Conformal Field Theory Problems

Chapter 1

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Problem 1.1. *Using only the quantities $\hbar$, G_N, and c, construct quantities that have the units of length, mass, and time. Compute the corresponding Planck length, Planck mass, and Planck time, using SI units.*

In SI units

$$\begin{aligned} G_N &= 6.67 \times 10^{-11} \text{ kg}^{-1} \text{ m}^3 \text{ s}^{-2} , \\ \hbar &= 1.05 \times 10^{-34} \text{ kg m}^2 \text{ s}^{-1} , \\ c &= 3.00 \times 10^{8} \text{ m s}^{-1} . \end{aligned}$$

The quantities requested are then

$$\begin{aligned} \ell_P = \sqrt{\frac{G_N \hbar}{c^3}} &= 1.61 \times 10^{-35} \,\text{m} , \\ m_P = \sqrt{\frac{\hbar c}{G_N}} &= 2.17 \times 10^{-8} \,\text{kg} , \\ t_P = \sqrt{\frac{G_N \hbar}{c^5}} &= 5.37 \times 10^{-44} \,\text{s} . \end{aligned}$$

Problem 1.2. *Another proposed source of extra physics is extra dimensions. Assume that we live not in a four dimensional world but a $(4+p)$-dimensional one where the extra dimensions are all extremely small circles of length ℓ.*

a) Noting that the dimensionality of G_N is different in $(4+p)$ dimensions, what is the new expression for the Planck energy E_P in terms of $\hbar$, c, and G_N?

b) Find a relationship between G_N and the observed 4d value G_N^{4d}. Given the observed 4d value for G_N^{4d}, how small must ℓ be in order to have $E_P = 1$ TeV? Are there some values of p that you can rule out?

a) Coulomb's law is modifed in $p+4$ dimensions. The potential energy stored by two point masses is $\frac{G_N m_1 m_2}{r^{1+p}}$ which means that Newton's constant has units $[G_N] = \text{kg}^{-1}$ m^{3+p} s^{-2}, and the Planck energy should be instead

$$E_P = \left(\frac{\hbar^{p+1} c^{p+5}}{G_N} \right)^{\frac{1}{p+2}} .$$

b) The 4d Newton's constant should be $G_N^{4d} \sim G_N/\ell^p$. In order for G_N^{4d} to take its usual value, we find that

$$\ell \sim \frac{\hbar^{1+\frac{1}{p}} c^{1+\frac{5}{p}}}{E_P^{1+\frac{2}{p}}} (G_N^{4d})^{-\frac{1}{p}}$$

For numbers of extra dimensions one through six,

$$\ell \sim 3 \times 10^{13}\,\mathrm{m}, 2\,\mathrm{mm}, 1 \times 10^{-8}\,\mathrm{m}, 2.2 \times 10^{-11}\,\mathrm{m}, 5.3 \times 10^{-13}\,\mathrm{m}, 4.5 \times 10^{-14}\,\mathrm{m}\ .$$

It looks like we can rule out $p = 1$ and 2 from this back of the envelope calculation. Note superstring theory predicts $p = 6$.

Problem 1.3. *Consider an interacting scalar field*

$$S = -\int \mathrm{d}^d x \left((\partial_\mu \phi)(\partial^\mu \phi) + g\phi^n \right)\ . \tag{1.1}$$

where n is a positive integer. For what pairs (n, d) can the coupling g be dimensionless?

The model $d = 3$ and ϕ^6 is claimed to be in the same universality class as the tricritical Ising model. The model $d = 4$ and ϕ^4 is in the same universality class as the Ising model. It is sometimes called the Wilson-Fisher theory or Landau-Ginzburg theory. Then there is $d = 6$ and ϕ^3.

Problem 1.4. *Consider the Lagrangian for a Dirac spinor in d dimensions*

$$S = -\int \mathrm{d}^d x \left(\frac{i}{2} \bar{\psi} \gamma^\mu \partial_\mu \psi + g(\bar{\psi}\psi)^n \right) \tag{1.2}$$

What is the scaling dimension of ψ? (You may assume the conjugate spinor $\bar{\psi}$ has the same scaling dimension as ψ. Moreover, the gamma matrices are dimensionless.) For what (n, d) can g be made dimensionless, assuming n is a positive integer? Considering now also the scalar field of the previous problem. In what dimensions do $\phi\bar{\psi}\psi$ and $\phi^2\bar{\psi}\psi$ lead to classically marginal couplings?

The scaling dimension of the free fermion is $\Delta_\psi = \frac{d-1}{2}$. We can make $(\bar{\psi}\psi)^n$ dimensionless only in $d = 2$. These types of model are often referred to as Gross-Neveu or Thirring models. (The difference depends on how gamma matrices are sprinkled through the interaction.) The $\phi\bar{\psi}\psi$ and $\phi^2\bar{\psi}\psi$ terms are sometimes called Yukawa couplings, and the associated theory a Yukawa theory. $\phi\bar{\psi}\psi$ is classically marginal in $d = 4$ while $\phi^2\bar{\psi}\psi$ is classically marginal in $d = 3$.

Problem 1.5. *Start with the assumption that the supersymmetry transformation Q squares to the momentum operator $Q^2 \sim P$ and moreover converts fermions into bosons and bosons into fermions. Try to guess how Q acts on ϕ and ψ, purely based on dimensional analysis.*

From $Q^2 \sim P$, we have that $\Delta_Q = \frac{1}{2}$. Since $\Delta_\phi = \Delta_\psi + \frac{1}{2}$, we have $Q\phi \sim \psi$. We also need a way to turn fermions back into bosons: $Q\psi \sim \partial_\mu \phi$.

Problem 1.6. *Consider QED in d dimensions*

$$S = -\int \mathrm{d}^d x \left(\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{i}{2} \bar{\psi} \gamma^\mu (\partial_\mu + igA_\mu) \psi \right) \tag{1.3}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. What is the scaling dimension of g? What is special about $d = 4$?

We have that $\Delta_{A_\mu} = \frac{d-2}{2}$, like for a free scalar. Furthermore $\Delta_g = d - 2\Delta_\psi - \Delta_{A_\mu} = 2 - \frac{d}{2}$. In other words, the coupling is classically marginal in $d = 4$ dimensions. (Quantum effects will typically lead to it running with energy scale.) In $d > 4$, the coupling is irrelevant, while in $d < 4$ it is relevant.