Example 1: Conserved charges & asymptotic symmetries of AdS_3

i) definition of AdS_{d+1}, $\forall d$ & Penrose diagram

ii) the Fefferman-Graham expansion

iii) AdS_3 w/ Dirichlet (or Brown-Henneaux) bnd. cond

- mass & angular momentum of stationary sols. in global AdS_3 : vacuum, conical defect space-times, black holes
- ASG.

i) AdS_{d+1} : maximally symmetric space-time of constant negative curvature $\frac{(d+1)(d+2)}{2}$ K.V.

$$R_{\mu\nu\rho\sigma} = -\frac{1}{\ell^2}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right)$$

- isometry group $SO(d,2)$ (for Lorentzian AdS_{d+1}) = same isometry as $\mathbb{R}^{d,2}$

- simplest construction : embed into $\mathbb{R}^{d,2}$, w/ metric

$$ds^2 = \tilde{\eta}_{MN}\, dX^M dX^N \qquad \tilde{\eta}_{MN} = \begin{pmatrix} - & & & & \\ & - & & & \\ & & + & & \\ & & & \ddots & \\ & & & & + \end{pmatrix} \qquad \text{manifest } SO(d,2)$$

& restrict to the hyperboloid $\tilde{\eta}_{MN} X^M X^N = -\ell^2$

- a metric on AdS_{d+1} can be obtained by writing down an explicit coord. system that solves this constraint, e.g.

global coordinates : $\begin{cases} X^0 = \ell \cosh\rho \cos\tau \\ X^1 = \ell \cosh\rho \sin\tau \end{cases}$ $\qquad X^i = \ell \sinh\rho\, \Omega_i$ w/ $\sum_{i=2}^{d+1} \Omega_i^2 = 1$

$$ds^2 = \ell^2\left(-\cosh^2\rho\, d\tau^2 + d\rho^2 + \sinh^2\rho\, \underbrace{d\Omega^2_{d-1}}_{\text{unit } S^{d-1}}\right)$$

& $SO(d)$ manifest

after decompactifying the τ coordinate

- AdS space-time is non-compact : how can we represent it?

- tool : conformal compactification
  Given a manifold $\mathcal{M}$ w/ metric $g_{\mu\nu}$, one looks for a conformal isometry
  $\varphi : (\mathcal{M}, g) \to (\tilde{\mathcal{M}}, \tilde{g})$, i.e. a change of coordinates $\ni$ $\tilde{g}_{\mu\nu} = \Omega^2(x)\, g_{\mu\nu}$, chosen so that $\varphi(\infty)$ is now @ finite distance in $\mathcal{M}'$. The bnd. of $\mathcal{M}'$ is called conformal infinity

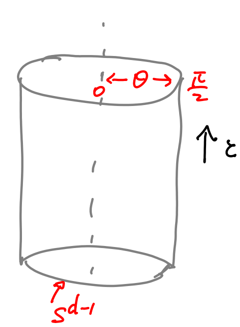
- since the coordinate ranges are compact, $\mathcal{M}'$ can be drawn. Moreover, since $g' = \Omega^2 g$, its causal structure (determined by the trajectories of null rays) is the same as that of $\mathcal{M}$, even though distances can be severely distorted => Penrose diagram

- $\tilde{\mathcal{M}}$ can also provide a coordinate-invariant way of charaterizing the asympt. structure of $\mathcal{M}$ (coord-invar way of char. asymptotic falloffs of various fields)

- to construct the Penrose diagram for AdS_{d+1} :

  - introduce a new compact radial coord. θ via $\cosh\rho = \frac{1}{\cos\theta}$ $\left(\rho \in [0,\infty) \Rightarrow \theta \in [0, \frac{\pi}{2})\right)$

$$ds^2 = \frac{\ell^2}{\cos^2\theta}\left(-d\tau^2 + \underbrace{d\theta^2 + \sin^2\theta\, d\Omega^2_{d-1}}_{\text{metric on half } S^d \,\approx\, \text{disk}}\right)$$



$$d\tilde{s}^2 = \cos^2\theta\, ds^2 \simeq \text{disk} \times \mathbb{R}_t$$

$\Rightarrow$ Penrose diagram of AdS_{d+1} = infinite solid cylinder w/ bnd. $S^{d-1} \times \mathbb{R}_\tau$

Exercise 1) Show that lightrays shot from the center of AdS take a finite coordinate time to reach the bnd. Compute it.

You can assume, for simplicity, that all motion takes place in the radial plane.

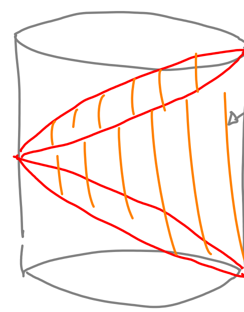
2) Show that massive particles' geodesics never reach the boundary.

- we thus find that AdS behaves "like a box". Dirichlet bnd. cond. @ the timelike bnd. are natural for massive fields, whereas they can be imposed for massless ones.

(N.B. In AdS, one can have causal fields of slightly negative m^2. For these fields in the "conformal window", one has a choice between Dirichlet, Neuman or mixed bnd. conditions)

- there are other useful coord. systems on AdS, e.g. Poincaré

$$ds^2 = \frac{\ell^2}{z^2}\left(\underbrace{\eta_{\mu\nu}\, dx^\mu dx^\nu}_{\mathbb{R}^{d-1,1}} + dz^2\right)$$



which only cover a patch of the AdS spacetime
(useful for making the Lorentz & dilatation symmetry along the bnd. manifest. The conformal bnd. is at $z=0$)

ii) The Fefferman - Graham expansion

- the above analysis was for vacuum AdS. We would now like to gain an understanding of the general asymptotic behaviour of fields near the AdS boundary, which will determine the natural boundary conditions to impose

- we do this by choosing a convenient coordinate system, namely the Fefferman-Graham gauge (good near the boundary @ $z=0$)

$$ds^2 = \ell^2 \frac{dz^2}{z^2} + g_{\mu\nu}(x^\mu, z)\, dx^\mu dx^\nu$$

· the solution to Einstein's eqns ($R_{MN} = -\frac{d}{\ell^2} g_{MN}$) order by order in the z expansion is

$\log z$ in even $d \neq 2$

$$g_{\mu\nu}(z, x^\mu) = \frac{g^{(0)}_{\mu\nu}(x^\mu)}{z^2} + g^{(2)}_{\mu\nu} + z^2 g^{(4)}_{\mu\nu} + \dots + z^{d-2} g^{(d)}_{\mu\nu} + \dots$$

arbitrary — det. by deriv.s of $g^{(0)}$ — new data (partially constrained) — det by $g^{(0)}, g^{(d)}$

· see e.g. Henningson & Skenderis 9806087; de Haro, Solodukhin & Skenderis 0002230

· $g^{(0)}_{\mu\nu}$ → representative of the class of metrics induced on the conformal bnd ($g^{(0)} \sim \Omega^2 g^{(0)}$)

· if $g^{(0)}_{\mu\nu} \propto \eta_{\mu\nu}$ then the spt is asymptotically AdS; if not, then asympt. locally AdS

· a similar expansion holds for matter fields → free near AdS bnd.

· this extremely simple form of the expansion is very special to AdS & this particular coordinate system (see e.g. Poole, Skenderis, Taylor 1812.05369 for AdS asymptotics in "Bondi" gauge)

AdS$_3$

· we will study the AdS$_3$ case in more detail

$$ds^2 = \ell^2 \frac{dz^2}{z^2} + \left(\frac{g^{(0)}_{\alpha\beta}}{z^2} + g^{(2)}_{\alpha\beta} + \dots \right) dx^\alpha dx^\beta$$

w/ $g^{(0)}$ arbitrary & $\quad g^{(0)\alpha\beta} g^{(2)}_{\alpha\beta} = -\frac{\ell^2}{2} R[g^{(0)}]$; $\nabla^{(0)}_\alpha g^{(2)\alpha\beta} = \nabla^{(0)}_\beta g^{(2)\alpha}{}_\alpha$ — constraints due to asympt. Einstein eqns.

"holographic Ward identities"

· in presence of matter fields, the ... are not universal ($G_{MN} = 8\pi G\, T_{MN}$ w/ bnd. conditions on the matter so that it carries finite energy)

- in pure gravity ($R_{MN} = -\frac{2}{\ell^2} g_{MN}$ everywhere), the expansion terminates at 2nd order

$$ds^2 = \ell^2 \frac{dz^2}{z^2} + \left(\frac{g^{(0)}_{\alpha\beta}}{z^2} + g^{(2)}_{\alpha\beta} + z^2 g^{(4)}_{\alpha\beta} \right) dx^\alpha dx^\beta \quad \text{w/} \quad g^{(4)}_{\alpha\beta} = \frac{1}{4} g^{(2)}_{\alpha\gamma} g^{(0)\gamma\delta} g^{(2)}_{\delta\beta}$$

Skenderis & Solodukhin 9910023

reason : in 3d the Weyl tensor vanishes identically $\Rightarrow R_{\mu\nu\rho\sigma}$ is det. by $R_{\mu\nu} = -\frac{2}{\ell^2} g_{\mu\nu}$. (by Einstein's eqns) as $R_{\mu\nu\rho\sigma} \propto (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho})$
$\Rightarrow$ all sols to pure 3d gravity are locally diffeomorphic to AdS_3

iii) AdS_3 w/ Dirichlet bnd. cond

- bnd. conditions : fix $g^{(0)}_{\alpha\beta} = \eta_{\alpha\beta}$

- more concretely, the bnd. metric is

$$ds^2_{(0)} = d\varphi^2 - dt^2 = dx^+ dx^- \qquad x^\pm = \varphi \pm t$$

The asympt. constraints on $g^{(2)}$, for $\eta_{\alpha\beta} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$ in null coord, are

$$\mathrm{tr}\, g^{(2)} = 4 g^{(2)}_{+-} = 0 \qquad \nabla^\alpha g^{(2)}_{\alpha\beta} = 0 \quad \Rightarrow \quad \partial_+ g^{(2)}_{--} = \partial_- g^{(2)}_{++} = 0$$

$$\Rightarrow \quad g^{(2)}_{++} = \mathcal{L}(x^+) \quad , \quad g^{(2)}_{--} = \bar{\mathcal{L}}(x^-)$$

- the phase space of asympt AdS_3 gravity w/ Dirichlet bnd. cond is parametrized by two arbitrary f. of $x^\pm = \varphi \pm \tau$, $\mathcal{L}(x^+)$ & $\bar{\mathcal{L}}(x^-)$, as well as the matter data we have not specified, though it is important these matter fields can be consistently added

- the bulk diffeomorphisms that asymptotically preserve this form of the metric must asympt. become conformal Killing vect. of the bnd. metric, $\eta_{\alpha\beta}$

- the CKV are given by $\partial_\alpha \chi_\beta + \partial_\beta \chi_\alpha = \eta_{\alpha\beta}\, \partial_\gamma \chi^\gamma$

$$\Rightarrow \partial_+ \chi_+ = \partial_- \chi_- = 0 \qquad \Rightarrow \qquad \chi^+ = 2\chi_- = \underbrace{\chi^+(x^+)}_{\chi_L} \;, \; \chi^- = \underbrace{\chi^-(x^-)}_{\chi_R}$$

- the bulk diffeomorphisms whose bnd. restriction is χ and which respect FG gauge (can be relaxed) are given by

$$\xi_L = \chi_L(x^+)\,\partial_+ + \frac{1}{2}\chi_L'(x^+)\, z\,\partial_z - \frac{\ell^2 z^2}{2}\chi_L''(x^+)\,\partial_- + \dots$$

$$\xi_R = \chi_R(x^-)\,\partial_- + \frac{1}{2}\chi_R'(x^-)\, z\,\partial_z - \frac{\ell^2 z^2}{2}\chi_R''(x^-)\,\partial_+ + \dots$$

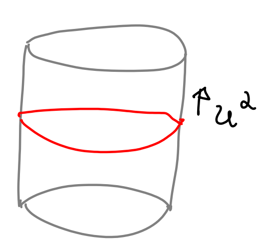
- plugging this into the expression for the charges, we obtain

$$Q_{\xi_L} = \frac{1}{8\pi G \ell}\int_0^{2\pi} d\varphi\, \mathcal{L}(x^+)\,\chi_L(x^+) \;; \qquad Q_{\xi_R} = -\frac{1}{8\pi G\ell}\int^{2\pi} d\varphi\, \bar{\mathcal{L}}(x^-)\,\chi_R(x^-)$$

- conserved b/c $\omega(\delta g, \delta_\xi g) \to 0$ asympt

iv) Let us now use this to compute the mass & angular momentum of various space-times

$$\partial_t = \partial_{x^+} - \partial_{x^-} \qquad\qquad \partial_\varphi = \partial_{x^+} + \partial_{x^-}$$



$$M = \frac{1}{8\pi G\ell}\int d\varphi\, \left[\mathcal{L}(x^+) + \bar{\mathcal{L}}(x^-)\right]$$

$$J = \frac{1}{8\pi G\ell}\int d\varphi\, \left[\mathcal{L}(x^+) - \bar{\mathcal{L}}(x^-)\right]$$

only z.m. survive

choice & set $\underline{0}$ of eng to be $\mathcal{L} = \bar{\mathcal{L}} = 0$

a) global AdS_3 : $ds^2 = \ell^2(-\cosh^2\rho\, d\tau^2 + d\rho^2 + \sinh^2\rho\, d\varphi^2)$ $\qquad \varphi \sim \varphi + 2\pi$

· to use the formulae previously derived, put this into Fefferman-Graham form

choose $c=2$

$$d\rho = -\frac{dz}{z} \Rightarrow z = e^{-\rho} \Rightarrow \cosh\rho = \frac{1}{2}\left(\frac{z}{c} + \frac{c}{z}\right), \quad \sinh\rho = \frac{1}{2}\left(\frac{c}{z} - \frac{z}{c}\right)$$

$$ds^2 = \ell^2\left(\frac{dz^2}{z^2} - d\tau^2\left(\frac{z}{4} + \frac{1}{z}\right)^2 + d\varphi^2\left(\frac{z}{4} - \frac{1}{z}\right)^2\right) = \ell^2\frac{dz^2}{z^2} + \underbrace{\ell^2\frac{d\varphi^2 - d\tau^2}{z^2}}_{g^{(0)}/z^2} \underbrace{- \ell^2\frac{d\tau^2 + d\varphi^2}{2} + \ldots}_{g^{(2)}}$$

· from here we read off $\mathcal{L}(x^+) = -\frac{\ell^2}{4} = \bar{\mathcal{L}}(x^-)$

· the mass & angular momentum of global AdS_3 are

$$M = \frac{1}{8\pi G \ell}\int_0^{2\pi} d\varphi\ 2\times\left(-\frac{\ell^2}{4}\right) = -\frac{\ell}{8G} = -\frac{c}{12}$$

This can be identified w/ the negative Casimir energy of the CFT vacuum on the cylinder (since $c = \frac{3\ell}{2G}$)

$J = 0$ $\qquad$ global AdS_3 has no angular momentum

b) conical defect spacetimes

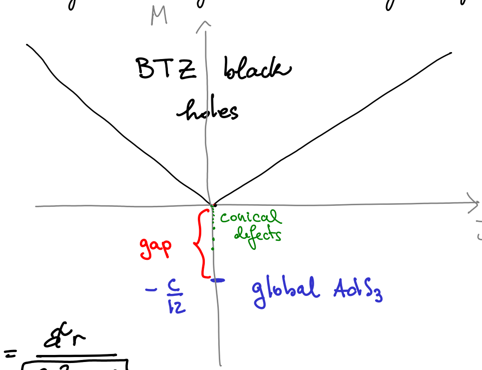
$$ds^2 = \ell^2\left[-(r^2+1)d\tau^2 + \frac{dr^2}{r^2+1} + r^2 d\varphi^2\right]$$

same metric as global AdS_3 ($r = \sinh\rho$)

but now w/ the identification $\varphi \sim \varphi + 2\pi\alpha$, $\alpha < 1$ (conical defect)

· to find the correct stress tensor, we need φ/bnd to have the usual identification 2π. We thus let $\varphi = \alpha\tilde{\varphi}$ $\quad r = \frac{1}{\alpha}\tilde{r}$ $\quad \tau = \alpha\tilde{\tau}$

Spectrum of stationary sols in 3d. gravity



$$\Rightarrow \; ds^2 = \ell^2\left[-(\tilde{r}^2+\alpha^2)\,d\tilde{\tau}^2 + \frac{d\tilde{r}^2}{\tilde{r}^2+\alpha^2} + \tilde{r}^2 d\tilde{\varphi}^2\right]$$

- to find the FG coefficients → change to FG gauge - $\frac{dz}{z} = \frac{d\tilde{r}}{\sqrt{\tilde{r}^2+\alpha^2}}$

$$\Rightarrow \ln z = -\ln\left(\tilde{r} + \sqrt{\tilde{r}^2+\alpha^2}\right) + \text{const} \quad \Rightarrow \quad \tilde{r} = \frac{1}{z} - \frac{\alpha^2}{4} z \, , \quad \tilde{r}^2 \simeq \frac{1}{z^2} - \frac{\alpha^2}{2} + \dots$$

$$ds^2 = \ell^2\left[-\left(\frac{1}{z^2} + \frac{\alpha^2}{2}\right) d\tilde{\tau}^2 + \frac{dz^2}{z^2} + \left(\frac{1}{z^2} - \frac{\alpha^2}{2}\right) d\tilde{\varphi}^2\right]$$

$$\Rightarrow \quad \mathcal{L} = \bar{\mathcal{L}} = -\frac{\alpha^2 \ell^2}{4} \quad \Rightarrow \quad M = -\frac{\alpha^2 \ell}{8G} \, , \quad J = 0$$

c) BTZ black holes (Bañados, Teitelboim & Zanelli)

$$ds^2 = -N^2(r)\, dt^2 + \frac{dr^2}{N^2(r)} + r^2 (d\varphi + N^\varphi dt)^2 \qquad N^2(r) = \frac{r^2}{\ell^2} - 8GM + \frac{16G^2J^2}{r^2}$$

$M\ell \geq |J|$ to avoid naked CTCs $\qquad N^\varphi = -\frac{4GJ}{r^2} \, , \; \varphi \sim \varphi + 2\pi\ell$

- solutions to pure 3d Einstein gravity w/ a negative cosmological constant.
- locally AdS_3 ; can be obtained as quotients of global AdS_3
- two isometries $(\partial_t, \partial_\varphi)$, even though locally $\exists$ 6 Killing vectors (only 2 are globally well-defined)

Exercise : Compute the mass & angular momentum of the BTZ black hole in terms of the parameters M, J using the Brown-York/ covariant formalism.*

* In the covariant formalism, the reference state can be taken to be the $M = J = 0$ b.h.

Asymptotic symmetries of AdS$_3$

- for $x^\pm$ compact, a natural basis of functions is

$$\chi^m_{\substack{L\\R}} = \pm e^{imx^\pm} = \pm e^{im(\varphi\pm t)}$$

- the Lie bracket algebra of the corresponding bulk asymptotic K.V. is

trivial

$$[\xi^m_{L/R}, \xi^n_{L/R}]_{L.B} = -i(m-n)\,\xi^{m+n}_{L/R} + \mathcal{O}(z^{\#})$$

$$[\xi^m_L, \xi^n_R]_{L.B} = 0 + \mathcal{O}(z^{\#})$$

Witt algebra
(same as classical 2d conformal algebra)

- since the conserved charges associated w/ any given $\chi_{L,R}$ is generically non-zero, the diffeomorphisms $\xi_{L,R}[\chi]$ written above represent asymptotic symmetries, which as explained are encoded by two arbitrary functions of x^+/x^- $\Rightarrow$ ∞ dim'l symmetry group which coincides w/ the 2d conformal group.

- using the representation thm, the Lie bracket algebra of the asymptotic diffeos is isomorphic to the Dirac bracket algebra of the conserved charges, up to a central extension

$$\{Q_\eta, Q_\xi\} = \delta_\xi Q_\eta = Q_{[\eta,\xi]_{L.B}} + K_{\eta,\xi}$$

need to compute the charge to evaluate this

- the commutator of the charges is

bnd. restriction of η_L

$$\{Q_{\eta_L}, Q_\xi\} = \delta_\xi Q_{\eta_L} = \frac{1}{8\pi G\ell}\int_0^{2\pi} d\varphi\; \delta_\xi \mathcal{L}(x^+)\, \chi_{\eta_L}(x^+)$$

- if ξ is a L.M. diffeomorphism, the change in the backgnd. value of $\mathcal{L}(x^+)$ under its action is (obtained from $\mathcal{L}_\xi g_{\mu\nu}$)

$$\delta_{\xi_L}\mathcal{L}(x^+) = 2\mathcal{L}(x^+)\,\chi'_{\xi_L}(x^+) + \chi_{\xi_L}(x^+)\,\mathcal{L}'(x^+) - \frac{\ell^2}{2}\chi'''_{\xi_L}(x^+)$$

and thus

$$\{Q_{\eta_L}, Q_{\xi_L}\} = \frac{1}{8\pi G\ell}\int_0^{2\pi} d\varphi\left[2\mathcal{L}\chi'_{\xi_L}(x^+) + \chi_{\xi_L}\mathcal{L}'(x^+) - \frac{\ell^2}{2}\chi'''_{\xi_L}(x^+)\right]\chi_{\eta_L}(x^+)$$

$$= \underbrace{\frac{1}{8\pi G\ell}\int_0^{2\pi} d\varphi\;\mathcal{L}(x^+)\left(\chi'_{\xi_L}(x^+)\chi_{\eta_L}(x^+) - \chi'_{\eta_L}(x^+)\chi_{\xi_L}(x^+)\right)}_{Q_{[\eta_L,\xi_L]_{LB}}} - \underbrace{\frac{\ell}{16\pi G}\int_0^{2\pi} d\varphi\,\chi'''_{\xi_L}(x^+)\chi_{\eta_L}(x^+)}_{\text{central charge } K_{\eta_L\xi_L}}$$

(check of the repr. thm.)

- if ξ is a R.M. diffeo, then $\delta_{\xi_R}\mathcal{L}(x^+) = 0 \Rightarrow \{Q_{\eta_L}, Q_{\xi_R}\} = 0$

- taking $\chi_{\eta_L} = e^{imx^+}$ $\quad \chi_{\xi_L} = e^{inx^+}$, then

$$\{Q_L^m, Q_L^n\} = -i(m-n)\,Q_L^{m+n} + \frac{i\ell n^3}{16\pi G}\int_0^{2\pi} d\varphi\,\underbrace{e^{i(m+n)x^+}}_{2\pi\delta_{m+n,0}}$$

$$= -i(m-n)\,Q_L^{m+n} - \frac{i\ell}{8G}m^3\delta_{m+n,0}$$

- passing from Dirac brackets to commutators $[\,,\,] = i\hbar\{\,,\,\}_{D.B}$

  classical $\longrightarrow$ semi-classical

  and letting $Q_L^m = \hbar L_m + \#\,\delta_{m,0}$, we find

$$\boxed{[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}}$$

Virasoro algebra w/

$$\boxed{c = \frac{3\ell}{2G}} \gg 1$$

$(\ell_{AdS} \gg l_{Planck})$

conformal anomaly

- this is the quantum symmetry algebra of a 2d CFT

- obtained from a purely classical GR calculation

Remarks : in higher dimensional AdS, the ASG = $SO(d,2)$ ≈ isometries of the AdS vacuum solution. This is a rather common coincidence

- for AdS_3, the isometries of the vacuum are $SO(2,2) \simeq SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ which is just a small subgroup (generated by $L_0, L_{\pm 1}$ & $\bar{L}_0, \bar{L}_{\pm 1}$) of the ∞-dim'l ASG.

- thus, in general ASG ≠ vacuum isometries. The symmetries of the theory are encoded in the ASG. (will see this again for flat space)

- it is also consistent to consider AdS_3 gravity w/ mixed bnd. cond for the metric (= non-linear combination of $g^{(0)}$ & $g^{(2)}$ held fixed) This change in the def^n of the grav. theory has the effect of making the dual QFT non-local