Resummation

LECTURE PLAN

① COLLIDER PHENOMENOLOGY

- COLLIDING PARTICLES
- PARTON-MODEL FOR THE DRELL-YAN PROCESS
- BREIT-WIGNER & NWA

② QUANTUM CHROMO DYNAMICS

- A REVIEW OF QCD
- RADIATIVE CORRECTIONS to DY
- DGLAP EVOLUTION

③ RESUMMATION

- SOFT-COLLINEAR FACTORISATION, IRC SAFETY
- THE TRANSVERSE MOMENTUM OF THE Z BOSON

④ JET PHYSICS

- WHY JETS?
- JET DEFINITIONS

⑤ JET SUBSTRUCTURE

- GROOMING and TAGGING
- MACHINE LEARNING (briefly)

INFRA-RED & COLLINEAR (IRC) SAFETY

- Let's consider a generic process that at LO has n-partons in the final state (DY: $n=0$)

$$\sigma^{NLO} = \sigma^{LO} + \sigma^{REAL} + \sigma^{VIRT}$$

$$= \int d\phi^{(n)} |M_0|^2 + \alpha_s \left[\int d\phi^{(n+1)} R + \int d\phi^{(n)} V \right]$$

separate divergent, but the sum is finite (after UV & PDF ren.)

- What happens if we want to measure an observable

$$O = O(\{q_i\})$$

final-state momenta (partons vs hadrons)

- we insert measurement functions (eg $\delta(\ldots)$, $\Theta(\ldots)$, etc.)

$$\sigma(O) = \int d\phi^{(n)} |M_0|^2 O^{(n)} + \alpha_s \left[\int d\phi^{(n+1)} R\, O^{(n+1)} + \int d\phi^{(n)} V O^{(n)} \right]$$

Cancellation of IRC singularities still holds $\iff$

$$O^{(n+1)} \to O^{(n)}$$ when
- two partons become collinear
- one parton becomes soft.

IRC SAFETY (Sterman - Weinberg)

$$O(q_1, q_2, \ldots, q_j, \ldots, q_n) \to O(q_1, \ldots, q_n) \quad \text{if } q_j \ll Q$$

$$O(q_1, \ldots, q_i, \ldots, q_j, \ldots, q_n) \to O(q_1, \ldots, q_{i+j}, \ldots, q_n) \quad \text{if } q_i // q_j$$

other definitions are possible: eg KOMISKE, METODIEV, THALER (2020)

For instance:

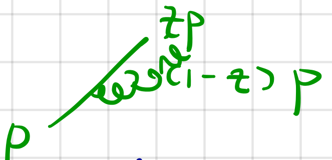
- counting the number of QCD particles (particle multiplicity) is NOT IRC safe

- the invariant mass of a QCD system is IRC safe:

$$m^2 = (q_1 + q_2 + \cdots q_n)^2$$

KLN theorem should ensure cancellation of all IR sing. in $|M|^2$, so why do we have uncancelled collinear sing. from ISR?

→ the concept of partonic x-sec is not IRC safe



so $\hat{\sigma}(zp)$ even if $\vartheta \to 0$

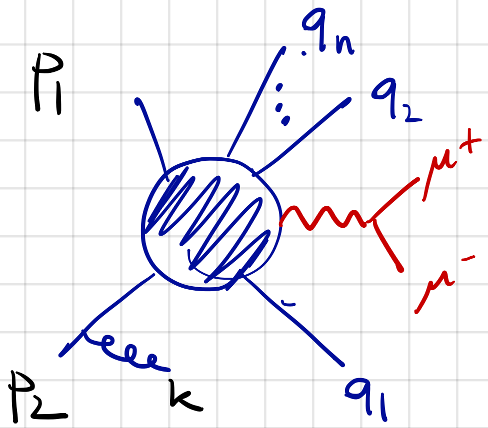
- other way of saying it: we demand a proton in the intial-state.

• So for IRC safe observables we can employ P.T.

• But is PT going to be reliable? Do we have to worry about potentially large logs?

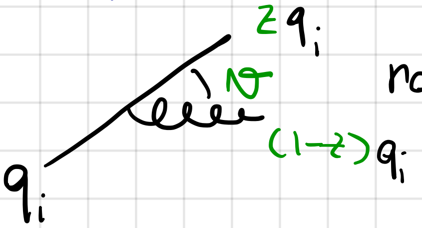
• In order to answer these questions, we have to look at matrix elements in the soft/collinear limits.

QCD matrix elements in the collinear limit

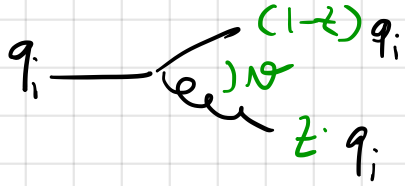


$$|M^{(n+1)}|^2 \, d\phi^{(n+1)} \simeq |M^{(n)}|^2 \, d\phi^{(n)} \frac{\alpha_s}{\pi} \frac{d\vartheta^2}{\vartheta^2} \, dz \, P_{ji}(z)$$

$(p_2+k)^2 = 2 p_2 \cdot k \to 0$



note that for FSR (timelike) the opposite convention is often used:

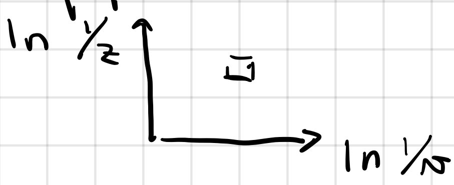


- emissions from different legs simply factorise
- emissions from the same legs give rise to a convolutional structure, which fully diagonalizes upon Mellin transform.

SOFT-COLLINEAR LIMIT:

$z \to 1$: $\quad \frac{\alpha_s}{\pi} C_i \frac{d\vartheta^2}{\vartheta^2} \frac{dz}{1-z}$ $\quad$ or $\quad$ $z \to 0$: $\quad \frac{\alpha_s}{\pi} \frac{d\vartheta^2}{\vartheta^2} \frac{dz}{z}$

QCD matrix elements in the most singular soft and collinear limit populate the $(\ln \frac{1}{z}, \ln \frac{1}{\vartheta})$ plane UNIFORMLY

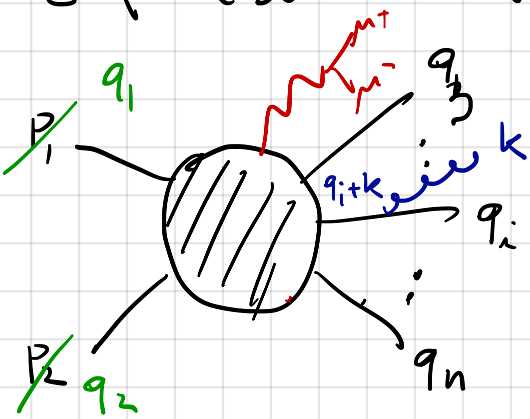


equal areas $\Leftrightarrow$ equal emission probability (more later)

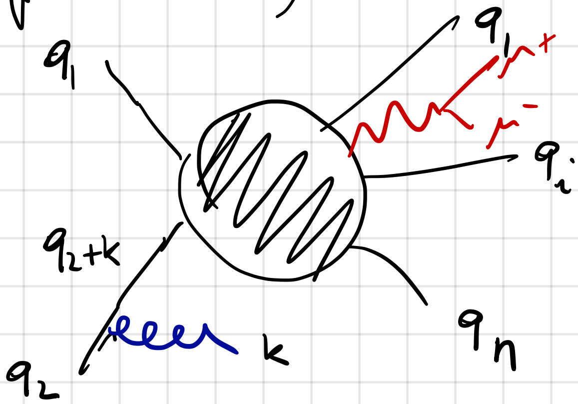
NOTE: collinear factorisation written in terms of squared matrix elements, ie "probability". This can be seen as a semi-classical process. This observation is the starting point of MARKOVIAN ALGORITHM know as PARTON SHOWERS.

QCD-matrix elements in the soft limit

- What about the behaviour of QCD matrix elements in the soft (but not necessarily collinear) limit?



(a) FSR



(b) ISR

- it's useful to label all "hard" momenta as q_i, $i=1,\dots,n$

(a) FSR

$$\mathcal{M}_i^{(n+1)} = \varepsilon^*_\mu(k)\,\bar{u}_i(q_i)\,(-i g_s \gamma^\mu t^a_i)\,\frac{i(\not{k}+\not{q}_i)}{(k+q_i)^2+i\varepsilon}\,\bar{\mathcal{M}}^{(n)}$$

$$\overset{k \ll q_i}{\simeq} \varepsilon^*_\mu(k)\,\bar{u}(q_i)\,\frac{(g_s \gamma^\mu t^a)\not{q}_i}{2k\cdot q_i+i\varepsilon}\,\bar{\mathcal{M}}^{(n)} =$$

$$= \varepsilon^*_\mu(k)\, \frac{g_s}{2q_i\cdot k + i\varepsilon}\, \bar u(q_i)\left[\underbrace{-\cancel{q}_i\gamma^\mu}_{=0 \text{ for Dirac equation}} + 2q_i^\mu\right] t^a\, \bar{\mathcal{M}}^{(n)}$$

$$= \varepsilon^*_\mu(k)\, g_s \frac{\cancel{2}\, q_i^\mu}{\cancel{2}\, q_i\cdot k + i\varepsilon/2}\, \bar u(q_i)\, t^a\, \bar{\mathcal{M}}^{(n)}$$

$\rightsquigarrow \varepsilon$

EIKONAL APPROXIMATION

- note that the spin dependence fully factorises, while the colour matrix remains trapped (much easier in QED)

ISR $$\mathcal{M}_i^{(n+1)} \simeq \frac{q_i\cdot\varepsilon^*}{-q_i\cdot k + i\varepsilon}\, g_s\, u(q_i)\, t^a\, \bar{\mathcal{M}}^{(n)}$$

abstract notation: $|n+1\rangle_i = g_s \dfrac{q_i\cdot\varepsilon^*}{q_i\cdot k}\, \mathbb{T}^a_i\, |n\rangle$ [Catani Ciafaloni Marchesini; Catani, Seymour]

colour operators

$\mathbb{T}^a_i \to t^a$, $\quad \mathbb{T}^a_i \to i f^a_{bc}$

$$\sum_{\text{spin pol.}}{}' \langle n+1|n+1\rangle = \sum_{\text{spin pol}}{}' \left|\sum_i{}' |n+1\rangle_i\right|^2$$

- it's not difficult to see that all $\langle n+1 | n+1 \rangle_i$ vanish in the soft limit because the sum over the gluon polarisations give $\sim g_{\mu\nu}' \, q_i^\mu q_i^\nu = 0$
- only interference terms survive

$$\sum_{\substack{\text{spin}\\ \text{pol}}}' \langle n+1 | n+1 \rangle = 4\pi\alpha_s \sum_{i,j}' \frac{q_i \cdot q_j}{(q_i \cdot k)(q_j \cdot k)} \langle m | \mathbb{T}_i^a \mathbb{T}_j^a | m \rangle$$

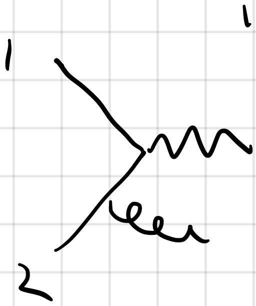
$$\mathbb{T}_i^a \mathbb{T}_j^a \equiv \mathbb{T}_i \cdot \mathbb{T}_j$$

"sum over dipole" formula

- we have achieved some sort of factorisation, but it's much more complicated than in the collinear case.
- colour conservation $\sum_i' \mathbb{T}_i^a | m \rangle = 0$ allows us to simplify things in certain cases

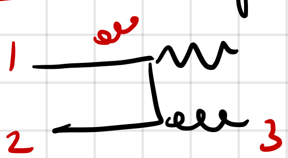
eg $n=2$ $\quad \mathbb{T}_1 + \mathbb{T}_2 = 0$

ie $\cancel{2}\,\mathbb{T}_1 \cdot \mathbb{T}_2 = -\mathbb{T}_1^2 - \mathbb{T}_2^2 = -\cancel{2}\,C_F$



$$\mathbb{T}_1 \cdot \mathbb{T}_2 = -C_F.$$

EXERCISE Diagonalize the colour algebra for $n=3$



or

- In QED, it's a textbook exercise (see eg Weinberg) to show that soft photon emissions factorise and exponentiate.
- in QCD things are complicated by colour (as we've seen) but also by non-Abelian splittings.
- Netherthereless, non-Abelian exponentiation (in a matrix/operator form) can be achieved.

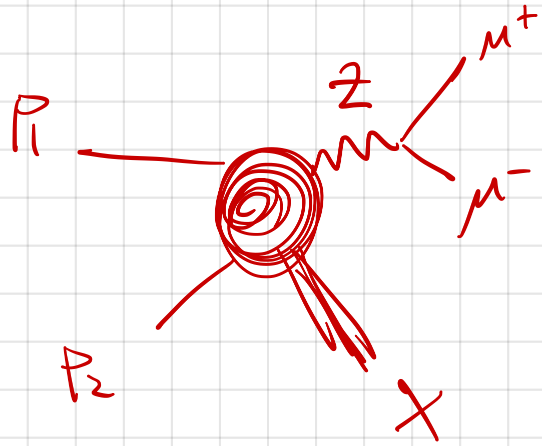
- Because of these complications, in the remaining part of these lectures, we'll mostly work in the collinear limit and only comment about the large angle soft when necessary.
- However, the fun is precisely there: is something can go wrong, it does so in the soft limit where all sort of interesting effects happen

FACTORISATION BREAKING, NON-GLOBAL LOGS, COHERENCE VIOLATION are all SOFT EFFECTS AT the FOREFRONT of QCD RESARCH.

TRANSVERSE MOMENTUM OF THE Z boson

· Now, I've been promising you an actual differential observable.

· Let's go back to our DY process: We would like an observable with the following features



a) "clean" ie easy to measure involving only $\mu^+\mu^-$ system

b) able to probe QCD dynamics

c) IRC SAFE

Let's consider an on-shell Z: $q^2 = m_Z^2$. Lab. frame:

$$q = (m_T \operatorname{ch} y,\ q_{Tx},\ q_{Ty},\ m_T \operatorname{sh} y)\ ;\ m_T = \sqrt{m_Z^2 + q_T^2}$$

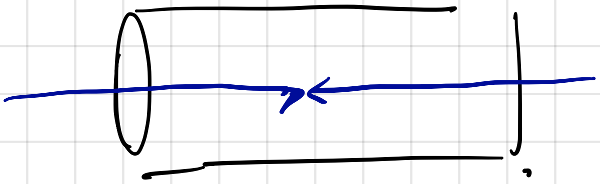
$$q^2 = (m_Z^2 + q_T^2)\operatorname{ch}^2 y - q_T^2 - (m_Z^2 + q_T^2)\operatorname{sh}^2 y = m_Z^2 \ \checkmark$$

i) Rapidity of the Z boson $\quad y = \frac{1}{2} \ln \frac{q^0 + q^3}{q^0 - q^3}$

ii) transverse momentum of the Z boson q_T.

We go for $q_T = |\vec{q}_T| = \sqrt{q_{Tx}^2 + q_{Ty}^2}$

- At $O(\alpha_s^0)$ in collinear fact the incoming quarks are directed along the beam (the "z" direction)



$p_1 + p_2 = q$ momentum conservation dictates $q_T = 0$ (irrespectively of the partonic com frame)

- Thus, a measure of q_T directly probes QCD radiation in X ✓ b)

- We can recontruct q_T from $p^{\mu+}, p^{\mu-}$ ✓ a)

- Is q_T IRC SAFE?

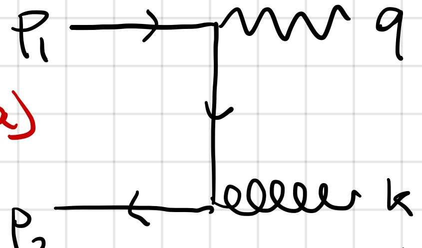
$$\vec{q}_T = \sum_{i=1}^{n} \vec{k}_{Ti}$$

linear in the momenta → collinear safe

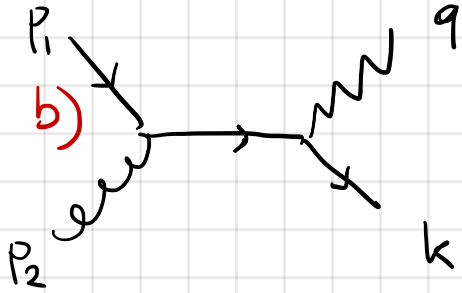
if $\vec{k}_{Ti}$ goes soft, it gives a vanishing contribution to the sum.

YES ✓ c)

· In order to compute the q_T-distribution at $\mathcal{O}(\alpha_s)$, for $q_T > 0$, we only need to consider two processes

a) 

(plus crossed)

and b) 

while  contribute as $\delta^{(2)}(\vec{q}_T)$

. What is the measurement function in this case?

$$\mathcal{O}^{(1)}(\vec{k}_T) = \delta^{(2)}(\vec{q}_T + \vec{k}_T)$$

[indeed $\mathcal{O}^{(1)} \to \mathcal{O}^{(0)} = \delta(\vec{q}_T)$ in soft/coll limit]

· The parton $\dfrac{d\hat{\sigma}}{dq_T}$ distributions can be obtained by integrating: $|\mathcal{M}_a|^2 \delta^{(2)}(\vec{q}_T + \vec{k}_T)$ &

$|\mathcal{M}_b|^2 \delta^{(2)}(\vec{q}_T - \vec{k}_T)$

· We are interested in their $q_T \to 0$ limit. The behaviour is different in the two cases:

$$\frac{d\hat{\sigma}^{(a)}}{dq_T^2} \sim \frac{\alpha_s C_F \sigma_0}{\pi} \int \frac{d\vartheta^2}{\vartheta^2} \int_0^1 \frac{dz}{1-z} \int_0^{2\pi} \frac{d\phi}{2\pi} \, \delta(q_T^2 - k_T^2)$$

$1-z$, ϑ, z

$\delta = (1-z)\vartheta$

$z = 1 - \delta/\vartheta < 1 - \delta$

can be made more precise

$$\sim \sigma_0 \frac{\alpha_s C_F}{\pi} \int_0^1 \frac{d\delta^3}{\delta^2} \int_0^{1-\sqrt{\delta^2}} \frac{dz}{(1-z)} \delta(q_T^2 - \delta^2 Q^2)$$

$$\sim \sigma_0 \frac{1}{q_T^2} \frac{\alpha_s C_F}{\pi} \int_0^{1-\sqrt{q_T^2/Q^2}} \frac{dz}{(1-z)}$$

$$\Downarrow \frac{d\hat{\sigma}^{(a)}}{dq_T^2} \sim \sigma_0 \left[-\frac{\alpha_s}{2\pi} \frac{1}{q_T^2} \ln \frac{q_T^3}{Q^2} + \dots \right]$$

double-logarithmic behaviour associated to soft and collinear behaviour.

- The (b) contribution gives only single-logs.
- The $O(\alpha_s)$ contribution has the following structure (in N-space for convenience) for $q_T^2 \to 0$:

$$\frac{1}{\sigma_0} \frac{d\hat{\Sigma}_1^{(1)}(N, Q^2, q_T^2)}{dq_T^2} = \frac{\alpha_s}{2\pi} \frac{1}{q_T^2} \left[a_1 \ln \frac{Q^2}{q_T^2} + b_1(N) + c_1\left(N, \frac{q_T^2}{Q^2}\right) \right] + O(\alpha_s^2)$$

vanishes for $q_T^2 \to 0$.

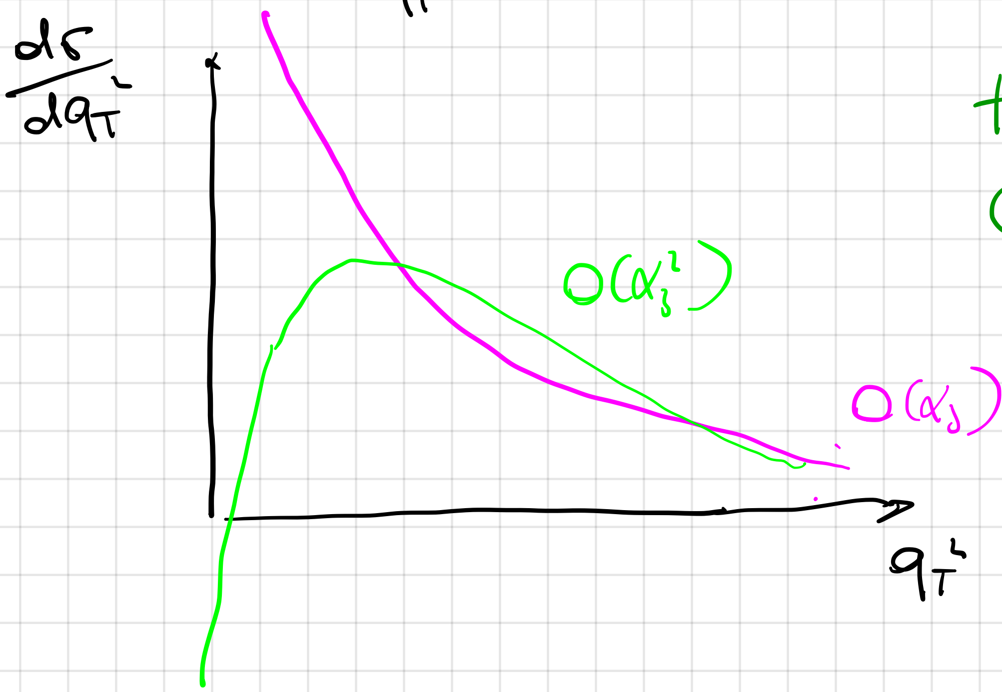
- Thus IRC safe observables can be computed in PT but IRC singularity cancellation leaves behind logs of the observable.
- The situation persists at higher orders:

$$\frac{1}{\sigma_0}\frac{d\hat{\Sigma}}{dq_T^2}(N, q_T^2, Q^2; \alpha_s) \underset{q_T^2 \to 0}{\sim} \frac{1}{q_T^2}\sum_{n=1}^{+\infty}\left(\frac{\alpha_s}{2\pi}\right)^n a_n \ln^{2n-1}\frac{Q^2}{q_T^2}$$

$$+ \frac{1}{q_T^2}\sum_{n=1}^{+\infty}\left(\frac{\alpha_s}{2\pi}\right)^n b_n(N) \ln^{2n-2}\frac{Q^2}{q_T^2}$$

$$+ \dots$$

- so in the region $q_T^2 \ll Q^2$, even if $\alpha_s \ll 1$
  $\alpha_s^n \ln^{2n-1}\frac{Q^2}{q_T^2} \sim 1$, invalidating the pert. expansion



the integral is finite (because of $\delta(q_T^2)$) but the pert. expansion is rubbish.

- We need to re-organise the pert. expansion so that we can account for large logs to all-orders: RESUMMATION

- the new expansion will be in powers of α_s, at fixed $\alpha_s^n \ln^{n+1} q_T^2/Q^2$

· Let's sketch the resummation (beautiful 1979 paper by Parisi and Petronzio).

· We consider the emission of n-collinear gluons (DGLAP evolution will give us collinear quark, as well as contribution (b)).

$$\frac{1}{\sigma_0}\frac{d^2\hat{\Sigma}}{d\vec{q}_T}\left(N, \frac{q_T^2}{Q^2}; \alpha_s\right) \simeq \sum_{n=0}^{+\infty} \frac{1}{n!} \prod_{i=1}^{n} \int \frac{dk_{T_i}^2}{k_{T_i}^2} \int dz_i \int \frac{d\phi_i}{2\pi}$$

$$\frac{\alpha_s^{CMW}(k_{T_i})}{2\pi} P_{qq}(z_i) z_i^{N-1} \Big[\underbrace{\delta^{(2)}\Big(\vec{q}_T + \sum_{i=1}^{n} \vec{k}_{T_i}\Big)}_{\text{REAL}} - \underbrace{\delta^{(2)}(\vec{q}_T)\,\delta_{(1-z)}}_{\text{VIRTUAL}} \Big]$$

needed to fully capture single logs

$$\Theta\left(1 - z_i - \frac{k_{T_i}}{Q}\right)$$

· everything is factorised but the measurement function. However we can use

$$\delta^{(2)}\Big(\vec{q}_T + \sum_{i=1}^{n} \vec{k}_{T_i}\Big) = \frac{1}{4\pi^2} \int d^2 b\, e^{i\vec{b}\cdot\vec{q}_T} \prod_{i=1}^{n} e^{i\vec{b}\cdot\vec{k}_{T_i}}$$

$$\simeq \frac{1}{4\pi^2} \int d^2\vec{b}\, e^{i\vec{b}\cdot\vec{q}_T} \exp\left\{ \int \frac{dk_T^2}{k_T^2} \frac{\alpha_s^{CMW}(k_T^2)}{2\pi} \int dz \int \frac{d\phi}{2\pi} P_{qq}(z) \left[z^{N-1} e^{i\vec{b}\cdot\vec{k}_T} - 1 \right] \Theta\left(1 - z - \frac{k_T}{Q}\right) \right\}$$

$$= \frac{1}{4\pi^2} \int d^2\vec{b}\, e^{i\vec{b}\cdot\vec{q}_T} \exp\left[-R(b)\right] \quad \text{with } b = |\vec{b}|$$

- R has various names: $\left\{ \begin{array}{l} \text{RADIATOR} \\ \text{RESUMMED EXPONENT} \\ \text{SUDAKOV FORM FACTOR} \end{array} \right.$

it resums large logs.

- We can evaluate it to a given log-accuracy: NLL

$$R^{NLL}(b) = \int_{\bar{b}^{-2}}^{Q^2} \frac{dk_T^2}{k_T^2} \frac{\alpha_s}{\pi}(k_T) \left[\left(A^{(1)} + \underbrace{A^{(2)}}_{\text{from CMW}} \frac{\alpha_s(k_T)}{\pi} \right) \ln \frac{Q^2}{k_T^2} + B^{(1)} + 2\gamma_{qq}(N) \right]$$

$$\bar{b} = \left(\frac{b e^{\gamma_E}}{2} \right)$$

$A^{(i)}$ cusp anomalous dimension

$A^{(1)} = C_F$ or C_A

$B^{(1)}$ is related to the end-point of the quark/gluon splitting function.

γ_{qq} can be used to DGLAP evolve the PDFs from $\mu^2 = Q^2$ to $\mu^2 = (\bar{b})^{-2}$

- The integrals over the r.c. should be performed to the required accuracy (in our case with β_0 & β_1)

$$R(b) = L g_0(\alpha_s L) + g_1(\alpha_s L) + \dots$$

$$L = \ln(\bar{b}^2 Q^2)$$

$\to$ logs of q_T are mapped into logs of $\bar{b}$

· The final (hadron-level) resummed formula looks like

$$\frac{d\sigma}{dQ^2 dq_T^2}(\tau, Q^2, q_T^2; \alpha_s) = \sum_{a,b} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^{+\infty} db \, \frac{b}{2} J_0(b q_T)$$

$$W_{ab}^{res}(x_1 x_2 S, Q, b; \alpha_s)$$

$$f_a(x_1, \bar{b}^{-2}) \, f_b(x_2, \bar{b}^{-2})$$

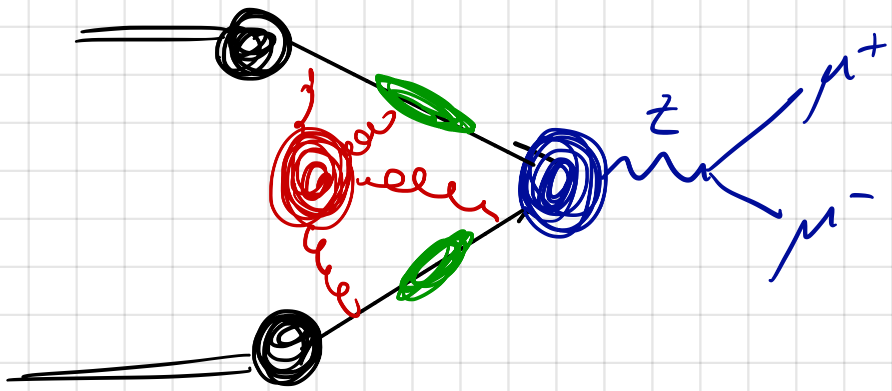
with

collinear functions

$$W_{ab}^{res} = \sum_c \int_0^1 dz_1 \int_0^1 dz_2 \, C_{ca}(z_1; \alpha_s(1/\bar{b}^2)) \, C_{cb}(z_2; \alpha_s(1/\bar{b}^2))$$

hard function

$$\sigma^0_{c\bar{c}\to Z}(Q^2) \, H_c(\alpha_s(Q^2)) \, \underbrace{e^{-R_c(Q, b; \alpha_s)}}_{S_c \text{ sudakov}}$$



· the integral over b is not well-defined (Landau pole)

· in any case the region of very large b ($q_T \sim \Lambda_{QCD}$) will be sensitive to NON-PERT corrections.

· be careful: extension to $gg \to H$ NOT trivial!

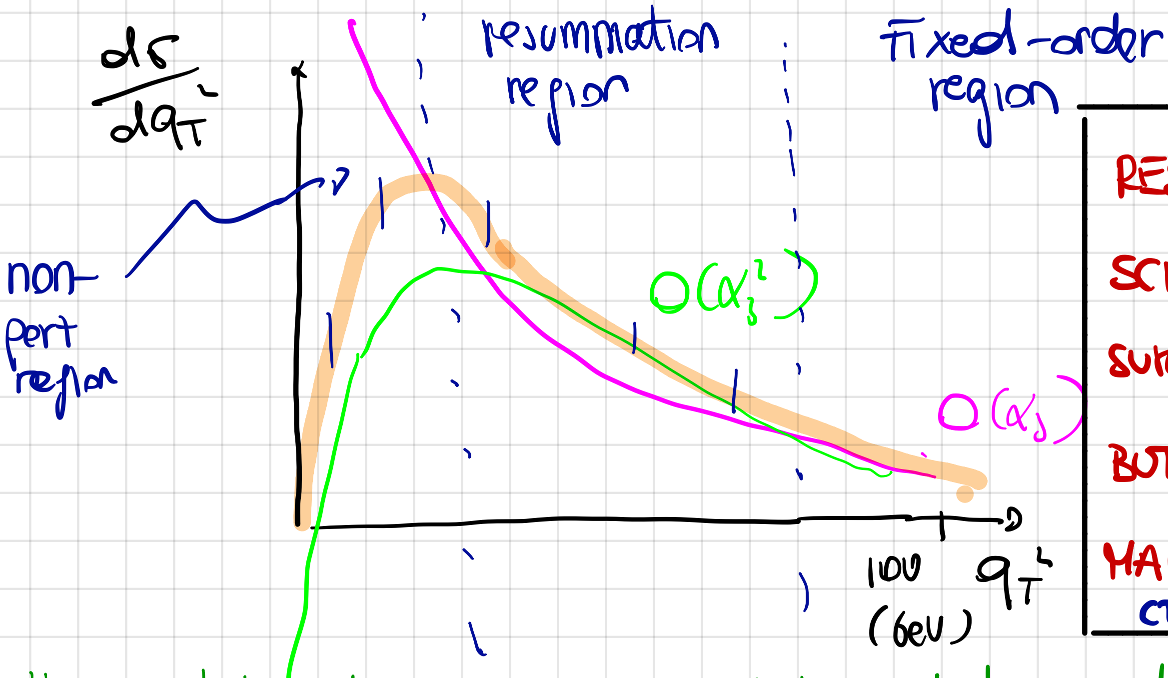
FIXED ORDER (relative to Born)		RESUMMATION (logs of b)	
α_s	LO	$\alpha_s^n L^{n+1}$	LL
α_s^2	NLO	$\alpha_s^n L^n$	NLL
α_s^3	NNLO	$\alpha_s^{n-1} L^n$	NNLL
⋮		⋮	

accurate when $q_T^2 \sim Q^2$, fails when $q_T^2 \ll Q^2$

accurate when $(\Lambda_{QCD}^2 \ll) q_T^2 \ll Q^2$, fails when $q_T^2 \sim Q^2$

BEST of BOTH WORLDS: MATCHING

$$\frac{d\sigma^{MATCH}}{dq_T^2} = \frac{d\sigma^{NNLO}}{dq_T^2} + \frac{d\sigma^{N^3LL}}{dq_T^2} - \text{double counting}$$



RESUMMATION JUST SCRATCHES THE SURFACE OF QCD. BUT IT MAKES A MARK! G. STERMAN CTEQ 2006

- the outstanding precision of the data really pushed the theory community to reach NNLO + N^{3}LL.