Conformal Field Theory Problems

Chapter 6

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Problem 6.1. *Show that a special conformal transformation with $b^\mu = (1, \vec{0})$ turns the plane $x^1 = \frac{1}{2}$ into a sphere centered at the origin of radius one. Furthermore, show that reflection about the plane $x_1 = \frac{1}{2}$ becomes inversion after the special conformal transformation.*

The special conformal transformation has the form $x'^\mu = \frac{x^\mu - b^\mu x^2}{1-2b\cdot x + b^2 x^2}$. If we plug in $x = (\frac{1}{2}, X^i)$ and $b = (1, \vec{0})$, we find $1 - 2b\cdot x + b^2 x^2 = \frac{1}{4} + X^2$ and

$$x'^\mu = \frac{1}{\frac{1}{4} + X^2}\left(\frac{1}{4} - X^2, X^i\right)$$

Regarding inversion, we consider $x_+^\mu = (\frac{1}{2} + \delta x, X^i)$ and its reflection $x_-^\mu = R(x_+^\mu) = (\frac{1}{2} - \delta x, X^i)$. We find

$$x'^\mu_\pm = \frac{1}{\left(\frac{1}{2} \pm \delta x\right)^2 + X^2}\left(\left(\frac{1}{4} - \delta x^2\right) - X^2, X^i\right)$$

for the image of the original and reflected points. The important algebraic identity here is that

$$\left(\frac{1}{4} - \delta x^2 - X^2\right)^2 + X^2 = \left(\left(\frac{1}{2} + \delta x\right)^2 + X^2\right)\left(\left(\frac{1}{2} - \delta x\right)^2 + X^2\right)$$

This identity allows us to establish that $I(x'^\mu_+) = \frac{x'^\mu_+}{|x'_+|^2} = x'^\mu_-$.

Problem 6.2. *By studying $\langle\phi|K^2P^2|\phi\rangle$ for scalar primary ϕ, demonstrate that the conformal dimension must satisfy the quadratic constraint $\Delta(2(\Delta+1) - d) \geq 0$.*

The identity follows by moving the K_μ generators to the right, using the commutation relations, so that they can annihilate when acting on $|\phi\rangle$.

Problem 6.3. *Use the explicit representation of $M_{\mu\nu}$ from problem 2.2 for spinors and vectors to show that Δ is bounded below by $\frac{d-1}{2}$ and $d-1$ respectively.*

For the vector, we have the representation $(M_{\mu\nu})^\lambda{}_\rho = i\delta_{\mu\rho}\delta_\nu^\lambda - i\delta_\mu^\lambda\delta_{\nu\rho}$. Note we have replaced the Minkowski tensor with the Kronecker delta because we are working in a Euclidean setting. We therefore want the quantity $\delta_{\mu\nu}\delta_\rho^\lambda\Delta - \delta_{\mu\rho}\delta_\nu^\lambda + \delta_\mu^\lambda\delta_{\nu\rho}$ to be a positive matrix. One way to analyze this complicated looking object is to contract some indices. Let us contract λ with ν which tells us that $\delta_{\mu\nu}(\Delta - d + 1)$ needs to be positive.

The analysis for the spinor is similar. We have instead $(M_{\mu\nu})_\alpha{}^\beta = -\frac{i}{4}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)_\alpha{}^\beta$. Now we require $\Delta\delta_{\mu\nu} - \frac{1}{4}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$ to be positive. We multiply this expression by γ^μ and sum over the μ index. Note that $\gamma_\mu\gamma_\nu\gamma^\mu = -\gamma_\mu\gamma^\mu\gamma_\nu + 2\gamma_\mu\delta_\nu^\mu$ by the anti-commutation relations. The latter expression in turn simplifies to $(-d+2)\gamma_\nu$. Thus we find that $\Delta\gamma_\nu - \frac{1}{4}((-d+2)\gamma_\nu - d\gamma_\nu)$ must be a positive expression. In other words $\Delta > \frac{d-1}{2}$.

Problem 6.4. *Show that reflection positivity for* $\langle \phi\, \phi \rangle$ *and* $\langle \partial_r \phi\, \partial_r \phi \rangle$ *constrain* χ *to lie in the range* $-1 \leq \chi \leq 1$.