Conformal Field Theory Problems

Chapter 5

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Problem 5.1. *Continuing the small $|x|$ expansion of $C_\Delta(x,\partial)$, we find at next order*

$$C_\Delta(x,\partial)\phi(0) = \frac{c}{|x|^{\Delta_1+\Delta_2-\Delta}}\left(1+\alpha x^\mu \partial_\mu + \ldots\right)\phi(0) \ .$$

By acting with K^μ on boths sides, show that $\alpha = \frac{\Delta_1-\Delta_2+\Delta}{2\Delta}$.

We act first with K_μ on the right hand side, finding

$$\begin{aligned}[K_\mu, \phi_1(x)\phi_2(0)] &= [K_\mu,\phi_1(x)]\phi_2(0) + \phi_1(x)[K_\mu,\phi_2(0)] \\ &= \left[-2ix_\mu\Delta_1 - i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu)\right]\phi_1(x)\phi_2(0)\end{aligned}$$

Note that $[K_\mu,\phi_2(0)]=0$. Next we insert the ansatz for $C_\Delta(x,\partial)$, looking at the contribution of the $\phi(x)$ primary to the OPE of $\phi_1(x)$ and $\phi_2(0)$:

$$[K_\mu,\phi_1(x)\phi_2(0)]|_\phi = \left[-2ix_\mu\Delta_1 - i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu)\right]\frac{c}{|x|^{\Delta_1+\Delta_2-\Delta}}\left(1+\alpha x^\mu\partial_\mu+\ldots\right)\phi(0)$$

Acting with the derivatives, this expression simplifies to

$$[K_\mu,\phi_1(x)\phi_2(0)]|_\phi = \frac{-ic}{|x|^{\Delta_1+\Delta_2-\Delta}}((\Delta_1-\Delta_2+\Delta)x_\mu+\ldots)\phi(0) \ .$$

Now we are ready to look at the right hand side. It would be a bit simpler to put the vacuum to the right of $\phi(0)$ and thereby consider the direct action of the generators on states. For its pedagogical value, let us look instead at the commutator action. While K_μ annihilates $\phi(0)$, it does not annihilate $\alpha x^\mu\partial_\mu\phi(0) = i\alpha x^\mu[P_\mu,\phi(0)]$. We have to use the commutator of K_μ with P_ν. By the Jacobi identity, we have

$$[K_\mu,[P_\nu,\phi(0)]] = -[P_\nu,[\phi(0),K_\mu]] - [\phi(0),[K_\mu,P_\nu]] \ . \tag{5.1}$$

As $[K_\mu,\phi(0)]=0$, only the second term survives. Moreover, we have that $[M_{\mu\nu},\phi(0)]=0$, for scalars, leaving us with

$$[K_\mu,[P_\nu,\phi(0)]] = [[K_\mu,P_\nu],\phi(0)] = 2i\eta_{\mu\nu}[D,\phi(0)] = -2\eta_{\mu\nu}\Delta \ . \tag{5.2}$$

The right hand side of the OPE therefore gives us

$$[K_\mu,\phi_1(x)\phi_2(0)]|_\phi = \frac{-2ic\alpha}{|x|^{\Delta_1+\Delta_2-\Delta}}\Delta\, x_\mu\phi(0)$$

Comparing the rhs and lhs, we learn that $\alpha = \frac{\Delta_1-\Delta_2+\Delta}{2\Delta}$.

Problem 5.2. *Using the procedure detailed in class notes comparing the three point and two point functions, compute the first three terms in $C_\Delta(x,\partial)$.*

We expand the three point function for small x:

$$\frac{1}{|x|^{\Delta_1+\Delta_2-\Delta}|z|^{\Delta_2+\Delta-\Delta_1}|x-z|^{\Delta_1+\Delta-\Delta_2}} = \frac{1}{|x|^{\Delta_1+\Delta_2-\Delta}|z|^{2\Delta}}\Big[1+\frac{\Delta_1+\Delta-\Delta_2}{2}\left(\frac{2x\cdot z}{z^2}-\frac{x^2}{z^2}\right)+ + \frac{\Delta_1+\Delta-\Delta_2}{4}\left(\frac{\Delta_1+\Delta-\Delta_2}{2}+1\right)\left(\frac{2x\cdot z}{z^2}\right)^2+O(x^3)\Big]$$

We have used here a couple of intermediate results:

$$\begin{aligned}\frac{1}{|x-z|^{2\alpha}} &= \frac{1}{(x^2-2x\cdot z+z^2)^\alpha}\,,\\ \frac{1}{(1-\epsilon)^\alpha} &= 1+\alpha\epsilon+\frac{\alpha(\alpha+1)}{2}\epsilon^2+\dots\,. \end{aligned} \tag{5.3}$$

We expect to be able to compare this expression with something of the form

$$C_\Delta(x,\partial_y)\frac{1}{|z-y|^{2\Delta}} = \frac{1}{|x|^{\Delta_1+\Delta_2-\Delta}}\left(1+\alpha x\cdot\partial_y+\beta x^2(\partial_y)^2+\gamma(x\cdot\partial_y)^2+\dots\right)\frac{1}{|z-y|^{2\Delta}}$$

evaluated at $y=0$, where we need to compute the constants α, β, and γ. Let's compute the ∂_y derivatives acting on $|y-z|^{-2\Delta}$ and then evaluated at $y=0$:

$$\begin{aligned}\frac{\partial}{\partial y^\mu}\frac{1}{|y-z|^{2\Delta}} &= -\frac{2\Delta(y_\mu-z_\mu)}{|y-z|^{2\Delta+2}} \xrightarrow{y\to 0} \frac{2\Delta z_\mu}{|z|^{2\Delta+2}}\\ \frac{\partial}{\partial y^\nu}\frac{\partial}{\partial y^\mu}\frac{1}{|y-z|^{2\Delta}} &= -\frac{2\Delta}{|y-z|^{2\Delta+2}}\left(\eta_{\mu\nu}-(2\Delta+2)\frac{(y_\mu-z_\mu)(y_\nu-z_\nu)}{|y-z|^2}\right)\xrightarrow{y\to 0}\\ &\quad -\frac{2\Delta}{|z|^{2\Delta+2}}\left(\eta_{\mu\nu}-(2\Delta+2)\frac{z_\mu z_\nu}{z^2}\right)\end{aligned}$$

We find then that

$$\begin{aligned}C_\Delta(x,\partial_y)\frac{1}{|x-y|^{2\Delta}}\bigg|_{y=0} &= \frac{1}{|x|^{\Delta_1+\Delta_2-\Delta}|z|^{2\Delta}}\Big(1+\frac{2\alpha x\cdot z}{z^2}-2\beta\Delta(d-2\Delta-2)\frac{x^2}{z^2}+\\ &\quad -2\gamma\Delta\left(\frac{x^2}{z^2}-(2\Delta+2)\frac{(x\cdot z)^2}{z^4}\right)+\dots\Big)\end{aligned} \tag{5.4}$$

Comparing terms, we find

$$\begin{aligned}2\Delta\alpha &= \Delta_1+\Delta-\Delta_2\,,\\ 2\gamma\Delta(2\Delta+2) &= (\Delta_1+\Delta-\Delta_2)\left(\frac{\Delta_1+\Delta-\Delta_2}{2}+1\right)\,,\\ -2\beta\Delta(d-2\Delta-2)-2\gamma\Delta &= -\frac{\Delta_1+\Delta-\Delta_2}{2}\,.\end{aligned}$$

The solution is

$$\begin{aligned}\alpha &= \frac{\Delta+\Delta_1-\Delta_2}{2\Delta}\\ \beta &= -\frac{(\Delta+\Delta_1-\Delta_2)(\Delta-\Delta_1+\Delta_2)}{8\Delta(1+\Delta)\left(\Delta-\frac{d-2}{2}\right)}\\ \gamma &= \frac{(\Delta+\Delta_1-\Delta_2)(2+\Delta+\Delta_1-\Delta_2)}{8\Delta(1+\Delta)}\end{aligned}$$

The answer looks a bit nicer for $\Delta_1 = \Delta_2$.

Problem 5.3. *By explicitly computing the first few terms in a small z expansion, verify the form of the conformal block for $\ell = 0$ and $d = 4$ by comparing it against your previous small x expansion of $C_\Delta(x, \partial)$.*

This is a good problem for Mathematica. We have to be a little careful the order in which we expand things, as the coordinate choice is adapted for x_2 small. Let us use the relation between two and three point functions to take care of the $C_\Delta(x_{34}, \partial_4)$ factor because x_3 and x_4 will not be small:

$$C_\Delta(x_{34}, \partial_4)\frac{1}{|x_{24}|^{2\Delta}} = \frac{1}{|x_{34}|^{2\eta-\Delta}|x_{24}|^{\Delta}|x_{23}|^{\Delta}}$$

Now we act with the series expansion form of $C_\Delta(x_{12}, \partial_2)$:

$$\frac{1}{|x_{12}|^{2\eta-\Delta}}\left(1+\frac{1}{2}x_{12}\cdot\partial_2+\dots\right)\frac{1}{|x_{34}|^{2\eta-\Delta}|x_{24}|^{\Delta}|x_{23}|^{\Delta}} =$$
$$\frac{1}{|x_{12}|^{2\eta-\Delta}}\left(1-\frac{\Delta}{2}\left(\frac{x_{12}\cdot x_{24}}{|x_{24}|^2}+\frac{x_{12}\cdot x_{23}}{|x_{23}|^2}\right)+\dots\right)\frac{1}{|x_{34}|^{2\eta-\Delta}|x_{24}|^{\Delta}|x_{23}|^{\Delta}}$$

We should then identify the conformal block with $|x_{12}|^{2\eta}|x_{34}|^{2\eta}$ times this series expansion

$$G_\Delta(u,v) = \left(1-\frac{\Delta}{2}\left(\frac{x_{12}\cdot x_{24}}{|x_{24}|^2}+\frac{x_{12}\cdot x_{23}}{|x_{23}|^2}\right)+\dots\right)\frac{|x_{12}|^{\Delta}|x_{34}|^{\Delta}}{|x_{24}|^{\Delta}|x_{23}|^{\Delta}}$$

We want to compare this expression against the small z and $\bar{z}$ expansion of the explicit representation of $G_\Delta(u, v)$:

$$G_\Delta(u,v) = (z\bar{z})^{\frac{\Delta}{2}}\left(1+\frac{\Delta}{4}(z+\bar{z})+\frac{(z^2+\bar{z}^2)\Delta(3\Delta^2-4)+\Delta^4(z+\bar{z})^2}{32(\Delta^2-1)}+\dots\right) \tag{5.5}$$

We see by inspection that the $O(0)$ and $O(z, \bar{z})$ terms match. That's enough for me for now.