Asymptotic symmetries of flat space

- far away from sources, but on distances shorter than the cosmological scale, our universe is approximately flat ⇒ the study of Minkowski spacetime is highly relevant to the real world, when we would like to study an isolated system (e.g. a star, or a b.h.)

- other reasons Minkowski spt. is interesting are:
  - radiation can reach ∞
  - rich asymptotic structure = ∞-dim'l extension of the Poincaré gp. (BMS) & associated conservation laws
  - observable effects of the asymptotic symm. = memory effect
  - interesting links to QFT soft theorems (relate amplitudes w/ and w/o soft particles) ↦ insight into the infrared structure of gravity in Mink.
  - flat space holography, including microscopic description of non-extr. b.h.

- to discuss infinity in a coordinate-independent manner, we turn again to the conformal compactification, i.e. a map $\varphi: \mathcal{M} \to \tilde{\mathcal{M}}$ ∋ $(\varphi^* \tilde{g})_{\alpha\beta} = \Omega^2 g_{\alpha\beta}$ ∋ $\varphi(\infty)$ is at "finite distance" in $\tilde{\mathcal{M}}$, so ∞ can be meaningfully studied

- the conformal bnd. of $\mathcal{M}$ is then simply $\partial\tilde{\mathcal{M}}$

- notice that the new metric $\tilde{g}$ no longer satisfies Einstein's eqns, but the causal structure of the original metric is preserved, though distances are severely distorted

Asymptotic structure of (empty) Minkowski spacetime

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_2^2$$

- to construct Penrose diagram, introduce the null coord

$$u = t - r \quad \text{retarded time} \qquad (u,v) \in (-\infty, \infty)$$
$$v = t + r \quad \text{advanced time} \qquad \text{w/ } u \leq v$$

$$ds^2 = -du\,dv + \left(\frac{u-v}{2}\right)^2 d\Omega_2^2$$

- to make the coordinate range compact, let $u = \tan U$, $v = \tan V$
  $(U,V) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ $U \leq V$

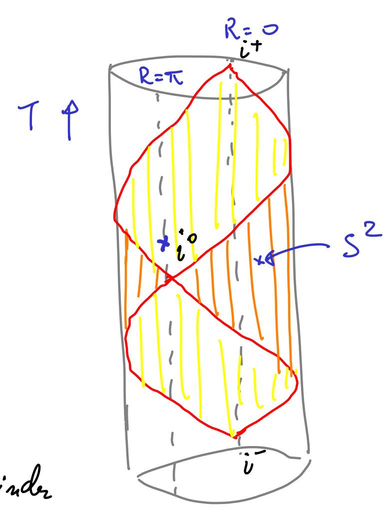
& $$u - v = \tan U - \tan V = \frac{\sin(U-V)}{\cos U \cos V}$$

$$\Rightarrow ds^2 = \frac{1}{4\cos^2 U \cos^2 V}\left(-4\,dU\,dV + \sin^2(U-V)\, d\Omega_2^2\right)$$

- letting $T = U + V \in (-\pi, \pi)$, $R = V - U \in (0, \pi)$ w/ $0 < R + |T| < \pi$

& dropping out the overall factor $(4\cos^2 U \cos^2 V)^{-1}$

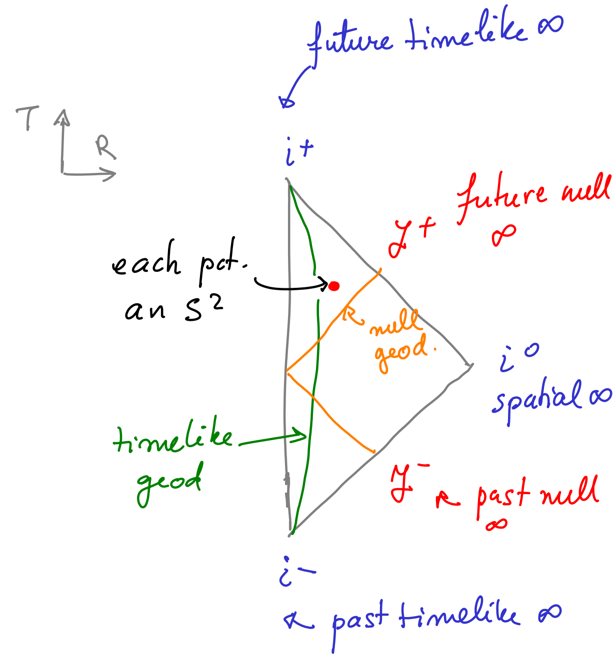
$$d\tilde{s}^2 = -dT^2 + \underbrace{dR^2 + \sin^2 R\, d\Omega_2^2}_{\text{locally, metric on } S^3}$$



$\Rightarrow$ Mink$_4$ is conformal to a patch of $\mathbb{R}_T \times S^3$
(Einstein static universe)

- do not confuse w/ AdS diagram, which is a solid cylinder

· unwrapping, we find

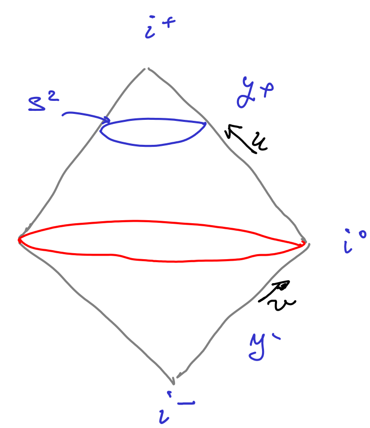


· future timelike ∞, i^+
$T=\pi$, $R=0$ point ($t\to\infty$ w/ r fixed)
endpt. of all maximally ext. timelike geodesics

· past timelike ∞, i^-
$T=-\pi$, $R=0$ point ($t\to-\infty$ w/ r fixed)
starting pt. of all timelike geodesics

· spatial ∞, i^0
$T=0$, $R=\pi$ point ($r\to\infty$ w/ t fixed)
endpt. of maximally ext. spacelike geodesics

Alternate drawing



misleadingly represents i^0 as a sphere (it is a point on $\tilde{M}$)

· future null ∞, $\mathscr{I}^+$ (null 3-surface $\sim S^2\times\mathbb{R}$)
$R=\pi-T$, or $V=\frac{\pi}{2}$ ($r\to\infty$ w/ $u=t-r$ fixed)
future endpoint of null geodesics

· past null ∞, $\mathscr{I}^-$ (null 3-surface)
$R=\pi+T$, or $U=-\frac{\pi}{2}$, $r\to\infty$ w/ $v=t+r$ fixed
starting point of null geodesics

· this analysis holds for empty Minkowski spt. In presence of matter

  · we can still define a notion of asymptotic flatness as we follow spatial/null curves towards i^0, $\mathscr{I}^\pm$

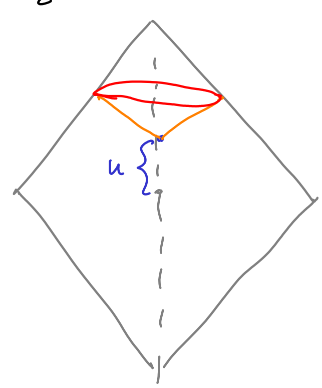
  · we cannot require the spt. to become flat @ late/early times → no requirements @ $i^\pm$

- lack of differentiability of the conformally rescaled manifold @ i^0,
  - the value of tensor fields depends on the spatial geodesic used to reach this point.
  - expected, since spatial curves diverge in real space, but they all end up @ the same point i^0 in the unphysical manifold $\tilde{M}$
  - for empty Minkowski, $\tilde{M}$ is only smooth near i^0 due to the isotropy & homogeneity of the Minkowski vacuum

- in the following, we will be especially interested in the asympt. str. of Minkowski spacetime near null infinity $\mathcal{I}^{\pm}$

- to reach $\mathcal{I}^+$, we take $t, r \to \infty$ w/ $u = t - r$ fixed
  retarded time

- metric in Bondi coord is $ds^2 = -du^2 - 2du\,dr + r^2 d\Omega_2^2$



- celestial sphere: located @ $r \to \infty$, u fixed = all directions towards which an obs. @ $r=0$ & $t=u$ can look

- we will be interested in the action of Minkowski isometries on the celestial sphere

- Poincaré gp: transl. + Lorentz. transf.
  - time transl. $t \to t+c \Rightarrow u \to u+c$

- spatial transl, e.g. $x^3 \to x^3 + \text{const.}$ $\quad x^3 = r\cos\theta \quad r = \sqrt{x_1^2+x_2^2+x_3^2}$

$$\partial_{x_3} = \frac{\partial r}{\partial x_3}\partial_r + \frac{\partial \theta}{\partial x^3}\partial_\theta = \cos\theta\,\partial_r\Big|_{t \text{ fixed}} - \frac{1}{r}\sin\theta\,\partial_\theta = -\underbrace{\cos\theta}_{Y^{m=0}_{\ell=1}}\partial_u + \cos\theta\,\partial_r - \frac{\sin\theta}{r}\partial_\theta$$

spherical harmonic

$\Rightarrow$ spatial transl. act as (θ,φ)-dep. transl. along $\mathcal{J}^+$ $\quad Y^m_{\ell=1}(\theta,\varphi)$

- the Lorentz gp. acts linearly on x^μ $\quad x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu \quad \Lambda \in O(1,3)$
- useful to remember that proper orthochronous subgp ($\Lambda^0{}_0 > 0$, $\det\Lambda = 1$)

$$L^\uparrow_+ \cong \frac{SL(2,\mathbb{C})}{\mathbb{Z}_2} \qquad \mathbb{X} = \begin{pmatrix} x^0+x^3 & x^1-ix^2 \\ x^1+ix^2 & x^0-x^3 \end{pmatrix} = x^0\sigma_0 + x^i\sigma_i$$

hermitean

$X \to S X S^\dagger$, w/ $S \in SL(2,\mathbb{C})$ preserves $\det X = -x^\mu x_\mu$

same as Lorentz transf Λ, 6 real param.

- to obtain the map between Λ & S, note that

$$x'^\mu \sigma_\mu = S x^\mu \sigma_\mu S^\dagger = \Lambda^\mu{}_\nu(S)\, x^\nu \sigma_\mu \quad \Rightarrow \quad S\sigma_\mu S^\dagger = \Lambda^\nu{}_\mu(S)\sigma_\nu$$

note S & $-S \to$ same Λ

- letting $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ w/ $ad-bc=1$ $(\in\mathbb{C})$

we can find the explicit general map $\Lambda(S)$ (see e.g. Oblak 1508.00920)

- examples:

$$R_3 = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} \qquad B_3 = \begin{pmatrix} e^{\chi/2} & 0 \\ 0 & e^{-\chi/2} \end{pmatrix}$$

rotation around x^3 axis $\qquad$ boost along x^3

• general map: $$\Lambda(S) = \begin{pmatrix} \frac{1}{2}\left(|a|^2+|b|^2+|c|^2+|d|^2\right) & -\mathrm{Re}(a\bar{b}+c\bar{d}) & -\mathrm{Im}(a\bar{b}+c\bar{d}) & \frac{1}{2}\left(|a|^2-|b|^2+|c|^2-|d|^2\right) \\ -\mathrm{Re}(\bar{a}c+\bar{b}d) & \mathrm{Re}(\bar{a}d+\bar{b}c) & -\mathrm{Im}(a\bar{d}-b\bar{c}) & -\mathrm{Re}(\bar{a}c-\bar{b}d) \\ \mathrm{Im}(\bar{a}c+\bar{b}d) & & & \\ \frac{1}{2}\left(|a|^2+|b|^2-|c|^2-|d|^2\right) & & & \end{pmatrix}$$

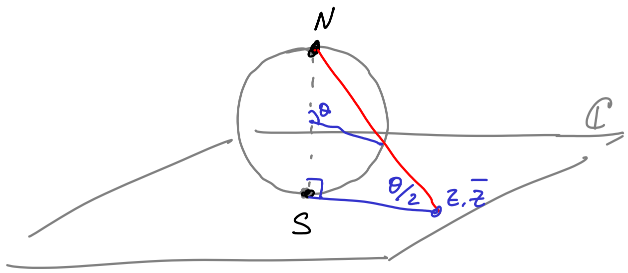
see. (2.30) in Oblak

• in part, we see that rotations ($SO(3)$) satisfy $\begin{cases} a\bar{b}+c\bar{d} = a\bar{c}+b\bar{d} = 0 \\ |a|^2 \pm |c|^2 = \pm|b|^2 + |d|^2 \end{cases}$

$\Rightarrow |a| = |d|,\ |b| = |c| \Rightarrow a = \bar{d},\ b = -\bar{c}$ w/ $|a|^2 + |b|^2 = 1$ (det. cond)

3 real param.

• to understand the action of the Lorentz gp on the celestial S^2, introduce stereographic coord. on S^2 (of radius $R = 1/2$)



$$z = \underbrace{e^{i\varphi}\cot\frac{\theta}{2}}_{\text{spherical coord}} = \underbrace{\frac{x_1+ix_2}{r-x_3}}_{\text{cartesian } S^2 \subset \mathbb{R}^3}$$

the metric on S^2 is $ds^2 = \dfrac{4\,dz\,d\bar{z}}{(1+z\bar{z})^2}$

• one can check that the previously introduced rotations act as

$$z \to \frac{az+b}{cz+d} \quad \text{on } z \qquad \text{w/ } d = \bar{a},\ c = -\bar{b}$$

use (2.30) in Oblak.

• to find the action of a general boost on the celestial S^2, we first find the transformation of $r = \sqrt{x_1^2+x_2^2+x_3^2}$ under Lorentz in the $r \to \infty$ limit

$$r \to r' = r\,\frac{|az+b|^2 + |cz+d|^2}{1+z\bar{z}} + O(1) = F(z,\bar{z})\cdot r + O(1)$$

so Lorentz transf preserve the $r\to\infty$ limit $\to$ angle-dependent rescaling

Exercise: Check that r is indeed left invariant by $SO(3)$ transf.

- $u' = t' - r' = G(z,\bar z)\, u + O\left(\frac{1}{r}\right)$ Note $O(r)$ term cancels out: important for consistency of action on $\mathcal{I}^+$; $r\to\infty$ w/ u finite

- $G(z,\bar z) = \dfrac{1}{F(z,\bar z)}$ b/c the interval $t^2 - r^2 = 2ur + u^2 \simeq 2ur$ as $r\to\infty$ is invariant

- $z' = \dfrac{x_1' + i x_2'}{r' - x_3'} = \dfrac{az+b}{cz+d} + O\left(\frac{1}{r}\right)$ conformal transf. of the celestial S^2

- indeed, under a conformal transf, the metric $g \to \Omega^2 g$

$$ds^2 = \frac{4\, dz\, d\bar z}{(1+z\bar z)^2} \quad \Rightarrow \quad z \to z' = f(z) \qquad \bar z \to \bar z' = f(\bar z)$$

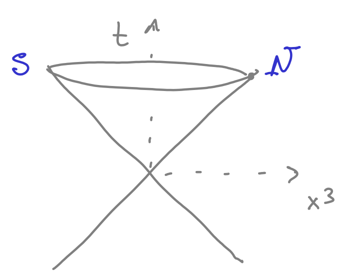
assuming $f(z)$ is meromorphic & injective (points e.g. $0,\infty$ are the image of a single pt)

$$\Rightarrow f(z) = \frac{az+b}{cz+d} \qquad \text{fix rescalings: } \quad ad - bc = 1 \qquad SL(2,\mathbb{C})$$

- thus, we find that Lorentz transf. act as conformal transf. of the celestial sphere

  - rotations $\rightarrow$ trivial $a = \bar d$, $b = -\bar c$

  - boosts, e.g. along x^3 $\quad a = e^{\chi/2} = d^{-1} \Rightarrow z' = e^{\chi} z$



$$z = e^{i\varphi} \cot\frac{\theta}{2}$$

$$z \nearrow \quad \Rightarrow \quad \theta \searrow$$

objects in boosted frame look closer to the direction of motion

Asymptotically flat space-times

- to define asympt. flat space-times, 2 options:

    - use conformal compactification to define asympt. flatness in a coordinate-invariant way (see e.g. Wald) & express falloffs in terms of the unphysical conformal factor (no cond on $i^{\pm}$, sufficiently lax @ i^0)

    → - use an adapted coord. system & specify fall-offs of the metric components (down-side → other interesting bnd. cond. may be missed)
    Bondi, van der Burg, Metzner & Sachs '62

- interested in asympt. str. near $\mathscr{I}^+$ (endpt. of null geodesics followed by e.g. E&M & gravitational radiation), well-described in Bondi gauge (avoids $\ln r$ terms in the asympt. exp.)

    - start by considering a family of null hypersurfaces, labeled by u = const. Using u as a coord., the spt. metric satisfies

$$g^{uu} = 0 \qquad (\text{b/c. normal } n^\lambda \propto g^{\lambda u} \text{ is null})$$

u is called retarded time

    - note that the normals n^μ to these hypersurfaces, $n_u = 1$ are tangent to null geodesics, since n^μ satisfies

$$n^\mu \nabla_\mu n_\nu = -n^\mu \Gamma^u_{\mu\nu} = n^\mu \nabla_\nu n_\mu = 0$$

- define the coord $(\theta, \varphi) = x^A$ to be constant along these null rays

$$n^\lambda \partial_\lambda x^A = 0 \quad \Rightarrow \quad g^{\lambda u} \partial_\lambda x^A = 0 \quad \Rightarrow \quad g^{uA} = 0$$

- letting the remaining coord to be r, the metric comp. obey

$$0 = g^{u\lambda} g_{\lambda r} = g^{ur} g_{rr} \quad \Rightarrow \quad g_{rr} = 0$$

$$0 = g^{u\lambda} g_{\lambda A} = g^{ur} g_{rA} \quad \Rightarrow \quad g_{rA} = 0 .$$

- finally, we choose r to be the luminosity distance $\partial_r \frac{\det g_{AB}}{r^2} = 0$

- these coord. are known as Bondi-Sachs coord, or Bondi gauge

- in them, the metric reads

$$ds^2 = g_{uu} du^2 + 2 g_{ur} du dr + 2 g_{uA} du dx^A + g_{AB} dx^A dx^B$$

- note Minkowski has $g_{uu} = g_{ur} = -1$, $g_{uA} = 0$ $\quad g_{AB} = r^2 \gamma_{AB}$ metric on unit S^2

- asympt. flatness $\mapsto$ approach Minkowski metric asymptotically $(r \to \infty, u \text{ fixed})$

  want bnd. cond lax enough $\ni$ spt. of interest are all included, but restrictive enough for charges to be finite

educated guess :
$$\begin{cases} g_{uu} = -1 + O\left(\frac{1}{r}\right) & g_{uA} = O(r^0) \\ g_{ur} = -1 + O\left(\frac{1}{r^2}\right) & g_{AB} = r^2 \gamma_{AB} + O(r) \end{cases}$$

$$ds^2 = -du^2 - 2du\,dr + r^2\gamma_{AB}\,dx^A dx^B \quad \text{(Minkowski)} \qquad \text{w/ } \gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2}$$

$$+ \frac{2m_B(u,x^A)}{r}du^2 + r\,C_{AB}\,dx^A dx^B + D^B C_{AB}\,du\,dx^A$$

cov. deriv. w.r.t. γ_{AB}.

$$+ \frac{1}{16r^2}C_{AB}C^{AB}\,du\,dr + \frac{1}{r}\left[\frac{4}{3}(N_A + u\,\partial_A m_B - \frac{1}{8}\partial_A(C_{BC}C^{BC})\right]du\,dx^A$$

$$+ \frac{1}{4}\gamma_{AB}C_{CD}C^{CD}\,dx^A dx^B + \text{subleading.}$$

(Ansatz consistent w/ Einstein's eqns, which will give some constraints)

- $m_B(u,x^A)$ = Bondi mass aspect (in e.g. Kerr $m_B = GM$, but in general it is a function)
  - angular density of energy @ $\mathcal{I}^+$
  - Bondi mass $M(u) = \int_{S^2} dx^A \sqrt{\gamma}\, m_B(u,x^A)$

    $\partial_u M(u) \leq 0$ for Einstein grav. + matter obeying NEC (null energy cond.)

    i.e., gravitational radiation carries energy away

    as $u \to -\infty$, Bondi mass = ADM energy = mass total spt

- $C_{AB}(u,x^A)$ w/ $\gamma^{AB}C_{AB} = 0$ traceless & symmetric => 2 polarizations
  - describes gravitational waves
  - from it, one constructs the Bondi news tensor : $N_{AB} = \partial_u C_{AB}$

    gravitational analogue of $F_{uA} = \partial_u A_A$ in E&M
  - $N_{AB}N^{AB} \propto$ energy flux across $\mathcal{I}^+$

- $N_A(u, x^A)$ = angular momentum aspect

  · angular density of angular momentum, $J(u) = \int_{S^2} d^2x^A \sqrt{\gamma}\, N_A$ total J.

- Einstein's eqns. further require that:

$$\partial_u m_B = \frac{1}{4} D^A D^B N_{AB} - T_{uu} \quad , \text{w/} \quad T_{uu} \equiv \frac{1}{8} N_{AB} N^{AB} + 4\pi \lim_{r\to\infty} \left(r^2 T_{uu}^{\text{matter}} \right)$$

(null)

$$\partial_u N_A = -\frac{1}{4} D^B \left(D_B D^C C_{AC} - D_A D^C C_{BC} \right) + u\, \partial_A \left(T_{uu} - \frac{1}{4} D^B D^C N_{BC} \right) - T_{uA}$$

$\uparrow$ matter + rad.

- thus, if we know $N_{AB}(u)$ & $T_{uu}^{\text{matter}}, T_{uA}^{\text{matter}}$ for all u, as well as the initial values as $u \to -\infty$ of m_B, C_{AB} & N_A $\Rightarrow$ we know the leading asympt. data m_B, C_{AB}, N_A for all u.

- we need to be careful about initial / final data on $\mathcal{I}^+$. $\nearrow \mathcal{I}^+_-$ At the end, want

$$N_{AB} \sim O\left(\frac{1}{|u|^{1+\varepsilon}} \right) \qquad \varepsilon > 0 \text{ as } |u| \to \infty \quad \& \quad m_B, N_A \text{ finite}$$

since N_A finite $\Rightarrow \partial_u N_A \to 0 \Rightarrow D^B \left(D_B D^C C_{AC} - D_A D^C C_{BC} \right) \Big|_{\mathcal{I}^+_-} = 0$

constr. on C_{AB}.

decomposing C_{AB} as

$$C_{AB} = -2 D_A D_C \underset{\text{scalar}}{C} + \gamma_{AB} D^2 C + \varepsilon_{(CA} D_{B)} D^C \underset{\text{pseudo-scalar}}{\Psi}$$

$C(u, x^A)$ = supertranslation memory field

$\Psi(u, x^A)$ = spin memory field $\leftarrow$ must vanish @ $\mathcal{I}^+_-$

Check!

To summarize, the leading asymptotic data @ $\mathscr{I}^+$ are encoded by

$$N_{AB}(u, x^A) \quad , \quad m_B(u \to -\infty, x^A) \, , \, C(u \to -\infty, x^A) \, , \, N_A(u \to -\infty, x^A).$$

+ subleading

- similar expr. hold for $\mathscr{I}^-$

Asymptotic symmetries

- diffeomorphisms that preserve the asymptotic form of the metric & generically lead to non-trivial conserved charges

- 2 steps : impose exact preservation of Bondi gauge $g_{rr} = g_{rA} = 0$, $\partial_r \det \frac{g_{AB}}{r^2} = 0$

  $\xi^\mu(x^\mu) \to 4$ functions of (u, x^A) & fixed r-dep.

  - preserve asympt. falloffs $(\mathcal{L}_\xi g)_{uu} = O(\frac{1}{r})$ etc.

  $\to$ 3 functions on S^2 : $T(x^A)$, $R^A(x^B)$

- we split the asympt. diffeos in 2 sets :

trivial

$$\text{BMS}_4 \text{ generators} \begin{cases} \xi_T = T(x^c)\partial_u - \frac{1}{r} D^A T(x^c) \partial_A + \frac{1}{2} D_A D^A T(x^c) \partial_r + \ldots & \text{supertranslations} \\ \xi_R = R^A(x^c)\partial_A + \frac{u}{2} D_A R^A(x^c) \partial_u - \frac{r+u}{2} D_A R^A(x^c) \partial_r + \ldots & \text{Lorentz transf/superrotations} \end{cases}$$

w/ $D_A R_B + D_B R_A = \gamma_{AB} D_C R^C \Rightarrow R^A =$ conformal Killing vector on S^2

R^A:
- if CKV is globally well-def 6 CKV $SO(2,2) \to$ Lorentz
- if singularities allowed $\Rightarrow R^A =$ arbitrary (anti)holomorphic f. : superrotations

- Lie bracket algebra : $[\xi_T, \xi_{T'}]_{L.B} = 0$ $\quad [\xi_R, \xi_{R'}]_{L.B} = \xi_{R''} = R^B D_B R^{A'} - R^{B'} D_B R^A$

$$[\xi_T, \xi_R]_{L.B} = \xi_{T'} = -R^A \partial_A T + \tfrac{1}{2} T D_A R^A$$

- supertranslations = angle-dependent transl along u (+ subleading)

hence the name

  - expand $T(x^A) = \sum_{\ell,m} C_{\ell m} Y_{\ell m}(x^A)$ spherical harmonics

  - translations (Poincaré) corresp to Y_{00} & Y_{1m} e.g. $\partial_{x^3} = -\cos\theta\,\partial_u + \ldots$

  - abelian ideal of $(bms)_4$ $\quad [\xi_T, \xi_{T'}]_{L.B} = 0 \quad [\xi_T, \xi_R]_{L.B} = \xi_{T'(T,R)}$

  - "conserved" charges = supermomenta : finite but non-integrable & not conserved (if $N_{AB} \neq 0$)

    physical flux

$$\delta\!\!\!/ Q_T = \delta\left(\frac{1}{4\pi G}\int_{S^2} d^2\Omega \sqrt{\gamma}\; T(x^A)\; m_B(u, x^A)\right) + \frac{1}{32\pi G}\int \underbrace{d^2\Omega \sqrt{\gamma}\, T(x^A) N^{AB} \delta C_{AB}}_{\text{non-integrable piece}}$$

  - charge algebra abelian $\{Q_T, Q_{T'}\} = 0$

  - in particular, supertranslations carry zero energy

  - act non-trivially on the asympt. data

$$\delta_T C_{AB} = T\partial_u C_{AB} - 2 D_A D_B T + \gamma_{AB} D_C D^C T \qquad \delta_T N_{AB} = T\partial_u N_{AB}$$

$$\delta_T m_B = T\partial_u m_B + \frac{1}{4}\left(N^{AB} D_A D_B T + 2 D_A N^{AB} D_B T\right)$$

see e.g. Flanagan & Nichols 1510.03386

$$\delta_T N_A = T D_A m_B + 3 m_B D_A T + \frac{1}{4} C_{AB} D^B D_D D^D T + \text{other terms bilinear in } C_{AB} \,\&\, T$$

- e.g if we start w/ Minkowski spt. $C_{AB} = m_B = N_A = 0$, then m_B (mass) & N_{AB} (rad) will continue to be zero after the supertransl. $\Rightarrow$ still a vacuum

- only shift is in $\delta_T C_{AB} = -2 D_A D_B T + \gamma_{AB} D^C D_C T$

$$\Rightarrow \quad \boxed{\delta_T \underbrace{C}_{\text{memory field}}(x^A) = T(x^A)}$$

except if $T \in$ Poincaré, for which $\delta_{T_p} C_{AB} = 0$

- $C(x^A)$ labels different zero energy configs ($\neq$ vacua in the quantum th.) Strominger '13

- fixing C $\leftrightarrow$ spontaneous breaking of supertransl. invariance
$\delta_T C = T(x^A)$ $\leftrightarrow$ transf. similarly to a Goldstone boson

- since supertranslations commute, the resulting backgnd has zero supertranslation charges (& in particular, zero energy), but it does have non-zero superrotation charge (in part, $J \neq 0$)

$$Q_{R^A} = \frac{1}{16\pi G} \int d^2\Omega \, R^A \left(2 \underbrace{N_A}_{\text{will become} \neq 0 \text{ @ quadratic order}} + \frac{1}{16} \partial_A (C_{BC} C^{BC}) \right)$$

these charges distinguish the different configs (states) created by acting w/ supertranslations

- note that Lorentz transf. do not commute w/ supertranslations $\Rightarrow$ no BMS-invariant def^n of angular momentum

Lorentz algebra & superrotations

$$\xi_R = R^A(x^C)\,\partial_A - \frac{r+u}{2}\, D_A R^A(x^C)\,\partial_r + \frac{u}{2}\, D_A R^A(x^C)\,\partial_u + \dots$$

w/ R^A = CKV on S^2 : $\quad D_A R_B + D_B R_A = \gamma_{AB} D_C R^C$

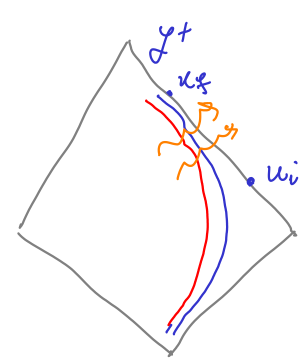
using $\gamma_{zz} = \gamma_{\bar z \bar z} = 0$, $\gamma_{z\bar z} = \dfrac{2}{(1+z\bar z)^2}$ $\Rightarrow$ $\partial_z R_z - \underbrace{\Gamma^z_{zz}}_{\partial_z \ln \gamma_{z\bar z}} R_z \propto \partial_z\big(\underbrace{R_z \gamma^{z\bar z}}_{R^{\bar z}}\big) = 0$

$\Rightarrow$ $R^z = R^z(z)$ & $R^{\bar z} = R^{\bar z}(\bar z)$

- globally defined $a + bz + cz^2$ & antiholom $\Rightarrow$ Lorentz generators
- in the case of CFT_2 , non-globally defined CKV $\to$ Virasoro alg (interesting & very powerful)
- if R^z meromorphic , e.g. $R^z \sim \dfrac{1}{z-w}$, $\partial_{\bar z} R^z = 2\pi\, \delta^{(2)}(z-w)$ singularities on the sphere $\sim$ cosmic strings piercing S^2 on $\mathcal{J}^+$
- ok if such superrot are allowed @ infinitesimal level, but if one wants to fully include them in the ASG, then $[\xi_T, \xi_R] \sim \xi_{T'}$ $\Rightarrow$ need meromorphic supertransl. as well , but these have ∞ charges for e.g. Kerr

The gravitational memory effect

$\exists$ physical, observable effects of BMS ?



- consider two inertial obs. near $\mathcal{J}^+$ w/ distance S^μ among them & tg. vect. $T^\mu \simeq \partial_u$ to their geodesics

- assume no radiation @ $\mathcal{I}^+$ for $u < u_i$ & $u > u_f$

- displacement memory effect = permanent change in the separation S^μ due to the passage of radiation

- to see this, consider the geodesic deviation eqn

$$\frac{D^2 S^\mu}{d\tau^2} = R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma \qquad \text{w/} \quad D = T^\mu \nabla_\mu$$

assuming $S^\mu = S^A$ $\Rightarrow$ $\partial_u^2 S^A = R^A{}_{uuB} S^B$

$$\Leftrightarrow \quad r^2 \gamma_{AB} \partial_u^2 S^B = -R_{AuBu} S^B = \frac{r}{2} \partial_u^2 C_{AB} S^B$$

- integrating, we find $\gamma_{AB} \Delta S^B = \frac{1}{2r} \Delta C_{AB} S^B + \mathcal{O}\left(\frac{1}{r^2}\right)$

$\Rightarrow$ if $\Delta C_{AB} \neq 0$ $\Rightarrow$ detectable displacement between obs (Zeldovich & Polnarev '74)

- integrating the u variation of the Bondi mass aspect, we find

$$\Delta m_B = \frac{1}{4} D^A D^B \Delta C_{AB} - \int_{u_i}^{u_f} du \, T_{uu}$$

- since the space-time is stationary outside (u_i, u_f), we can write

$$\Delta C_{AB} = -2 D_A D_B \Delta C + \gamma_{AB} D^2 \Delta C$$

(b/c Ψ vanishes @ $\mathcal{I}^+_\pm$)

$\Rightarrow$ ΔC obeys

$$-\frac{1}{4}(D^2+2) D^2 \Delta C = \Delta m_B + \int_{u_i}^{u_f} du \left[\frac{1}{8} \overbrace{N_{AB} N^{AB}}^{\text{grav. waves}} + 4\pi \overbrace{\lim_{r\to\infty} \left(r^2 T_{uu}^{\text{matt}}\right)}^{\text{null matter}} \right]$$

$G(x^A, x'^A)$

change or redistribution of mass

$$\Rightarrow \quad \Delta C(x^A) = 2 \int d^2 x^{A} \underbrace{G(x^A, x^{A'})} \int_{u_i}^{u_f} du \left(T_{uu}(x^{A'}, u) + \Delta m_B \right)$$

$$-\frac{1}{\pi} \sin^2 \frac{\Theta}{2} \ln \sin^2 \frac{\Theta}{2}$$ angle between x^A & $x^{A'}$

- thus, the passage of gravitational waves through $\mathcal{I}^+$ (or null matter, or Δm_B) makes the vacua before & after differ by a supertranslation

- this effect is highly non-local on S^2: if gravity waves pass through e.g. the north pole, then $\Delta C = 0$ @ the N & S poles & is largest near the equator

- $\exists$ also a spin memory effect $\rightarrow$ memory effect related to superrotations induced by momenta of angular momentum flux passing through $\mathcal{I}^+$ & physically measurable as a time delay between clockwise/counterclockwise lightrays (subleading in $1/r$)

Summary

- we discussed the asymptotic structure (& appropriate definition) of asymptotically flat (AF) spacetimes near $\mathcal{I}^+$

- we discovered a truly rich and beautiful story $\approx$ lame

- $\mathcal{I}^+$ is interesting b/c this is where gravitational waves & other radiation passes through

- the charges are not conserved and not integrable, but for a good physical reason (passage of radiation or null matter)

- surprisingly, the ASG of AF spacetimes near $\mathcal{I}^+$ is not the Poincaré gp. Rather, it is the infinite dim'l BMS gp

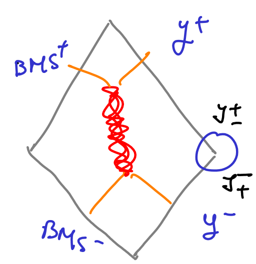
    - supertranslations = arbitrary angle-dependent translations along $\mathcal{I}^+$
      $\Rightarrow$ energy moments are conserved along $\mathcal{I}^+$ w/ no flux.
      – abelian symmetries

    - Lorentz transf. $\leftrightarrow$ globally well-defined conformal transf. on celestial S^2
      ~ may be extended to all S^2 CKV (superrotations) $\to$ Virasoro alg

- acting w/ supertranslations on the empty Minkowski spt $\Rightarrow$ different zero-energy configurations (because its (super) rotation charges are different)
  $\Rightarrow$ ∞ degeneracy of vacua in the quantum theory (insights into the infrared str. of gravity)

- the effect of supertranslations in principle measurable $\Rightarrow$ gravitational memory effect

- our analysis was performed @ $\mathcal{I}^+$; similar results hold @ $\mathcal{I}^-$
  $\to$ need to connect the two to have cons. laws valid @ all times / study symmetries of gravitational scattering



- for this we need to connect data on $\mathcal{I}^+_-$ w/ that on $\mathcal{I}^-_+$ = antipodal matching

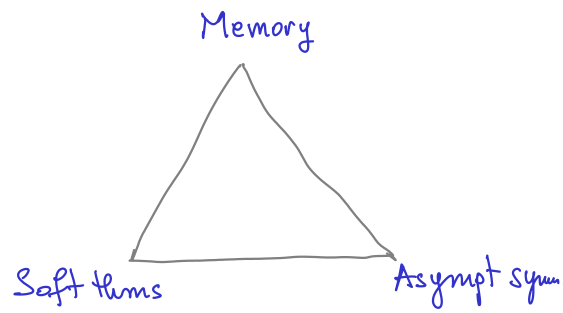
$BMS^+ \times BMS^- \to BMS^{gl.}$ $\Rightarrow$ energy is conserved @ every angle

= ∞-dim'l BMS symmetries of the S-matrix (Strominger '13)

$$Q_{BMS^+}\, S = S\, Q_{BMS^-}$$

- moreover, the Ward identities associated w/ these symmetries = Weinberg's soft graviton theorem

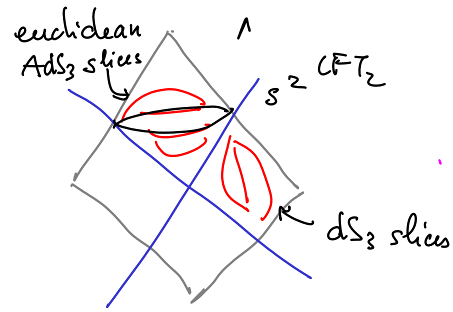
- the BMS triangle : very general relation



  - $\neq$ gauge theories (grav, YM ...)
  - $\neq$ soft thms (leading, subleading ...) which correspond to $\neq$ asympt. symmetries
  - higher dimensions

- very important for understanding IR structure of gravity & relations between results derived in the '60s, but also departure point for flat space holography

- holography → hyperbolic foliation → AdS_3/CFT_2?
(de Boer & Solodukhin 0303606)



- after reduction along non-compact direction → CFT_2 on celestial S^2?

- main observable = S-matrix (usually Fourier basis) needs to be rewritten in basis adapted to holography → Smatrix elem ↔ CFT_2 correlators

- made possible by underst. the BMS triangle