Conformal Field Theory Problems

Chapter 4

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Problem 4.1. *In the case of the free scalar field, the simple* $(\partial\phi)^2$ *action is not Weyl symmetric. However, if one adds the* $R\phi^2$ *term*

$$S = -\frac{1}{2}\int \mathrm{d}^d x \sqrt{-g}\left[(\partial_\mu\phi)(\partial^\mu\phi) + \xi R\phi^2\right] \tag{4.1}$$

where $\xi = \frac{(d-2)}{4(d-1)}$, *the action is Weyl symmetric. Verify this fact, assuming* $\phi \to \Omega^{\frac{d-2}{4}}\phi$ *and* $g_{\mu\nu} \to \Omega^{-1} g_{\mu\nu}$ *under Weyl rescaling.*

Let $\Omega = 1 + \omega$ where ω is assumed to be small. At linear order, the shift in the metric is then $\delta g_{\mu\nu} = -\omega g_{\mu\nu}$ and $\delta\phi = \frac{d-2}{4}\omega\phi$. We also need $\delta g^{\mu\nu} = \omega g^{\mu\nu}$ and $\delta\sqrt{-g} = -\frac{d}{2}\omega\sqrt{-g}$. Figuring out how the Ricci scalar shifts is a bit of a pain. One nice computer package for doing so is called xAct. The result is that

$$\delta\left(\sqrt{-g}R\right) = \sqrt{-g}\,\omega\left(-\frac{d-2}{2}R + (d-1)\Box\right) .$$

We find then that

$$\begin{aligned} \delta S &= -\frac{1}{2}\int \mathrm{d}^d x \Big(2(\partial^\mu\phi)(\partial_\mu\delta\phi)\sqrt{-g} + (\partial_\mu\phi)(\partial_\nu\phi)\delta(\sqrt{-g}g^{\mu\nu}) + \\ &\qquad +2\xi R\phi\delta\phi\sqrt{-g} + \xi\phi^2\delta(\sqrt{-g}R)\Big) \\ &= -\int \mathrm{d}^d x\sqrt{-g}\Big[\frac{d-2}{4}(\partial^\mu\phi)\partial_\mu(\omega\phi) - \frac{d-2}{4}\omega(\partial^\mu\phi)(\partial_\mu\phi) + \\ &\qquad + \frac{d-2}{4}\xi R\omega\phi^2 - \frac{d-2}{4}\xi R\omega\phi^2 + \frac{d-1}{2}\xi\omega\Box\phi^2\Big] . \end{aligned}$$

Integrating the first term by parts, we see that the first and second terms cancel against the last term, given the special value of ξ stated in the problem. The third and fourth terms cancel trivially.

Problem 4.2. *Compute the stress tensor in the flat space limit* $g_{\mu\nu} = \eta_{\mu\nu}$ *for the scalar field of problem 4.1 with the conformal coupling* $\xi = \frac{(d-2)}{4(d-1)}$. *Check that* $T^{\mu\nu}$ *is conserved and traceless on-shell in the flat space limit.*

Varying the Ricci scalar with respect to the metric is a bit of a pain. Using xAct, we find

$$\delta\left(\sqrt{-g}R\right) = \sqrt{-g}\,\delta g_{\mu\nu}\left[-\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right) + (\nabla^\mu\nabla^\nu - g^{\mu\nu}\Box)\right]$$

The remainder of the calculation is straightforward:

$$
\begin{aligned}
T^{\mu\nu} = 2\sqrt{-g}\frac{\delta S}{\delta g_{\mu\nu}} &= (\partial^\mu\phi)(\partial^\nu\phi) - \frac{1}{2}g^{\mu\nu}((\partial_\rho\phi)(\partial^\rho\phi) + \xi R\phi^2) \\
&\quad +\xi(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R)\phi^2 - \xi(\nabla^\mu\nabla^\nu - \eta^{\mu\nu}\Box)\phi^2 \ .
\end{aligned}
$$

In flat space, this result reduces to

$$
T^{\mu\nu} = (\partial^\mu\phi)(\partial^\nu\phi) - \frac{1}{2}\eta^{\mu\nu}(\partial_\rho\phi)(\partial^\rho\phi) - \xi(\partial^\mu\partial^\nu - \eta^{\mu\nu}\Box)\phi^2 \ . \tag{4.2}
$$

The last term is invisible from the standard derivation, using Noether's theorem and translation invariance. It is however a total derivative and corresponds to a standard "improvement" term.

The trace of the stress tensor is then

$$
T^\mu_\mu = \left(1 - \frac{d}{2}\right)(\partial\phi)^2 - \xi(1-d)\Box\phi^2 \ . \tag{4.3}
$$

The $(\partial\phi)^2$ terms cancel for the special value of ξ and the $\phi\Box\phi$ terms vanish by the equations of motion.

Similarly, for conservation, dropping immediately all terms proportional to $\Box\phi$, we find

$$
\partial_\mu T^{\mu\nu} = (\partial^\mu\phi)(\partial_\mu\partial^\nu\phi) - \eta^{\mu\nu}(\partial^\nu\partial_\rho\phi)(\partial^\rho\phi) - \xi(\Box\partial^\nu - \partial^\nu\Box)\phi^2 = 0 \ . \tag{4.4}
$$