• Exercises •

• Chapter 1:

Show that, up to a multiplicative factor, the spinor wave-functions and the polarization vector for, respectively, $s=1/2$ particles and anti-particles, and for $s=1$ particles, are given by the following expressions:

$$u_h = \begin{pmatrix} |\chi^h]_\alpha \\ |\chi^h\rangle^{\dot\alpha} \end{pmatrix} \quad ; \quad \bar{v}_h = v_h^\dagger \gamma^0 = \left([\chi^h|^\alpha , \langle \chi^h|_{\dot\alpha} \right)$$

$$\varepsilon^h_{\alpha\dot\alpha} = \frac{\sqrt{2}}{m} \, \tau^{(1,h)}_{a_1 a_2} \, |\chi^{a_1}]_\alpha \langle \chi^{a_2}|_{\dot\alpha}$$

where the τ tensors are:

$$\tau^{(1,1)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} ; \quad \tau^{(1,-1)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} ; \quad \tau^{(1,0)} = \begin{pmatrix} 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 \end{pmatrix}$$

and the angular and square symbols mean:

$$|\chi^h]_\alpha = \chi^h_\alpha \quad ; \quad [\chi^h|^\alpha = \varepsilon^{\alpha\beta} \chi^h_\beta$$

$$|\chi^h\rangle^{\dot\alpha} = \bar{\chi}^{h,\dot\alpha} \quad ; \quad \langle \chi^h|_{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta} \bar{\chi}^{h,\dot\beta}$$

$$\bar{\chi}^a_{\dot\alpha} = -\left[\varepsilon_{ab} \chi^b_{\dot\alpha} \right]^*$$

where two possible explicit representations of the χ spinors are:

$$\text{JW}: \left[\chi^+_{JW} , \chi^-_{JW} \right] = \begin{bmatrix} \omega_+ \cos\frac{\theta}{2} & -\omega_+ \sin\frac{\theta}{2} \\ \omega_- e^{i\phi} \sin\frac{\theta}{2} & \omega_+ \cos\frac{\theta}{2} \end{bmatrix}$$

where $\omega_\pm = \sqrt{p^0 \pm |\vec{p}|}$ and σ, φ are spherical coordinates for the 3-momentum $\vec{p}$.

Or, another representation is:

$$\text{LF}: \quad \left[\chi^+_{\text{LF}}, \chi^-_{\text{LF}} \right] = \begin{bmatrix} \frac{m}{\sqrt{p_+}} & -\frac{\bar{p}_T}{\sqrt{p_+}} \\ 0 & \sqrt{p_+} \end{bmatrix}$$

where the momentum is expressed in light-cone coordinates.

Hints:

- Prove first that there is only one non-vanishing element in the $\langle 0 | \psi_\alpha | m \mathbb{1}, h \rangle$, $\langle 0 | \bar{\psi}_{\dot{\alpha}} | m \mathbb{1}, h \rangle$, etc tensors that define the wave functions at rest, in the case where $h = s$ ($h = +\frac{1}{2}$, or $+1$) where the particle has maximal helicity.
  You can thus compute the $h = s$ wave-functions at rest, up to a constant

- Use that $| m \mathbb{1}, h \rangle$ offers a standard $SU(2)$ group representation to express each helicity state as the repeated action of J_- (the $SU(2)$ lowering operator) on the $h = s$ state.
  Since the commutator between J_- and the field operators $\psi_\alpha, \bar{\psi}_{\dot{\alpha}}$ etc is known, this enables you to express $h < s$ wave functions in terms of the known $h = s$ wave function.

• Chapter 2 :

Compute the amplitudes for the decay of a massive spin-one particle into massless spin-½ particle and anti-particle. Using the vertex:

$$= i g_L \gamma^\mu \frac{1-\gamma^5}{2} + i g_R \gamma^\mu \frac{1+\gamma^5}{2}$$

you should find:

$$i\mathcal{M}_R^{h=\pm1} = -i\sqrt{2}\, m g_R\, e^{\pm i\phi}\, \frac{1\pm\cos\theta}{2}$$

$$i\mathcal{M}_R^{h=0} = -i\sqrt{2}\, m g_R\, e^{\theta}\, \frac{\sin\theta}{\sqrt{2}}$$

$$i\mathcal{M}_L^{h=\pm1} = -i\sqrt{2}\, m g_L\, e^{\pm i\phi}\, \frac{1\mp\cos\theta}{2}$$

$$i\mathcal{M}_L^{h=0} = +i\sqrt{2}\, m g_L\, e^{\theta}\, \frac{\sin\theta}{\sqrt{2}}$$

when the decaying particle is at rest and its spin $S_z = h$ is measured along the z axis. "θ" and "ϕ" are spherical coordinates for the final state fermion (and not anti-fermion). "L" and "R" refer to the chirality of the fermions.

- Chapter 3 :

Derive the PDF of the electron (parton) in the electron (particle) by following the steps described in the lecture. Specifically:

- Use light-cone coordinates:

$$p = \left(P_+ , \frac{m_e^2}{P_+} , \vec{0} \right)$$

$$K = \left((1-x) P_+ , \frac{|P_T|^2}{(1-x) P_+} , \vec{P}_T \right)$$

and define the on-shell q momentum as:

$$\bar{q} = \left(x P_+ , \frac{m_e^2 + |P_T|^2}{x P_+} , -\vec{P}_T \right)$$

- From writing the numerator of the fermion propagator as:

$$\not{q} + m_e \simeq \not{\bar{q}} + m_e = \sum_h u_h(\bar{q}) \bar{u}_h(\bar{q})$$

one obtains a formula for the splitting amplitude:

$$\mathcal{M}_S^{(h)} = e \, \bar{u}_h(\bar{q}) \gamma^\mu u(P) \, \varepsilon_\mu^*(K)$$

Evaluate it, taking $m_e = 0$, for all combinations of helicities (h, the incoming electron and outgoing photon helicities) for which it is non-vanishing.

- Square the factorized scattering amplitude and integrate over the photon phase-space.