Conformal Field Theory Problems

Chapter 2

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$$[\delta_1, \delta_2] x^\mu = (\omega^\mu_{1\,\lambda} a_2^\lambda - \omega^\mu_{2\,\lambda} a_1^\lambda) + (\omega^\mu_{1\,\lambda} \omega^\lambda_{2\,\nu} - \omega^\mu_{2\,\lambda} \omega^\lambda_{1\,\nu}) x^\nu \ . \tag{2.1}$$

$$\delta x^\mu = i a^\nu P_\nu (x^\mu) + \frac{i}{2} \omega^{\nu\lambda} M_{\nu\lambda}(x^\mu) \ . \tag{2.2}$$

$$\begin{aligned} [P_\mu, P_\nu] &= 0 \ , \\ [P_\mu, M_{\nu\lambda}] &= i\eta_{\mu\nu} P_\lambda - i\eta_{\mu\lambda} P_\nu \ , \\ [M_{\mu\nu}, M_{\lambda\rho}] &= i\eta_{\mu\lambda} M_{\nu\rho} - i\eta_{\nu\lambda} M_{\mu\rho} - i\eta_{\mu\rho} M_{\nu\lambda} + i\eta_{\nu\rho} M_{\mu\lambda} \ . \end{aligned} \tag{2.3}$$

Problem 2.1. *Reproduce the result (2.1) using P_μ and $M_{\nu\lambda}$ and in particular (2.2) and the commutator algebra (2.3).*

$$\begin{aligned} [\delta_1, \delta_2] &= -a_1^\mu a_2^\nu [P_\mu, P_\nu] - \frac{1}{2} \omega_1^{\mu\nu} a_2^\lambda [M_{\mu\nu}, P_\lambda] - \frac{1}{2} a_1^\lambda \omega_2^{\mu\nu} [P_\lambda, M_{\mu\nu}] - \frac{1}{4} \omega_1^{\mu\nu} \omega_2^{\lambda\rho} [M_{\mu\nu}, M_{\lambda\rho}] \\ &= -\frac{i}{2} \omega_1^{\mu\nu} a_{2\mu} P_\nu + \frac{i}{2} \omega_1^{\mu\nu} a_{2\nu} P_\mu + \frac{i}{2} \omega_2^{\mu\nu} a_{1\mu} P_\nu - \frac{i}{2} \omega_2^{\mu\nu} a_{1\nu} P_\mu \\ &\quad + \frac{i}{4} (\omega_1^{\mu\lambda} \omega_{2\lambda}{}^\nu - \omega_1^{\lambda\mu} \omega_{2\lambda}{}^\nu + \omega_1^{\lambda\mu} \omega_2{}^\nu{}_\lambda - \omega_1^{\mu\lambda} \omega_2{}^\nu{}_\lambda) M_{\mu\nu} \\ &= i(\omega_1^{\mu\nu} a_{2\nu} - \omega_2^{\mu\nu} a_{1\nu}) P_\mu + \frac{i}{2} (\omega_1^{\mu\lambda} \omega_{2\lambda}{}^\nu - \omega_2^{\mu\lambda} \omega_{1\lambda}{}^\nu) M_{\mu\nu} \end{aligned}$$

Problem 2.2. *For a vector representation, one takes*

$$(M_{\mu\nu})^\lambda{}_\rho = i\eta_{\mu\rho} \delta^\lambda_\nu - i\delta^\lambda_\mu \eta_{\nu\rho} \ . \tag{2.4}$$

(Notice that the indices μ and ν take a dual role, labeling both the Lorentz generator and its matrix components.) For the spinor representation, one takes instead

$$(M_{\mu\nu})_\alpha{}^\beta = -\frac{i}{2} (\gamma_{\mu\nu})_\alpha{}^\beta = -\frac{i}{4} (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu)_\alpha{}^\beta \ , \tag{2.5}$$

where $(\gamma_\mu)_\alpha{}^\beta$ are the Dirac γ-matrices, $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$. Verify that these two representations of the Lorentz group obey the commutation relations (2.3).

For the vector, let's verify the first term in the commutation relations. The remaining three should follow by symmetry.

$$\begin{aligned} (M_{\mu\nu})^\sigma{}_\tau (M_{\lambda\rho})^\tau{}_\theta &= (i\eta_{\mu\tau}\delta^\sigma_\nu - i\delta^\sigma_\mu \eta_{\nu\tau})(i\eta_{\lambda\theta}\delta^\tau_\rho - i\delta^\tau_\lambda \eta_{\rho\theta}) \\ &= -\eta_{\mu\rho}\eta_{\lambda\theta}\delta^\sigma_\nu + \ldots \end{aligned}$$

Next consider

$$(M_{\lambda\rho})^\sigma{}_\tau (M_{\mu\nu})^\tau{}_\theta = (i\eta_{\lambda\tau}\delta^\sigma_\rho - i\delta^\sigma_\lambda \eta_{\rho\tau})(i\eta_{\mu\theta}\delta^\tau_\nu - i\delta^\tau_\mu \eta_{\nu\theta})$$

In particular I want to combine the second of each of the two terms in the product:

$$(M_{\lambda\rho})^\sigma{}_\tau (M_{\mu\nu})^\tau{}_\theta = \ldots - \eta_{\rho\mu}\eta_{\nu\theta}\delta^\sigma_\lambda$$

We find then that

$$\begin{aligned} [M_{\mu\nu}, M_{\lambda\rho}]^\sigma{}_\theta &= i\eta_{\mu\rho}(i\delta^\sigma_\nu \eta_{\lambda\theta} - i\delta^\sigma_\lambda \eta_{\nu\theta}) + \ldots \\ &= -i\eta_{\mu\rho}(M_{\nu\lambda})^\sigma{}_\theta + \ldots \end{aligned}$$

For the spinor, we need the relation

$$\gamma^{\mu\nu}\gamma^{\lambda\rho} = \gamma^{\mu\nu\lambda\rho} + \eta^{\mu\rho}\gamma^{\nu\lambda} + \eta^{\nu\lambda}\gamma^{\mu\rho} - \eta^{\mu\lambda}\gamma^{\nu\rho} - \eta^{\nu\rho}\gamma^{\mu\lambda} + \eta^{\mu\rho}\eta^{\nu\lambda} - \eta^{\mu\lambda}\eta^{\nu\rho} \ .$$

The hard way to demonstrate this relation is to use $\{\gamma^\mu, \gamma^\nu\} = \eta^{\mu\nu}$. The easy way is to test the various special cases. From this relation, we deduce that

$$[\gamma_{\mu\nu}, \gamma_{\lambda\rho}] = 2\eta_{\nu\lambda}\gamma_{\mu\rho} - 2\eta_{\mu\lambda}\gamma_{\nu\rho} + 2\eta_{\mu\rho}\gamma_{\nu\lambda} - 2\eta_{\nu\rho}\gamma_{\mu\lambda} \ .$$

Multiplying through by $(-\frac{i}{2})^2$ yields the desired relation.

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = f(x)\eta_{\mu\nu} \ . \qquad (2.6)$$

Problem 2.3. *Verify that* $b_{\mu\nu} = \lambda\eta_{\mu\nu} + \omega_{\mu\nu}$ *and* $c_{\mu\nu\rho} = \eta_{\mu\rho}b_\nu + \eta_{\mu\nu}b_\rho - \eta_{\nu\rho}b_\mu$ *are the only solutions for* $b_{\mu\nu}$ *and* $c_{\mu\nu\rho}$ *consistent with (2.6).*

Problem 2.4. *Verify that the infinitesimal versions of the transformations in Figure 1 recover* a_μ, $b_{\mu\nu}$ *and* $c_{\mu\nu\rho}$.

Problem 2.5. *Demonstrate that an inversion followed by a translation followed by a further inversion is equivalent to a special coordinate transformation.*

We look at

$$x^\mu \to \frac{x^\mu}{x^2} \to \frac{x^\mu}{x^2} + a^\mu \to \frac{\frac{x^\mu}{x^2} + a^\mu}{\left(\frac{x^\nu}{x^2} + a^\nu\right)\left(\frac{x_\nu}{x^2} + a_\nu\right)} = \frac{\frac{x^\mu}{x^2} + a^\mu}{\frac{1}{x^2} + 2\frac{a\cdot x}{x^2} + a^2} = \frac{x^\mu + a^\mu x^2}{1 + 2a\cdot x + a^2} \ .$$

So we just need to make the identification $a^\mu = -b^\mu$.

- translations: $x'^{\mu} = x^{\mu} + a^{\mu}$
- Lorentz: $x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$
- dilatations (scale transformations): $x'^{\mu} = \lambda x^{\mu}$
- special conformal transformations: $x'^{\mu} = \frac{x^{\mu} - b^{\mu} x^2}{1 - 2b\cdot x + b^2 x^2}$

Figure 1: The finite versions of the generators of the conformal symmetry group.

Problem 2.6. *Compute the commutator of P^2 with K_μ and D. What happens to a massive particle state $|p\rangle$ (where $P^2|p\rangle = m^2|p\rangle$, $m^2 \neq 0$) under the infinitesimal special conformal transformation K_μ?*

Both $[D, P^2]$ and $[K_\mu, P^2]$ are nonzero. Therefore P^2 is not a Casimir of the conformal group. Using these nonzero commutation relations, it is straightforward to see that $K_\mu|p\rangle$ is not an eigenstate of P^2 and does not have a well defined mass.

Problem 2.7. *If $\mu, \nu = 0, \ldots, d-1$, then define $J_{\mu\nu} = M_{\mu\nu}$ along with $J_{\mu,d} = \frac{1}{2}(P_\mu - K_\mu)$, $J_{\mu,d+1} = \frac{1}{2}(P_\mu + K_\mu)$, and $J_{d,d+1} = D$, along with the constraint that $J_{ab} = -J_{ba}$ is antisymmetric. Show that the commutators of these generators are the same as for a $(d+2)$-dimensional orthogonal group, with metric signature $(2, d)$, i.e. $SO(2, d)$.*

One needs to check that

$$
\begin{aligned}
[J_{\mu\nu}, J_{\rho d}] &= i(\eta_{\nu\rho} J_{\mu d} - \eta_{\mu\rho} J_{\nu d}) \ , \\
[J_{\mu\nu}, J_{\rho,d+1}] &= i(\eta_{\nu\rho} J_{\mu,d+1} - \eta_{\mu\rho} J_{\nu,d+1}) \ , \\
[J_{\mu d}, J_{\nu,d+1}] &= -i\eta_{\mu\nu} J_{d,d+1} \ .
\end{aligned}
$$

Problem 2.8. *Write out the consistency relations (2.6) in $d = 2$ in the coordinate system $x_\pm = x \pm t$. What can you conclude about the allowed form of ϵ_μ?*

We need to take a little time to do the coordinate transformations. Note that

$$
\partial_x = \frac{\partial x_+}{\partial x}\frac{\partial}{\partial x_+} + \frac{\partial x_-}{\partial x}\frac{\partial}{\partial x_-} = \partial_+ + \partial_- \ .
$$

Similarly $\partial_t = \partial_+ - \partial_-$. The transformation rule for ϵ_μ is similar, $\epsilon_x = \epsilon_+ + \epsilon_-$ and $\epsilon_t = \epsilon_+ - \epsilon_-$.

Now let's write out

$$
\begin{aligned}
0 &= \partial_t \epsilon_x + \partial_x \epsilon_t = (\partial_+ - \partial_-)(\epsilon_+ + \epsilon_-) + (\partial_+ + \partial_-)(\epsilon_+ - \epsilon_-) = 2(\partial_+\epsilon_+ - \partial_-\epsilon_-) \ , \\
-f(x) &= 2\partial_t \epsilon_t = 2(\partial_+ - \partial_-)(\epsilon_+ - \epsilon_-) \ , \\
f(x) &= 2\partial_x \epsilon_x = 2(\partial_+ + \partial_-)(\epsilon_+ + \epsilon_-) \ .
\end{aligned}
$$

Adding the second two equations, we find $\partial_+\epsilon_+ + \partial_-\epsilon_- = 0$. Combining this result with the first equation, we find that $\partial_\pm\epsilon_\pm = 0$. In other words, any $\epsilon_\pm = f(x \mp t)$ will be a conformal transformation. This set of transformations is much larger than in $d > 2$.

An alternate approach is to transform the Minkowski tensor:

$$\begin{pmatrix} \frac{\partial t}{\partial x_+} & \frac{\partial x}{\partial x_+} \\ \frac{\partial t}{\partial x_-} & \frac{\partial x}{\partial x_-} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial t}{\partial x_+} & \frac{\partial t}{\partial x_-} \\ \frac{\partial x}{\partial x_+} & \frac{\partial x}{\partial x_-} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$$

where we need $x = \frac{1}{2}(x_+ + x_-)$ and $t = \frac{1}{2}(x_+ - x_-)$. In the new coordinate system, we see directly that $\partial_\pm\epsilon_\pm = 0$.

Problem 2.9. *Compute $\Omega(x)$ for the (finite) special conformal transformations.*

$$\Omega(x) = (1 - 2b \cdot x + b^2x^2)^2$$

Problem 2.10. *Verify that a special conformal transformation that does not preserve the location of a defect will transform the defect into a spherical configuration.*

See the answer to problem 6.1.