EXAMPLE **HOLOGRAPHIC TWO-POINT FUNCTION**

see e.g. Kiritsis "String theory in a nutshell" ch. 14.8
or Ammon, Erdmenger "Gauge/Gravity duality" ch. 5.4-5.5

$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle_{\text{vacuum}}$ of $\mathcal{O}$ w/ Δ $\longleftrightarrow$ massive bulk scalar Φ w/ $m^2R^2 = \Delta(\Delta-d)$

$EAdS_{d+1}$: $ds^2 = \dfrac{R^2}{z^2}\left(dz^2+dx^2\right)$; $dx^2 = \delta_{\mu\nu}\,dx^\mu dx^\nu$

boundary for $z\to 0$: $\mathbb{R}^d$

+ for simplicity $\underline{R=1}$.

To compute the 2pt fct (2 functional derivatives wrt the source), we can neglect interactions and only keep terms that are <u>quadratic</u> in the Euclidean action:

$$S_E = \frac{1}{2}\int d^dx\,dz\,\sqrt{g}\left(g^{MN}\partial_M\Phi\,\partial_N\Phi + m^2\Phi^2\right)$$

$\Rightarrow$ EoM (KG in EAdS) : for the Fourier transform $\Phi(z,k)$

$$z^2\partial_z^2\Phi - (d-1)\,z\,\partial_z\Phi - z^2k^2\Phi - m^2\Phi = 0$$

This can be solved exactly.

N.B. This is not always possible. The general method uses a <u>perturbative analysis close to the boundary</u>, expanding Φ in powers of $z\to 0$ and solving the eq. iteratively order by order.

In this case the solutions are (MODIFIED) BESSEL FCTS:

$$\Phi_c(z,k) = C_1(k)\, z^{d/2} K_\nu(|k|z) + C_2(k)\, z^{d/2} I_\nu(|k|z)$$

w/ $|k| = \sqrt{\delta_{\mu\nu} k^\mu k^\nu}$

- $z \to \infty$ (AdS interior) since in Euclidean $|k| \geq 0$

$$\Phi_c(z,k) \sim C_1(k)\, z^{\frac{d-1}{2}} e^{-z} + C_2(k)\, z^{\frac{d-1}{2}} \underbrace{e^{z}}_{\text{diverges!}}$$

Requiring regularity in the interior (the saddle pt cannot be singular, otherwise it would not contribute to the path integral)

$$\Longrightarrow \boxed{C_2(k) = 0 \quad \forall k}$$

OBS In Lorentzian, the two solutions oscillate in the interior and we can consider a linear combination of the two.

- $z \to 0$ (AdS boundary)

$$\Phi_c(z,k) \sim C_1(k)\, z^{\frac{d}{2}-\nu} = C_1(k)\, z^{d-\Delta}$$

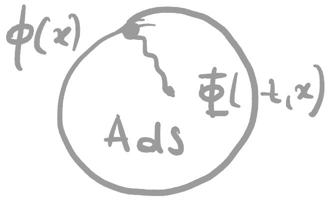
$$\Longrightarrow \quad C_1(k) = \phi(k)$$ bdy value and source for $\mathcal{O}$

In coord. space:

$$\Phi_c(z,x) \sim \phi(x)\, z^{d-\Delta} \qquad w/ \quad m^2 = \Delta(\Delta-d)$$

Let's rewrite the entire sol. explicitly in terms of $\phi(x)$ (arbitrary fct on $\mathbb{R}^d$)

To do it, <u>def</u>: **BULK TO BOUNDARY PROPAGATOR**



which propagates the field in the bulk from its asymptotic value.

Is is def as:

$$\begin{cases} (\Box - m^2)\, K(\overbrace{z,x}^{\text{bulk}}; \underbrace{x'}_{\text{bdy}}) = 0 \\[2ex] \lim\limits_{z\to 0} z^{\Delta-d} K(z,x;x') = \delta^d(x-x') \end{cases}$$

$\Rightarrow$ allows to write:

$$\Phi_c(z,x) = \int_{\partial AdS} d^d x'\, K(z,x;x')\, \phi(x')$$

$$\Longrightarrow K(z,x;x') = C_\Delta \left(\frac{z}{z^2 + |x-x'|^2} \right)^{\Delta} \qquad \text{(CHECK!)}$$

$$w/ \quad C_\Delta \equiv \frac{\Gamma(\Delta)}{\pi^{d/2}\, \Gamma(\Delta - \frac{d}{2})}.$$

Given $\Phi_c(z,x)$ as a fct of BCs $\phi(x)$, we evaluate the <u>on-shell</u> action:

$$S_E[\Phi_c] = \frac{1}{2}\int d^dx\, dz \sqrt{g}\left(g^{MN}\partial_M \Phi_c \partial_N \Phi_c + m^2 \Phi_c^2\right)$$

$$= \frac{1}{2}\int d^dx\, dz \left\{ -\sqrt{g}\,\Phi_c \underbrace{(\Box - m^2)\Phi_c}_{=0 \text{ on-shell}} + \partial_M\left(\sqrt{g}\, g^{MN} \Phi_c \partial_N \Phi_c\right)\right\}$$

$$= \frac{1}{2}\int d^dx \left(\sqrt{g}\, g^{zz} \Phi_c \partial_z \Phi_c\right)\Big|_{z=0}^{z=\infty} \quad \begin{array}{l} z=\infty \text{ term} = 0 \text{ regularity in} \\ \text{the interior}\end{array}$$

$$= -\frac{1}{2}\int_{z=0} d^dx\, z^{1-d}\, \Phi_c \partial_z \Phi_c$$

Since $\Phi_c \sim z^{d-\Delta}$ at the boundary: the argument of the integral goes like $z^{\cancel{-d+1}}\, z^{\cancel{d}-\Delta}\, z^{d-\Delta-\cancel{1}} = z^{d-2\Delta}$

$\Longrightarrow$ <u>diverges</u> for $\Delta > \frac{d}{2}$!

$\Longrightarrow$ Introduce a <u>regulator $z = \epsilon$</u>:

$$S_E^{reg}[\Phi_c]\Big|_{z=\epsilon} = -\frac{1}{2}\int_{z=\epsilon} d^dx\, z^{1-d}\, \Phi_c \partial_z \Phi_c =$$

$$= -\frac{1}{2}\int d^dx_1\, d^dx_2\, \phi(x_1)\phi(x_2) \int_{z=\epsilon} d^dx\, z^{1-d} \times$$

$$\times\, K(z,x;x_1)\, \partial_z K(z,x;x_2)$$

Use:

$$K(z,x;x') \xrightarrow[z\to 0]{} z^{d-\Delta}\delta^d(x-x') + z^{\Delta}\frac{C_\Delta}{|x-x'|^{2\Delta}} + \cdots$$

and observe:

$$\int_{z=\epsilon} d^d x \; z^{1-d} K(x,z;x_1)\,\partial_z K(x,z;x_2) =$$

$$= \underbrace{\epsilon^{1-d}\,\epsilon^{d-\Delta}(d-\Delta)\,\epsilon^{d-\Delta-1}}_{\sim\, \epsilon^{d-2\Delta} \text{ diverges for } \Delta > \frac{d}{2}} \delta^d(x_1-x_2) +$$

$$+ \underbrace{\cancel{\epsilon^{1-d}}\,\cancel{\epsilon^{d-1}}}_{\sim \text{ finite}}(\cancel{\Delta} + d - \cancel{\Delta})\frac{C_\Delta}{|x_1-x_2|^{2\Delta}} + \text{subleading in } \epsilon$$

OBS IR divergences in the bulk perspective

$\updownarrow$ UV/IR correspondence

UV divergences of QFT correlators associated to contact terms.

$\Longrightarrow$ Add a COUNTERTERM to remove the divergence and def:

$$S_E^{ren}[\Phi_c] = \lim_{\epsilon\to 0}\left\{ S_E^{reg}[\Phi_c]\Big|_{z=\epsilon} - S_{CT}[\Phi_c]\Big|_{z=\epsilon}\right\}$$

Here S_{CT} needs to contain:

$$S_{CT}[\Phi_c] \supset \frac{d-2}{2}\, \epsilon^{d-2\Delta} \int d^dx \; \phi(x)^2 \sim$$

$$\underset{\sim}{\underline{\text{covariantly}}} \quad \frac{d-\Delta}{2} \int_{z=\epsilon} d^dx \sqrt{h}\; \Phi_c(z,x)^2$$

w/ $h_{\mu\nu} = \frac{\delta_{\mu\nu}}{\epsilon^2}$ induced metric at $z=\epsilon$.

OBS We are working on a slice at $\underline{z=\epsilon}$, thus to preserve covariance, counterterms need to be built w/ fields at $z=\epsilon$ (scalars, induced metric, Ricci, etc.). They also have to be <u>local</u>, as in QFT (there cannot be for instance radial derivatives).

N.B. We could also add other <u>FINITE CTs</u>, it would correspond to a different <u>renormalization scheme</u> in the QFT.

! CT removes the divergences and contains also <u>finite terms</u>:

$$\Phi_c(z,x)\Big|_{z=\epsilon} = \int_{z=\epsilon} d^dx'\, K(z,x;x')\, \phi(x') =$$

$$= \epsilon^{d-\Delta}\phi(x) + \int d^dx'\, \epsilon^{\Delta}\, C_\Delta \frac{\phi(x')}{|x-x'|^{2\Delta}} + \dots$$

$$\Rightarrow S_{CT}[\Phi_c]\Big|_{z=\epsilon} = \frac{d-\Delta}{2}\int d^d x \; \epsilon^{-d}\Big\{ \epsilon^{2(d-\Delta)}\phi(x)^2 +$$

$$+ 2\epsilon^d \phi_0(x)\int d^d x' \, C_\Delta \frac{\phi(x')}{|x-x'|^{2\Delta}} + \dots \Big\}$$

$$\Rightarrow S^{ren}_{\epsilon}[\Phi_c] = -\frac{1}{2}\int d^d x_1 \, d^d x_2 \, \phi(x_1)\,\phi(x_2) \times$$

$$\times \, C_\Delta \, \frac{d-2d+2\Delta}{|x_1-x_2|^{2\Delta}} =$$

$$= \frac{1}{2}(d-2\Delta)\, C_\Delta \int d^d x_1 \, d^d x_2 \, \frac{\phi(x_1)\,\phi(x_2)}{|x_1-x_2|^{2\Delta}}$$

$$\Longrightarrow \boxed{\langle \mathcal{O}(x_1)\,\mathcal{O}(x_2)\rangle = \frac{c}{|x_1-x_2|^{2\Delta}}}$$

as in CFT ✓

+ Expectation value in presence of a source:

$$\boxed{\langle \mathcal{O}(x_1)\rangle_\phi = (2\Delta-d)\int d^d x_2 \, \frac{C_\Delta\,\phi(x_2)}{|x_1-x_2|^{2\Delta}}}$$

which reconsidering:

$$\Phi_c(z,x) = \int_{\partial AdS} d^d x' \, K(z,x;x') \, \phi(x')$$

and $K(z,x;x') \xrightarrow[z\to 0]{} z^{d-\Delta}\delta^d(x-x') + z^{\Delta} \dfrac{C_\Delta}{|x-x'|^{2\Delta}}$

we see being precisely:

$$\boxed{\langle \mathcal{O}(x) \rangle_\phi = 2\nu \, B(x)}$$

w/ $\nu = \sqrt{\frac{d^2}{4} + m^2 R^2}$

$B(x)$: normalizable mode

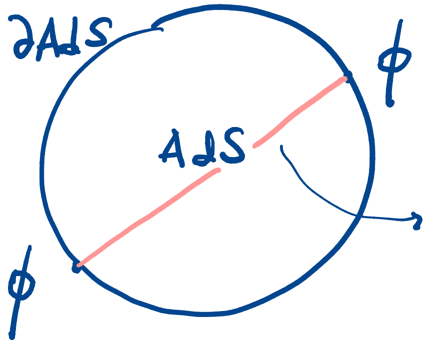
as anticipated. The normalizable mode fixes the expectation value of the dual operator.

OBS 1 This is just the simplest example of a systematic procedure known as HOLOGRAPHIC RENORMALIZATION, which allows to evaluate the renormalized on-shell action for any field (scalar, metric, gauge fields) in AAdS solutions.

See K. Skenderis et al: ex lecture notes hep-th/0209067.

OBS 2

$\langle \mathcal{O}\mathcal{O} \rangle$: can be computed as Feynman diagram in AdS w/ extremes on ∂AdS

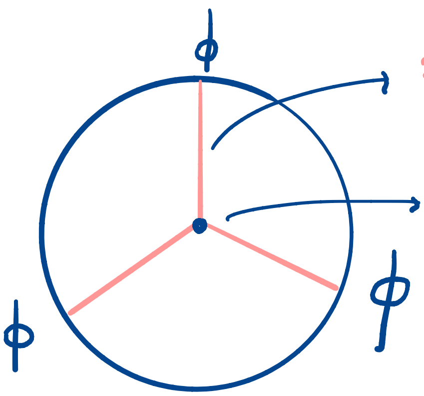


ϕ source for op. $\mathcal{O}$

BOUNDARY TO BOUNDARY PROPAGATOR

Feynman's diagrams for tree-level correlators in AdS are known as WITTEN DIAGRAMS.

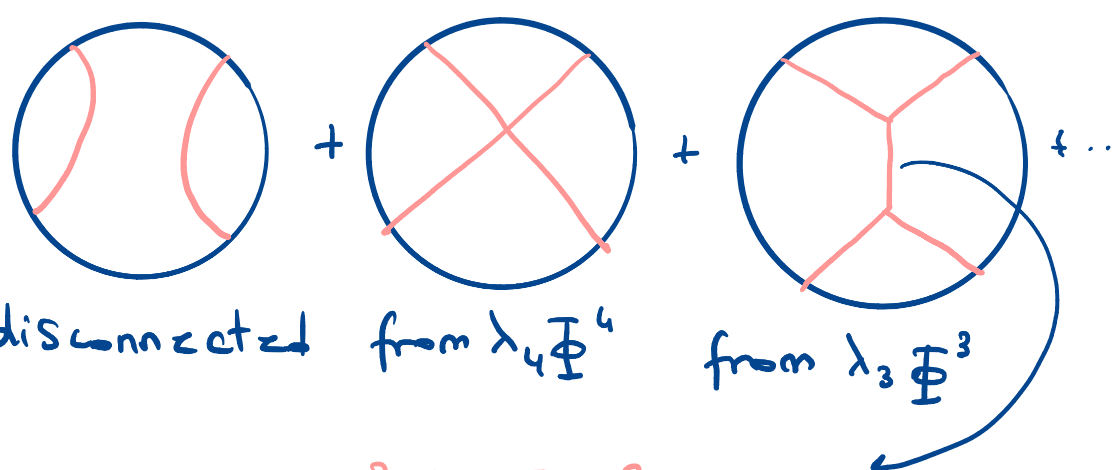
$\langle \mathcal{O}\mathcal{O}\mathcal{O} \rangle$:



BULK TO BOUNDARY PROPAGATOR

bulk interactions governed by the interaction terms that appear in (super)gravity $\mathcal{L}$.

$\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O} \rangle$:



disconnected   from $\lambda_4 \Phi^4$ from $\lambda_3 \Phi^3$

BULK-TO-BULK PROPAGATOR