-\hat{\tau})_+} + \frac{3}{2}\delta(1-\hat{\tau})\right]} + D_{q\bar{q}}(\hat{\tau}) \Bigg\}$$

- this the famous DGLAP splitting function

$$P_{qq}(z) = C_F\left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right]$$

- $D_{q\bar{q}}(z)$ has no $1/\varepsilon$ poles but it's a distribution.

- If we sum real and virtual, we see that the singularities $\frac{1}{\varepsilon^2}$ cancel but the cancellation of $1/\varepsilon$ pole is incomplete.

- We can trace this back to the collinear region.

- This is a feature of initial-state radiation. Final-state QCD radiation instead enjoys full cancellation:

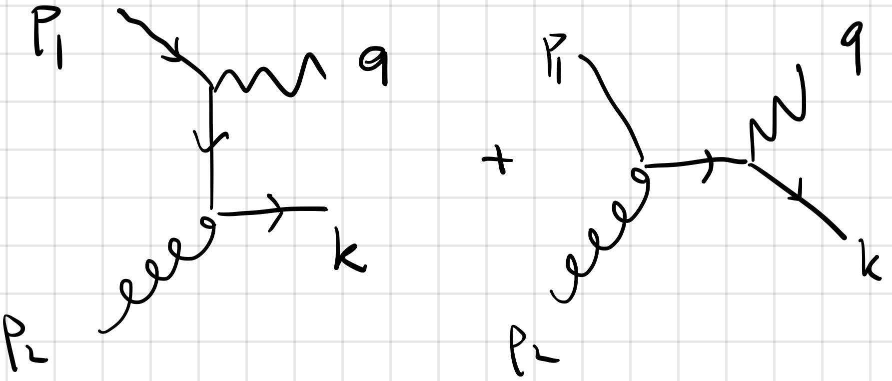
has NO IR poles.

KLN theorem

- before dealing with these uncancelled collinear singularity, we note the in the spirit of being inclusive, there is one more contribution we should include.

- The QCD-improved parton model states that all QCD quanta are present in the proton: quarks and gluons.

- So we should also include this possibitily:



[while  is $\mathcal{O}(\alpha_s^2)$]

this leads to

$$\sim |M^{(0)}|^2 \frac{4\alpha_s}{\pi} \left(\frac{\mu^2}{q^2}\right)^{\varepsilon/2} \frac{4\pi^{\varepsilon/2}}{\Gamma(1-\varepsilon)} \left[-\frac{1}{\varepsilon} P_{qg}(\hat{z}) + D_{qg}(\hat{z}) \right]$$

$$P_{qg}(z) = T_R \left[z^2 + (1-z)^2 \right]$$

RENORMALISATION OF PDFs and DGLAP EVOLUTION

$$Q^2 \frac{d\sigma}{dQ^2}^{NLO} = \sigma_0 \left\{ \int_0^1 dz_1 \int_0^1 dz_2 \int_0^1 d\hat{\tau}\; \delta(z_1 z_2 \hat{\tau} - \tau) \right.$$

$$\boxed{\frac{1}{\bar{\varepsilon}} = \frac{1}{\varepsilon} + \ln(4\pi e^{-\gamma_E})}$$

$$\left\{ \sum_q{}' e_q^2 \left[\delta(1-\hat{\tau}) + \frac{\alpha_s}{2\pi} \left(-\frac{1}{\bar{\varepsilon}} P_{qq}(\hat{\tau}) \right.\right.\right.$$

$$\left.\left. + 2 P_{qq}(\hat{\tau}) \ln \frac{Q^2}{\mu_F^2} + C_F D_{q\bar{q}}(\hat{\tau}) \right] \right.$$

$$f_q(z_1)\, f_{\bar{q}}(z_2) + (q \leftrightarrow \bar{q})$$

$$\sum_{i=q,\bar{q}}{}' e_q^2 \frac{\alpha_s}{2\pi} \left[P_{qg}(\hat{\tau}) \left(-\frac{1}{\bar{\varepsilon}} + 2 \log \frac{Q^2}{\mu_F^2} \right) \right.$$

$$\left. + T_R D_{qg}(\hat{\tau}) \right] f_g(z_1)\, f_i(z_2) + (z_1 \leftrightarrow z_2) \Big\}$$

• we now introduce "renormalised PDFs"

$$f_i^{REN}(x, \mu_F^2) \equiv f_i(x) - \frac{\alpha_s(\mu_F^2)}{2\pi} \frac{1}{2\bar{\varepsilon}} \int_x^1 \frac{dz}{z}$$

$$\left[P_{iq}\left(\frac{x}{z}\right) f_q(z) + P_{ig}\left(\frac{x}{z}\right) f_g(z) \right] + O(\alpha_s^2)$$

$$= f_i(x) - \frac{\alpha_s}{2\pi} \frac{1}{2\bar{\varepsilon}} \int_0^1 d\alpha \int_0^1 dz\; \delta(z\alpha - x)$$

$$\left[P_{iq}(\alpha)\, f_q(z) + P_{ig}(\alpha)\, f_g(z) \right]$$

so we have to evaluate

$$\int_0^1 dz_1 \int_0^1 dz_2 \int_0^1 d\hat{\tau} \int_0^1 d\alpha \int_0^1 dz \;\; \delta(z_1 z_2 \hat{\tau} - \tau)\, \delta(1-\hat{\tau})\, \delta(z\alpha - z_1)$$

$$P_{iq}(\alpha)\; f_q(z)\, f_q(z_2)$$

$$= \int dz \int dz_2 \int d\alpha \;\; P_{iq}(\alpha)\, f_q(z)\, f_{\bar{q}}(z_2) \int_0^1 dz_1 \;\; \delta(z_1 z_2 - \tau)$$

$$\delta(z\alpha - z_1)$$

$$= \int dz \int dz_2 \int d\alpha \;\; \frac{1}{z_2}\, \delta\left(z\alpha - \frac{\tau}{z_2}\right) P_{iq}(\alpha)\, f_q(z)\, f_{\bar{q}}(z_2)$$

$$= \int dz \int dz_2 \int d\alpha \;\; \delta(z z_2 \alpha - \tau)\; P_{iq}(\alpha)\, f_q(z)\, f_{\bar{q}}(z_2)$$

which indeed has the same form as the $\frac{1}{\varepsilon}$ contr.

- Thus in terms of renormalised PDFs we have

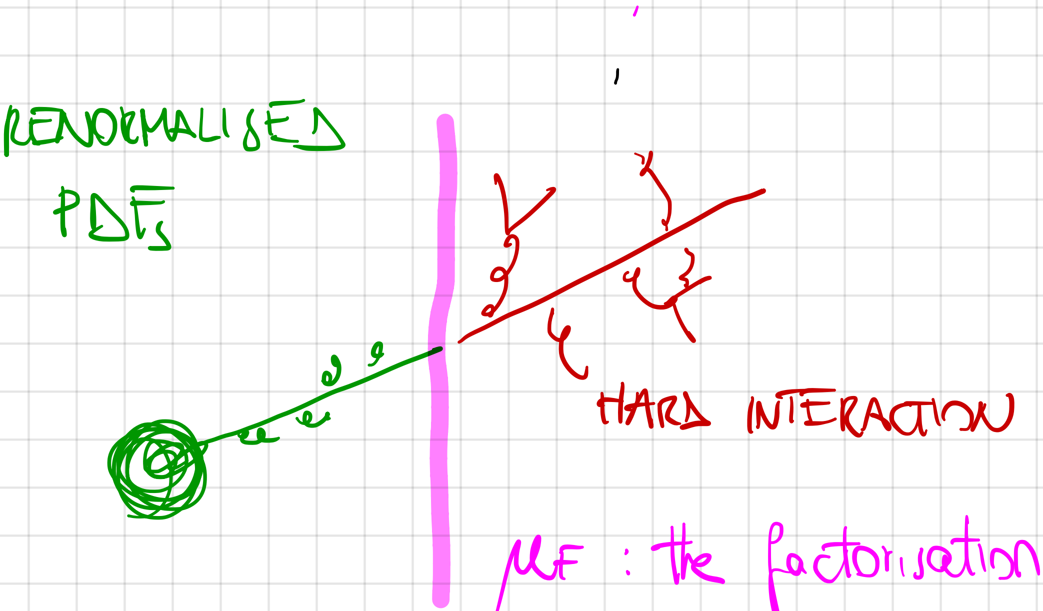
$$a\frac{d\sigma}{dQ^2} = \sigma_0 \int_0^1 dz_1 \int_0^1 dz_2 \int_0^1 d\hat{\tau} \;\; \delta(z_1 z_2 \hat{\tau} - \tau)$$

$$\sum_{ij} \left[\delta_{iq}\, \delta_{i\bar{q}}\; \delta(1-\hat{\tau}) + \frac{\alpha_s(\mu_R^2)}{\pi} \right.$$

$$\left. C_{ij}\left(\hat{\tau}, \frac{Q^2}{\mu_R^2}, \frac{Q^2}{\mu_F^2}\right) \right] f_i^{REN}(z_1, \mu_F^2)$$

$$f_j^{REN}(z_2, \mu_F^2)$$

$$+ O(\alpha_s^2)$$



μ_F : the factorisation scale tells us "where the proton begins".

- The coefficients of $\frac{1}{\varepsilon}$ collinear poles are universal: so the RENORMALISED PDFs are still universal* too
- *this is true once we fix a factorisation scheme: $\overline{MS}$ PDFs, etc..
- Physical observables shoued not depend on μ_F

$$\mu_F^2 \frac{d}{d\mu_F^2}\left(\frac{d\sigma}{d\Phi}\right) = 0 \quad \left[\text{more precisely } O(\alpha_s^{n+1})\right]$$

- thus order by order the μ_F dependence of the PDFs is cancelled by the μ_F dependence of the coefficient function $C(z,\mu_F)$.
- but $\mu_F^2 \frac{d}{d\mu_F^2} C_{ij} \sim \frac{\alpha_s}{2\pi} P_{ij}$!!! ...

- this leads to the famous (we assume one quark flavour for simplicity)

DGLAP EVOLUTION EQUATIONS

$$\mu_F^2 \frac{d}{d\mu_F^2} \begin{pmatrix} f_q(x,\mu_F^2) \\ f_g(x,\mu_F^2) \end{pmatrix} = \frac{\alpha_s(\mu_F^2)}{2\pi} \int_x^1 \frac{dz}{z} \begin{pmatrix} P_{qq}\left(\frac{x}{z}\right) & P_{qg}\left(\frac{x}{z}\right) \\ P_{gq}\left(\frac{x}{z}\right) & P_{gg}\left(\frac{x}{z}\right) \end{pmatrix} \begin{pmatrix} f_q(z,\mu_F^2) \\ f_g(z,\mu_F^2) \end{pmatrix}$$

- the initial-condition is non-perturbative, but the evolution is!

- STRATEGIES for FITTING PDFs

$$\underset{\substack{\uparrow \\ \text{measure}}}{\sigma} = \underset{\substack{\uparrow \\ \text{compute}}}{\hat{\sigma}} \otimes f_1 \otimes \underset{\substack{\nwarrow \\ \text{evolve to common scale} \\ \text{using DGLAP and FIT.}}}{f_2}$$

COLLINEAR FACTORISATION THEOREM

DGLAP evolution ↑

$$\sigma(\tau, Q^2) = \sigma_0 \sum_{ij} C\left(\frac{\tau}{z}, \frac{\mu_R^2}{Q^2}, \frac{\mu_F^2}{Q^2}; \alpha_s(\mu_R^2)\right) \otimes_{z_1 z_2} f_i^{\cancel{REN}}(z_1, \mu_F^2)\, f_j^{\cancel{REN}}(z_2, \mu_F^2)$$

↓

perturbative expansion in α_s

UV divergences : RENORMALISATION of Z_i's

COLL. divergences : RENORMALISATION of PDFs

- $C_{ij} = C^{(0)} + \alpha_s C^{(1)} + \alpha_s^2 C^{(2)} + \dots, \quad \alpha_s = \alpha_s(\mu_R^2)$

- $C^{(i)}$ contains logs of the ren. scale μ_R^2 / Q^2
- the ren. group evolution allows us to choose $\mu_R^2 = Q^2$
- the coeff. is free of large μ_R logs, which are resummed by the running coupling :

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu_R^2)}{1 + \alpha_s(\mu_R^2)\beta_0 \ln\frac{Q^2}{\mu_R^2}} \simeq \alpha_s \sum_{k=0}^{+\infty} \alpha_s^k \beta_0^k \ln^k \frac{Q^2}{\mu_R^2}$$

- Something very similar happens for logs of μ_F^2 / Q^2

DGLAP allows us to choose $\mu_F^2 = Q^2$

· MATHEMATICAL ASIDE: Mellin transform:

Theorem If $$h(x) = \int_x^1 \frac{dz}{z} f(z)\, g\left(\frac{x}{z}\right)$$

then $$\tilde{h}(N) = \tilde{f}(N)\, \tilde{g}(N)$$

PROOF

$$\tilde{h}(N) = \int_0^1 dx\, x^{N-1} \int_x^1 \frac{dz}{z} f(z)\, g\left(\frac{x}{z}\right)$$

$$= \int_0^1 \frac{dz}{z} \int_0^z dx\; x^{N-1} f(z)\, g\left(\frac{x}{z}\right) \qquad \left[\begin{array}{l} \frac{x}{z} = w \\ dx = z\,dw \end{array}\right]$$

$$= \int_0^1 \frac{dz}{\cancel{z}} \int_0^1 dw\, \cancel{z}\; w^{N-1} z^{N-1} f(z)\, g(w)$$

$$= \tilde{f}(N)\; \tilde{g}(N)$$

· So the DGLAP eqns can be written as

$$\mu^2 \frac{d}{d\mu^2} \begin{pmatrix} \tilde{f}_q(N,\mu^2) \\ \tilde{f}_g(N,\mu^2) \end{pmatrix} = \frac{\alpha_s(\mu^2)}{2\pi} \begin{pmatrix} \gamma_{qq}(N) & \gamma_{qg}(N) \\ \gamma_{gq}(N) & \gamma_{gg}(N) \end{pmatrix} \begin{pmatrix} \tilde{f}_q \\ \tilde{f}_g \end{pmatrix}$$

Where

$$\gamma_{ij}(N) = \int_0^1 dz\, z^{N-1} P_{ij}(z)$$ are the

LO anomalous dimensions.

- We got rid of convolutions, but we still have a matrix structure
- Furthermore, beyond LO, we have $\gamma_{ij}(N, \alpha_s(\mu^2))$: so the solution is a complicated path-ordered exponential.
- In order to understand the solution, let's consider the simple case of quarks only *

* if you're not happy, think about diagonalising the matrix.

$$\mu^2 \frac{\partial}{\partial \mu^2} q = \frac{\alpha_s(\mu^2)}{2\pi} \gamma(N)\, q(N,\mu^2)$$

$$\frac{\partial}{\partial \ln \mu^2} \ln q(N,\mu^2) = \frac{\alpha_s(\mu^2)}{2\pi} \gamma(N)$$

$$\ln \frac{q(N,Q^2)}{q(N,\mu_F^2)} = \int_{\mu_F^2}^{Q^2} \frac{d\mu^2}{\mu^2} \frac{\alpha_s(\mu^2)}{2\pi} \gamma(N)$$

$$q(N,Q^2) = q(N,\mu_F^2)\, e^{\int_{\mu_F^2}^{Q^2} \frac{d\mu^2}{\mu^2} \frac{\alpha_s(\mu^2)}{2\pi} \gamma(N)}$$

- the integral over the 1-loop running coupling can be performed (EXERCISE).
- to get an idea, let's work at fixed coupling $\alpha_s(Q^2)$

$$q(N,Q^2) = q(N,\mu_F^2) \exp\left[\frac{\alpha_s}{2\pi} \gamma(N) \ln \frac{Q^2}{\mu_F^2} \right]$$

$$= q(N,\mu_F^2) \left(\frac{Q^2}{\mu_F^2} \right)^{\frac{\alpha_s \gamma(N)}{2\pi}}$$

DGLAP resums collinear logs to all-orders in P.T.

RECAP SO FAR

We've explicitly considered $\frac{d\sigma}{dq^2}$ in DY and we've learnt that for INCLUSIVE cross-sections:

- we can separate partonic (hard) from PDFs
- in computing $\hat{\sigma}$ various divergences arise:
  - UV → REN.
  - IR
    - SOFT sing. cancels when adding real/virtual
    - FSR collinear sing. also cancel (not seen)
    - ISR collinear singularities must be re-absorbed in the PDFs, leaving behind potentially large collinear logs!. These can be resummed using DGLAP.

However, collider phenomenology is much more than inclusive observables!