Diffeomorphism - invariant theories

- the formulae presented so far are completely general; we will now specialize to the case of the gauge symmetry = diffeomorphism invariance

- will work in terms of forms: Lagrangian $\underset{\text{scalar}}{L} \to \underset{\text{d-form}}{\mathcal{L}^{(d)}} = \frac{1}{d!}\varepsilon_{\mu_1\ldots\mu_d} L \, dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_d}$

  currents : $J^\mu, \Theta^\mu \to J^{(d-1)}, \Theta^{(d-1)}$, etc

  Noether - Wald surface charge: $Q^{\mu\nu} \to Q^{(d-2)}$

- under a diffeo, $\delta_\xi X = \mathcal{L}_\xi X$

- repeat the previous manipulations in form language

$$\delta \mathcal{L}^{(d)} = E(\phi)\,\delta\phi + d\Theta^{(d-1)}(\delta\phi) \qquad \text{general variation}$$

$$\delta_\xi \mathcal{L} = dM_\xi = d(\underbrace{\xi\cdot\mathcal{L}}_{\xi^\alpha \varepsilon_{\alpha\mu_1\ldots\mu_{d-1}} L \, dx^{\mu_1}\wedge\ldots dx^{\mu_{d-1}}}) \qquad \text{under a diffeomorphism } \xi^\mu$$

interior product $(:\Omega_p \to \Omega_{p-1})$

$\xi\cdot\omega = i_\xi \omega = \omega(\xi,\ldots)$

this boundary term comes from $\delta_\xi L = \xi^\mu \partial_\mu L \qquad \delta_\xi \sqrt{g} = \nabla_\mu \xi^\mu \sqrt{g}$

then $\quad J_\xi(\phi) = \Theta(\delta_\xi\phi) - \xi\cdot\mathcal{L}$

- equating the first two variations, we find $E(\phi)\,\delta_\xi\phi = dS_\xi(\phi) = -dJ_\xi(\phi)$

  on-shell vanishing Noether current (for S_ξ)

$$\Rightarrow \quad S_\xi = -J_\xi - dQ_\xi \qquad (\text{off-shell}) \quad \Rightarrow \text{expr. for } Q_\xi^{(d-2)}$$

- notice dQ_ξ need not vanish on-shell $\to$ not clear it will give rise to a conserved charge

- need to find some related quantity, Q'_ξ, which does satisfy $dQ'_\xi = 0$ on-shell

- as we will show, a quantity related to the infinitesimal variation δQ_ξ between two field configurations is closed on-shell

Consider the charge difference between two nearby field configurations, ϕ & $\phi+\delta\phi$.

$$\delta S_\xi = -\delta J_\xi - \delta d Q_\xi = -\delta\big(\Theta(\underbrace{\delta_\xi\phi}_{\mathcal{L}_\xi\phi}) - \xi\cdot\mathcal{L}\big) - d\,\delta Q_\xi(\phi)$$

on-shell

$$= \underbrace{-\delta\Theta(\delta_\xi\phi) + \delta_\xi\Theta(\delta\phi)}_{\equiv\, \omega(\delta_\xi\phi,\,\delta\phi)} \underbrace{- \delta_\xi\Theta(\delta\phi) + \xi\cdot d\Theta(\delta\phi)}_{-d(\xi\cdot\Theta(\delta\phi))} - d\,\delta Q_\xi(\phi)$$

presymplectic current

identity $\mathcal{L}_\xi F = d(\xi\cdot F) + \xi\cdot dF$

(in comp $\mathcal{L}_\xi F_{\mu_1\ldots\mu_n} = \xi^\lambda\partial_\lambda F_{\mu_1\ldots\mu_n} + \partial_{\mu_i}\xi^\lambda F_{\ldots\lambda\ldots}$)

- $\omega(\delta_1\phi, \delta_2\phi) \equiv \delta_1\Theta(\delta_2\phi) - \delta_2\Theta(\delta_1\phi)$ antisymmetric

- since $S_\xi = 0$ on-shell, we have $\boxed{\omega(\delta_\xi\phi,\delta\phi) + dK_\xi(\phi,\delta\phi) = 0}$

  where $\boxed{K_\xi(\phi,\delta\phi) \equiv -\delta Q_\xi(\phi) - \xi\cdot\Theta(\delta\phi)}$

- the quantity $K_\xi(\phi,\delta\phi)$ is closed on-shell if $\omega(\delta_\xi\phi,\delta\phi)$ vanishes, which happens if $\delta_\xi\phi \approx 0$ in the region of interest $\Rightarrow$ it does give rise to a conserved quantity = charge difference between two nearby field configurations

- the surface charge difference $\boxed{\not\delta H_\xi(\delta\phi,\phi) \equiv \int_{\partial\Sigma} K_\xi(\phi,\delta\phi)}$

  where the symbol $\not\delta$ stands for the fact that in general $\not\delta Q_\xi$ is not δ(something) (unlike in E&M)

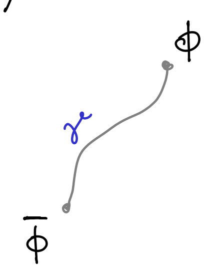
- to obtain the full charge, we need to integrate the charge difference in field space, from a reference backgnd. $\bar\phi$ to the desired backgnd. ϕ

final formula for the conserved charge

$$\boxed{H_\xi(\phi,\bar\phi) = \int_\gamma\int_{\partial\Sigma} K_\xi(\phi,\delta\phi) + N_\xi(\bar\phi)}$$

path in field space btw/ $\bar\phi$ & ϕ

integration ct. (unfixed by this formalism)



Example : G.R.

$$S = \frac{1}{16\pi G}\int d^dx \underbrace{R\sqrt{g}}_{L} \qquad \delta L = -\frac{1}{16\pi G} G^{\mu\nu}\underbrace{\delta g_{\mu\nu}}_{h_{\mu\nu}} + \partial_\mu \Theta^\mu \qquad \Theta^\mu = \frac{1}{16\pi G}\sqrt{g}\left(\nabla_\nu h^{\mu\nu} - \nabla^\mu h\right)$$

$$J^\mu = \Theta^\mu(\delta_\xi g_{\mu\nu}) - M^\mu_\xi = \frac{\sqrt{g}}{16\pi G}\left(\nabla_\nu(\nabla^\mu\xi^\nu + \nabla^\nu\xi^\mu) - 2\underbrace{\nabla^\mu\nabla_\nu}_{\nabla_\nu\nabla^\mu + R_\rho{}^{\nu\mu}{}_\nu\xi^\rho}\xi^\nu\right) - \frac{\sqrt{g}}{16\pi G} R\,\xi^\mu$$

$$= \frac{\sqrt{g}}{16\pi G}\left[\nabla_\nu\left(\nabla^\nu\xi^\mu - \nabla^\mu\xi^\nu\right) + \underbrace{2R^\mu{}_\nu\xi^\nu - R\,\xi^\mu}_{2G^\mu{}_\nu\xi^\nu}\right]$$

The on-shell vanishing Noether current is given by

$$E\,\delta_\xi g = -\frac{1}{16\pi G} G^{\mu\nu}\nabla_\mu\xi_\nu \times 2 = \nabla_\mu\Big(\underbrace{-\frac{1}{8\pi G} G^{\mu\nu}\xi_\nu}_{S^\mu}\Big) + \frac{1}{8\pi G}\underbrace{\nabla_\mu G^{\mu\nu}}_{0 \text{ by N.I.}}\xi_\nu$$

so we see that, indeed, $J^\mu = -S^\mu - \sqrt{g}\,\nabla_\nu \frac{Q^{\mu\nu}}{\sqrt{g}}$ Check! w/

$$\boxed{Q^{\mu\nu} = -\frac{\sqrt{g}}{16\pi G}\left(\nabla^\nu\xi^\mu - \nabla^\mu\xi^\nu\right)}$$

Komar term : used to compute mass, angular momentum of simple spacetimes

for more complicated situations, we need $K_\xi = \delta Q_\xi - \xi\cdot\Theta(\delta g)$ w/

$$K_\xi^{\mu\nu}(h,g) = \frac{\sqrt{g}}{16\pi G}\left(\xi^{[\mu}\nabla_\sigma h^{\nu]}{}_\sigma - \xi^{[\mu}\nabla^{\nu]}h + \xi_\sigma\nabla^{[\mu}h^{\nu]\sigma} + \frac{1}{2}h\,\nabla^{[\nu}\xi^{\mu]} - h^{\rho[\nu}\nabla_\rho\xi^{\mu]}\right)$$

- for some rather general formulae for K_ξ in Einstein gravity coupled to matter fields, see G. Compère 0902.1001

- to compute e.g. the mass of a Schwarzschild spacetime → consider backgnds of varying M (parameter in the Schwarzschild metric), so $g_{\mu\nu}(M) = \mathrm{Schw}(M)$ & $h_{\mu\nu} = g_{\mu\nu}(M+\delta M) - g_{\mu\nu}(M)$. Computing $\int K_\xi(h,g) \to \delta M$, which is trivially integrable

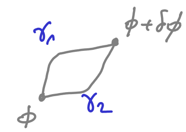
- since this is an algorithmic procedure, can compute conserved charges for arbitrary theories of gravity ($\forall$ matter, $\forall$ higher derivatives)

Properties

i) integrability : in order that the charge so defined to make sense, it should be *independent of the path* in field space used to define it.

This leads to an integrability condition, which infinitesimally reads:

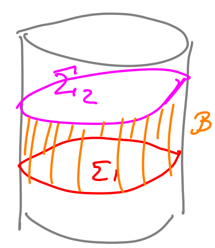
$$\delta_1 \int_{\partial\Sigma} k_\xi(\phi, \delta_2\phi) - \delta_2 \int_{\partial\Sigma} k_\xi(\phi, \delta_1\phi) = 0 \quad , \forall\, \delta_1\phi, \delta_2\phi$$



ii) conservation : remember that $\omega(\delta_\xi\phi, \phi) + dk_\xi(\phi, \delta\phi) = 0$
(valid on-shell)

· the charge $\not\delta Q_\xi = \int_{\partial\Sigma} k_\xi(\phi, \delta\phi)$ will be cons.

if $\omega(\delta_\xi\phi, \delta\phi)\big|_{\partial\Sigma} = 0$



a) if ξ is a Killing vector of the backgnd $\Rightarrow \delta_\xi\phi = 0$

$\Rightarrow \omega(\delta_\xi\phi, \delta\phi) = 0$ *everywhere*

$\Rightarrow \not\delta Q_\xi$ is not only conserved in time, but also can be evaluated by integration over *any* radial surface on a $t = \text{const.}$ slice, since

$$\int_{\partial\Sigma_1} k_\xi = \int_{\partial\Sigma_2} k_\xi + \int_{\Sigma_1\setminus\Sigma_2} \underbrace{dk_\xi}_{-\omega = 0} = \int_{\partial\Sigma_2} k_\xi$$

b) it is often sufficient that ξ be a Killing vector ($\delta_\xi\phi \to 0$) only *asymptotically*, w/ $\omega(\delta_\xi\phi, \delta\phi)\big|_{\substack{\partial\Sigma \\ r\to\infty}} = 0$. Then $\not\delta Q_\xi$ is conserved, but it needs to be evaluated as $r \to \infty$.

In this case, ξ is called an *asymptotic symmetry*

Observation : In general, the task of specifying boundary conditions on the behaviour of various fields near $\partial\Sigma$ so that it can be established that ω does not contribute to the charge for a general class of configurations is a complicated one, and is usually analysed on a case-by-case basis.

- a nice feature of the current formalism is that it provides a general formula for the conserved charges that is independent of the spacetime under consideration. The details of the spacetime enter only when specifying the asymptotic falloffs
- the vector field ξ^μ is an asymptotic symmetry provided that the charge δQ_ξ associated w/ it is generically non-zero. Otherwise, ξ^μ = trivial diffeomorphism. An asymptotic symmetry corresponds in fact to an equivalence class of vector fields, which differ by the addition of a trivial diffeo

Asymptotic symmetry group :

- consider a space-time where the metric & other fields, collectively denoted as ϕ, satisfy a set of boundary conditions, $\mathcal{C}$ ← defines phase space
- a vector ξ^μ is an allowed diffeomorphism if $\mathcal{L}_\xi \phi \in \mathcal{C}$, $\forall\, \phi \in \mathcal{C}$
- the algebra of these vectors is given by the Lie bracket $[\xi_a, \xi_b]_{L.B} = C_{ab}^{\ \ c}\, \xi_c$
- it is assumed the bnd. cond. have been chosen $\ni$ $Q_\xi[\phi]$ are finite, integrable and conserved for all allowed diffeomorphisms (usually, this means ξ_{allowed} asymptotically satisfy the Killing eqn.)
- a diffeomorphism is called trivial if $Q_\xi[\phi] = 0$, $\forall\, \phi \in \mathcal{C}$.

• the asymptotic symmetry group

$$\boxed{ASG \equiv \frac{\text{Allowed diffeos}}{\text{Trivial diffeos}}}$$

equivalence classes of diffeos that act non-trivially on states (true symm.)

Remarks : if Q_ξ are not finite / integrable / conserved → need more stringent bnd cond.

• usually, the bnd. conditions are determined by the generic falloffs of solutions to the e.o.m. in the given spacetime (universal near ∞)

• however, many times the falloffs are difficult to characterize → → choose $\mathcal{C}$ as lax as possible $\ni$ Q_ξ is still finite

Note these are very general guidelines. The appropriate bnd. cond. to impose should be analysed on a case-by-case basis, in function of the physical situation @ hand. For example,

• sometimes, the lack of integrability/conservation is physical, e.g. when there is flux passing through ∂M

• other times, it may be resolved by adding a counterterm

• sometimes, the charges can be made finite, integrable by restricting the phase space ↦ are these restricted bnd. cond. physically motivated?

• should also check that the symplectic form is positive definite

iii) charge algebra. One can define a commutator of the charges via

$$\{\mathcal{H}_\chi, \mathcal{H}_\xi\} \equiv \delta_\xi \mathcal{H}_\chi = \int_{\partial\Sigma} \mathcal{K}_\chi [\delta_\xi \phi, \phi]$$

- one can prove the following representation theorem: the algebra of the conserved charges $\mathcal{H}_\xi$ associated w/ $\xi \in ASG$ is given by (see e.g. Geoffrey's lecture notes)

$$\{\mathcal{H}_\chi, \mathcal{H}_\xi\} = \underbrace{\mathcal{H}_{[\chi,\xi]_{L.B}}}_{\approx \text{ Lie bracket of the diffeos}} + \underbrace{\mathbb{K}_{\chi;\xi}[\bar{\phi}]}_{\text{central extension}}$$

cannot be absorbed into the normalization of the charges (shift N_ξ)

$\approx$ the algebra of the conserved charges = Lie bracket algebra of the associated diffeos up to a central extension

- if the diffeos are field-dependent → modified Lie bracket (Barnich, Troessaert 1001.1541)

iv) residual ambiguities : to what extent is the charge δQ_ξ fixed?

- adding a bnd. term to the Lagrangian $\mathcal{L} \to \mathcal{L} + d\mu$ does not change the e.o.m, but changes $\Theta \to \Theta + \delta\mu$. Note ω is not affected.

- since Θ is defined via $\delta\mathcal{L} = E\,\delta\phi + d\Theta$, it is ambiguous up to a boundary term $\Theta \to \Theta + dY^{(d-2)}(\delta\phi)$

under this $\omega^{(d-1)} \to \omega + dB^{(d-2)}$, w/ $B(\delta_1\phi, \delta_2\phi) = \delta_1 Y(\delta_2\phi) - \delta_2 Y(\delta_1\phi)$

- the Noether current $J_\xi = \Theta(\delta_\xi\phi) - \xi\cdot\mathcal{L}$ is then affected as

$$J_\xi \to J_\xi + \underbrace{\delta_\xi \mu}_{\xi\cdot d\mu + d(\xi\cdot\mu)} + dY(\delta_\xi\phi) - \xi\cdot d\mu = J_\xi + d\left(Y(\delta_\xi\phi) + \xi\cdot\mu\right)$$

- using the def^n $J_\xi = -S_\xi - dQ_\xi$, we see that

$$Q_\xi \to Q_\xi - Y(\delta_\xi\phi) - \xi\cdot\mu - dZ$$ ← new ambiguity

- finally, $\mathcal{K}_\xi(\phi,\delta\phi) = -\delta Q_\xi - \xi\cdot\Theta(\delta\phi)$, so

$$\mathcal{K}_\xi \to \mathcal{K}_\xi + \delta Y(\delta_\xi\phi) + \cancel{\xi\cdot\delta\mu} + d\delta Z - \cancel{\xi\cdot\delta\mu} - \xi\cdot dY$$

since $\delta_\xi Y(\delta\phi) = d(\xi\cdot Y(\delta\phi)) + \xi\cdot dY(\delta\phi)$, we find

$$\mathcal{K}_\xi \to \mathcal{K}_\xi + \underbrace{B(\delta\phi,\delta_\xi\phi)}_{\text{irrelevant for exact symm}} + d\big(\underbrace{\delta Z + \xi\cdot Y(\delta\phi)}_{\text{only contributes if } \partial\Sigma \text{ has bnd}}\big)$$

so that $\omega(\delta_\xi\phi,\delta\phi) + d\mathcal{K}_\xi(\delta\phi) = 0$

Summary:

- I presented a very general, covariant formalism for computing conserved charges in gauge theories

- necessity for lower-form currents clearly encoded ($J_\xi = -dQ_\xi$ on-shell)

- the formalism provides a bridge between the Lagrangian formulation (covariant) & a Hamiltonian one (physical degrees of freedom are clear) through the covariant construction of a symplectic form $\Omega = \int_{\Sigma} \omega$

- extremely powerful & <u>algorithmic</u>: given $\forall$ action (Einstein grav $+ R^{17}$ corr. $+$ matter, etc.) $\rightarrow K_\xi \rightarrow \delta\!\!\!/ Q_\xi$, up to the ambiguity $B(\delta\phi, \delta_\xi \phi)$ (not present for exact symm).

- however, technically heavy and not physically very transparent $\rightarrow$ "recipe" for computing conserved charges (using a computer)

- for consistency, need to check that $\delta\!\!\!/ Q_\xi$ is finite, conserved (i.e. $\omega(\delta_\xi \phi, \delta\phi)\big|_{\partial\Sigma} = 0$) and integrable. This is where the details – more precisely, the asymptotic str. – of the space-time under consideration will enter.

  The bnd. term Y ($\rightarrow B$) may be adjusted to achieve some of these properties.

- the set of asymptotic vector fields ξ^μ (modulo trivial diffeomorphisms) whose action respects the bnd. cond. on the fields are called large diffeomorphisms and represent true symmetries of the gravity theory (analogues of global symm)

<u>Why are ASG's interesting?</u>

i) true symmetries of the gravity theory (constrain observables)

ii) holography

Quantum Gravity on $\mathcal{M}$ = Field theory on $\partial\mathcal{M}$

asympt. symm $\longleftrightarrow$ global symm

- the ASG of a given gravitational theory in a given spacetime strongly depends on the asymptotic structure of the spacetime, but not so much on the particular theory under consideration (most important: gauge fields)

- this universality implies that ASG $\Rightarrow$ "type" of QFT that lives on the boundary (e.g. AdS/CFT)

- the ASG gives powerful constraints on the dual QFT, especially when ∞-dimensional (far from enough to fix QFT, but still useful)

- will discuss two examples:

    - AdS_3, ASG = Virasoro $\times$ Virasoro

        - same as symm. CFT_2 $\Rightarrow$ hint for an universal AdS_3/CFT_2 corresp.

    - $Mink_4$ ASG = BMS (Bondi, van der Burg, Metzner, Sachs)

        - infinite dimensional!

        - measurable effects of asympt. diffeos

        - interesting connections to soft theorems

        - ∞ degeneracy of the Minkowski vacuum