Conformal Field Theory Problems

Chapter 3

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Problem 3.1. *Verify that*

$$[K_\mu, \phi_I(x)] = -2ix_\mu \Delta\phi_I(x) - i(2x_\mu x^\nu \partial_\nu - x^2\partial_\mu)\phi_I(x) - 2x^\rho (M_{\rho\mu})_I{}^J \phi_J(x) \tag{3.1}$$

for special conformal transformations.

We follow the same strategy used in class for $[D, \phi_I(x)]$:

$$\begin{aligned}
[K_\mu, \phi_I(x)] &= [K_\mu e^{iP\cdot x}\phi_I(0)e^{-iP\cdot x}] = K_\mu e^{iP\cdot x}\phi_I(0)e^{-iP\cdot x} - e^{iP\cdot x}\phi_I(0)e^{-iP\cdot x}K_\mu \\
&= e^{iP\cdot x}\left(e^{-iP\cdot x}K_\mu e^{iP\cdot x}\phi_I(0) - \phi_I(0)e^{-iP\cdot x}K_\mu e^{iP\cdot x}\phi_I(0)\right)e^{-iP\cdot x} \\
&= e^{iP\cdot x}[\hat{K}_\mu, \phi_I(0)]e^{-iP\cdot x}
\end{aligned}$$

where $\hat{K}_\mu \equiv e^{-iP\cdot x}K_\mu e^{iP\cdot x}$. We then examine $\hat{K}_\mu$:

$$\begin{aligned}
\hat{K}_\mu &= \left(1 - iP\cdot x - \frac{1}{2}(P\cdot x)^2 + \ldots\right) K_\mu \left(1 + iP\cdot x - \frac{1}{2}(P\cdot x)^2 + \ldots\right) \\
&= K_\mu - i[P_\nu, K_\mu]x^\nu - \frac{1}{2}[P_\lambda, [P_\nu, K_\mu]]x^\lambda x^\nu + \ldots
\end{aligned}$$

The ellipsis contains higher order commutators of P_μ and vanishes.

$$\begin{aligned}
\hat{K}_\mu &= K_\mu - 2(\eta_{\mu\nu}D - M_{\mu\nu})x^\nu + i[P_\lambda, \eta_{\mu\nu}D - M_{\mu\nu}]x^\lambda x^\nu \\
&= K_\mu - 2(\eta_{\mu\nu}D - M_{\mu\nu})x^\nu + P_\lambda \eta_{\mu\nu}x^\lambda x^\nu + (\eta_{\lambda\mu}P_\nu - \eta_{\lambda\nu}P_\mu)x^\lambda x^\nu \\
&= K_\mu - 2x_\mu D + 2M_{\mu\nu}x^\nu + 2x_\mu x_\nu P^\nu - x^2 P_\mu \ .
\end{aligned}$$

We thus find that

$$[K_\mu, \phi_I(x)] = -2ix_\mu \Delta\phi_I(x) - i(2x_\mu x_\nu \partial^\nu - x^2\partial_\mu)\phi_I(x) + 2x^\nu (M_{\mu\nu})_I{}^J\phi_J(x)$$

Problem 3.2. *Verify that the rule*

$$\phi'(x') = \Omega^{\Delta/2}\phi(x) \ . \tag{3.2}$$

for the finite conformal symmetry transformations is consistent with the infinitesimal transformation rule (3.1) for the special conformal transformations K_μ.

From problem (2.9), we use that $\Omega = (1 - 2b\cdot x + b^2x^2)^2$. To linear order then

$$\phi'(x') = (1 - 2b\cdot x + b^2x^2)^{\Delta}\phi(x) \approx (1 - 2\Delta b\cdot x)\phi(x)\ .$$

Furthermore

$$\phi(x') = \phi\left(\frac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + x^2}\right) \approx \phi\left(x^\mu - b^\mu x^2 + 2x^\mu b\cdot x\right) \approx \phi(x) + (-b^\mu x^2 + 2x^\mu b\cdot x)\partial_\mu\phi(x)$$

For a scalar then

$$\delta\phi = \phi'(x') - \phi(x') \approx -2\Delta b\cdot x\phi(x) - (2x^\mu b\cdot x - b^\mu x^2)\partial_\mu\phi(x)$$

We make the identification $\delta\phi = -ib^\mu[K_\mu, \phi(x)]$.

Problem 3.3. Conformal Currents. *We have the action for a real, massless scalar field*

$$S = -\frac{1}{2}\int \mathrm{d}^d x\,(\partial^\mu\phi)(\partial_\mu\phi)\ .$$

In each part below, we ask you to consider a variation of the scalar field $\delta\phi(\lambda(x))$ which depends linearly on an infinitesimal parameter $\lambda(x)$. In each case, compute δS, putting it in the form

$$\delta S = \int \mathrm{d}^d x(\partial_\mu\lambda)J^\mu\ .$$

Verify that $\partial_\mu J^\mu = 0$ on-shell.

a) Translations:

$$\delta\phi = \lambda(x)v^\mu\partial_\mu\phi(x)\ ,$$

where v^μ is a constant vector of unit length.

b) Dilatations:

$$\delta\phi = \lambda(x)(\Delta_\phi + x^\mu\partial_\mu)\phi(x)\ .$$

c) Special conformal transformations:

$$\delta\phi = \lambda(x)v^\mu\left(2\Delta_\phi x_\mu + 2x_\mu x^\nu\partial_\nu - x^2\partial_\mu\right)\phi\ .$$

a)

$$\begin{aligned}
\delta S &= -\int \mathrm{d}^d x(\partial^\mu\phi)\partial_\mu(\lambda v^\nu\partial_\nu\phi) \\
&= \int \mathrm{d}^d x\left(-\frac{1}{2}\partial_\nu\left[\lambda v^\nu(\partial^\mu\phi)(\partial_\mu\phi\right] - (\partial^\mu\lambda)v^\nu(\partial_\mu\phi)(\partial_\nu\phi) + \frac{1}{2}(\partial_\nu\lambda)v^\nu(\partial^\mu\phi)(\partial_\mu\phi)\right) \\
&= \int \mathrm{d}^d x\,(\partial^\mu\lambda)v^\nu\left((\partial_\mu\phi)(\partial_\nu\phi) - \frac{1}{2}\eta_{\mu\nu}(\partial^\rho\phi)(\partial_\rho\phi)\right)\ .
\end{aligned}$$

Thus we find

$$J^\mu_T = v^\nu\left((\partial_\mu\phi)(\partial_\nu\phi) - \frac{1}{2}\eta_{\mu\nu}(\partial^\rho\phi)(\partial_\rho\phi)\right)\ .$$

b)

$$\begin{aligned}
\delta S &= -\int \mathrm{d}^d x (\partial^\mu \phi)\partial_\mu(\lambda(\Delta_\phi + x^\nu\partial_\nu)\phi) \\
&= -\int \mathrm{d}^d x (\partial^\mu \phi)\left[(\partial_\mu\lambda)(\Delta_\phi + x^\nu\partial_\nu)\phi + \lambda\Delta_\phi\partial_\mu\phi + \lambda\delta^\nu_\mu\partial_\nu\phi + \lambda x^\nu\partial_\nu\partial_\mu\phi\right] \\
&= -\int \mathrm{d}^d x \left[(\partial_\mu\lambda)(\partial^\mu\phi)(\Delta_\phi + x^\nu\partial_\nu)\phi + (\Delta_\phi+1)\lambda(\partial^\mu\phi)(\partial_\mu\phi) + \lambda(\partial^\mu\phi)x^\nu\partial_\mu\partial_\nu\phi\right] \\
&= -\int \mathrm{d}^d x \left[(\partial_\mu\lambda)(\partial^\mu\phi)(\Delta_\phi + x^\nu\partial_\nu)\phi + \frac{1}{2}\partial_\nu\left(\lambda(\partial^\mu\phi)x^\nu\partial_\mu\phi\right) - \frac{1}{2}(\partial_\nu\lambda)x^\nu(\partial^\mu\phi)(\partial_\mu\phi)\right] \\
&= -\int \mathrm{d}^d x (\partial_\mu\lambda)\left[(\partial^\mu\phi)(\Delta_\phi + x^\nu\partial_\nu)\phi - \frac{1}{2}\eta^{\mu\nu}x_\nu(\partial^\rho\phi)(\partial_\rho\phi)\right] .
\end{aligned}$$

So we find

$$J^\mu_D = -\Delta_\phi(\partial^\mu\phi)\phi - x^\nu(\partial^\mu\phi)(\partial_\nu\phi) + \frac{1}{2}x^\mu(\partial^\rho\phi)(\partial_\rho\phi)$$

For conservation, note

$$\partial_\mu J^\mu_D = -\Delta_\phi(\partial^\mu\phi)(\partial_\mu\phi) - (\partial^\mu\phi)(\partial_\mu\phi) - x^\nu(\partial^\mu\phi)(\partial_\mu\partial_\nu\phi) + \frac{d}{2}(\partial^\mu\phi)(\partial_\mu\phi) + x^\mu(\partial^\rho\phi)(\partial_\rho\partial_\mu\phi) = 0$$

c)

$$\begin{aligned}
\delta S &= -\int \mathrm{d}^d x (\partial^\mu\phi)\partial_\mu(\lambda v^\rho\left(2\Delta_\phi x_\rho + 2x_\rho x^\nu\partial_\nu - x^2\partial_\rho\right)\phi) \\
&= -\int \mathrm{d}^d x (\partial^\mu\phi)\Big[(\partial_\mu\lambda)v^\rho\left(2\Delta_\phi x_\rho + 2x_\rho x^\nu\partial_\nu - x^2\partial_\rho\right)\phi \\
&\quad +(\partial^\mu\phi)\lambda v^\rho(2\Delta_\phi\eta_{\rho\mu} + 2\eta_{\rho\mu}x^\nu\partial_\nu + 2x_\rho\partial_\mu - 2x_\mu\partial_\rho + 2\Delta_\phi x_\rho\partial_\mu + 2x_\rho x^\nu\partial_\nu\partial_\mu - x^2\partial_\rho\partial_\mu)\phi\Big] \\
&= -\int \mathrm{d}^d x (\partial^\mu\phi)v^\rho(\partial_\mu\lambda)\left(2\Delta_\phi x_\rho + 2x_\rho x^\nu\partial_\nu - x^2\partial_\rho\right)\phi \\
&\quad +2(\partial^\mu\phi)\lambda\left(\Delta_\phi v_\mu + \frac{d}{2}v\cdot x\partial_\mu + v\cdot x\, x\cdot\partial\partial_\mu - \frac{1}{2}x^2 v\cdot\partial\partial_\mu\right)\phi \\
&= -\int \mathrm{d}^d x (\partial^\mu\phi)v^\rho(\partial_\mu\lambda)\left(2\Delta_\phi x_\rho + 2x_\rho x^\nu\partial_\nu - x^2\partial_\rho\right)\phi \\
&\quad +\partial^\rho\left(\lambda\Delta_\phi v_\rho\phi^2 + \lambda\left(v\cdot x x_\rho - \frac{1}{2}x^2 v_\rho\right)(\partial^\mu\phi)(\partial_\mu\phi)\right) \\
&\quad -(\partial^\rho\lambda)\left(\Delta_\phi v_\rho\phi^2 + \left(v\cdot x x_\rho - \frac{1}{2}x^2 v_\rho\right)(\partial^\mu\phi)(\partial_\mu\phi)\right)
\end{aligned}$$

We thus identify the conserved current

$$\begin{aligned}
J^\mu_K &= -(\partial^\mu\phi)(2\Delta_\phi v\cdot x + 2v\cdot x\, x\cdot\partial - x^2 v\cdot\partial)\phi + v^\mu\Delta_\phi\phi^2 + \left(v\cdot x x^\mu - \frac{1}{2}x^2 v^\mu\right)(\partial\phi)\cdot(\partial\phi) \\
&= -v^\rho(2x_\rho x_\nu - \eta_{\rho\nu}x^2)T^{\nu\mu} - (\partial^\mu\phi)\Delta_\phi(2v\cdot x)\phi + v^\mu\Delta_\phi\phi^2 .
\end{aligned}$$

Problem 3.4. *For special conformal transformations, verify the remarkable property*

$$|x_1' - x_2'| = \frac{|x_1 - x_2|}{\gamma_1^{1/2}\gamma_2^{1/2}} \ , \tag{3.3}$$

where $\gamma_i = 1 - 2b \cdot x_i + b^2 x_i^2$.