Summary

① Take a pair of punctures with local coordinates w, w' (same or different surfaces)

② Sew them using $w w' = -q$

$\Rightarrow$ a new surface

For correlation fn. this is achieved by inserting:

$$\phi_r(0)\ \phi_s(0)\ \langle \phi_r^c | \phi_s^c \rangle\ q^{h_s}\ \bar{q}^{\bar{h}_s}$$

In w coordinate → $\phi_r(0)$

in w' coordinate → $\phi_s(0)$

Repeated application $\Rightarrow$ correlators on any Riemann surface.

In practice one can combine this with analyticity and OPE to find closed form expression for correlators.

$\Rightarrow$ All correlators are obtained from knowledge of conformal weights and 3-point fn. on sphere.

Procedure of sewing $\to$ plumbing fixture

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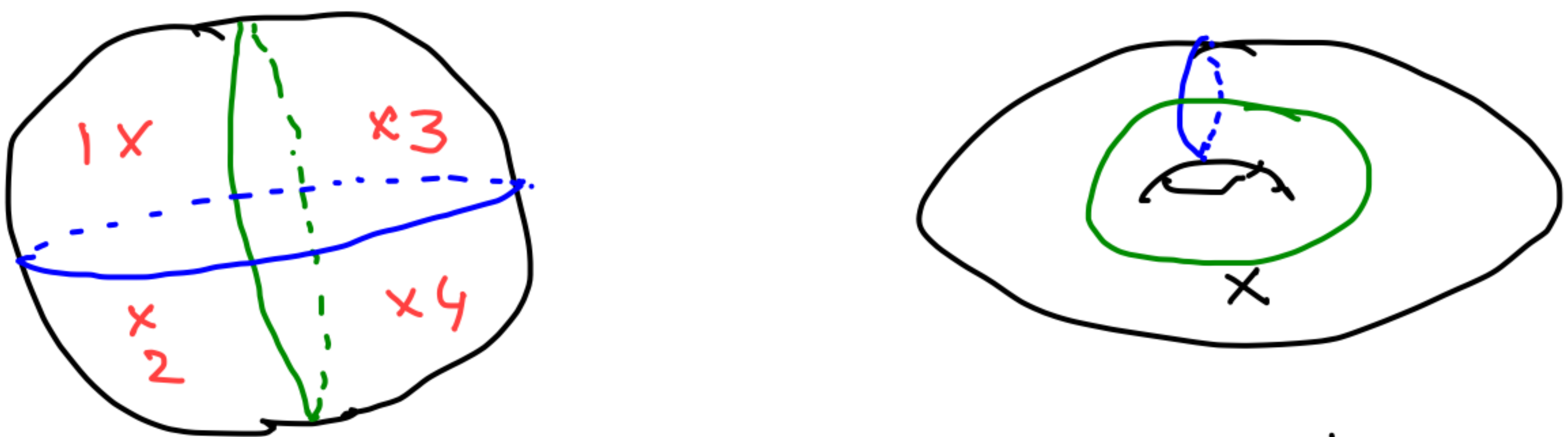
Consistency check

The same Riemann surface with punctures may be built from sphere 3-pt fn. in more than one way.



This is part of the consistency requirement of CFT

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In prinicipfle there are ∞ no. of such consistency checks.

$\Rightarrow$ # constraints is infinite.

Result: As long sphere 4 pt. fn. and torus one point fn. are consistent, all higher genus correlation fn.s are also consistent.

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Ex. Prove that on a genus g Riemann surface we need total ghost no. $-(6g-6)$ to get a non-zero correlator.

Hint: Use ghost no. of ϕ_r^c

$= 6 -$ ghost no. of ϕ_r

$$\langle \phi_r^c | \phi_s \rangle = \delta_{rs}$$

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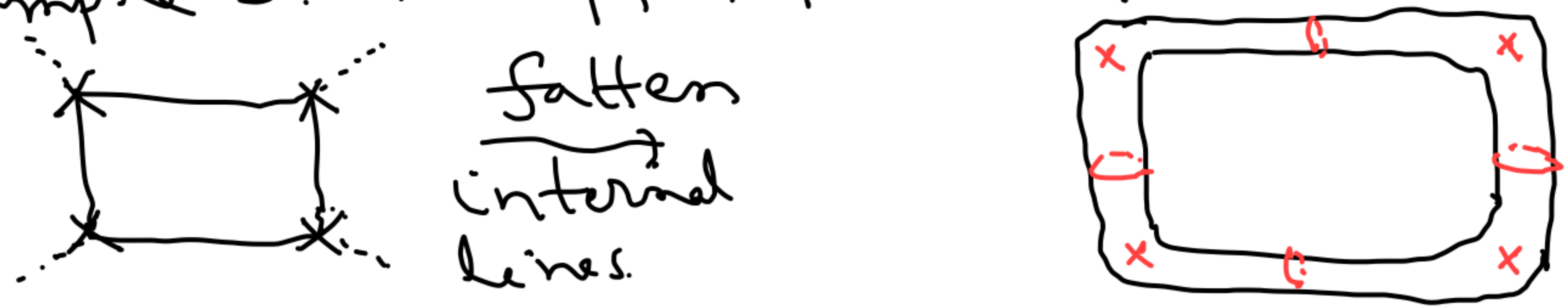
Q. Why are we interested in correlation fn. of higher genus surfaces?

A. A g-loop amplitude in string theory is expressed in terms of CFT correlators on genus g surface.

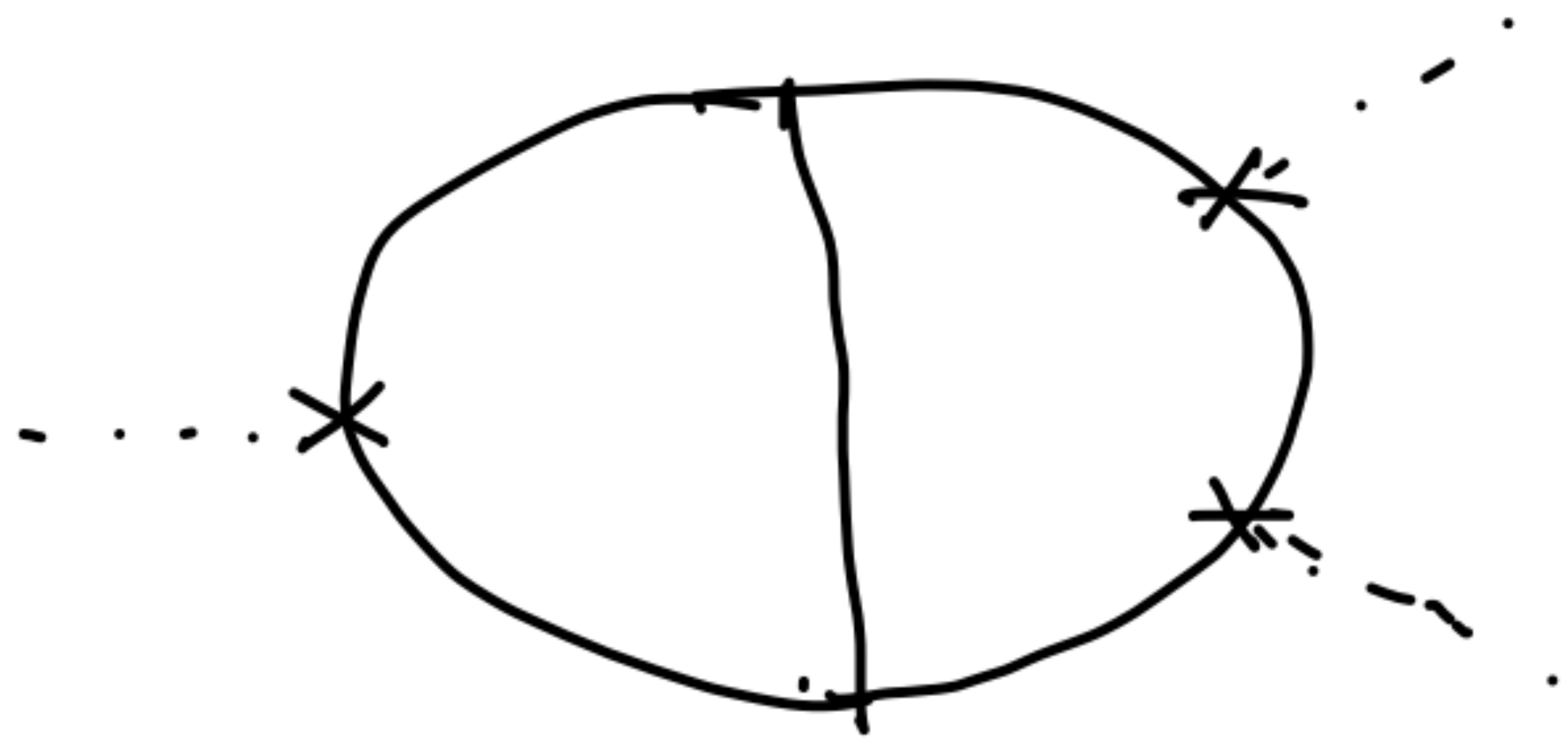
<u>Intuitive understanding</u>

Example I: 1 loop, 4 pt. amplitude in QFT

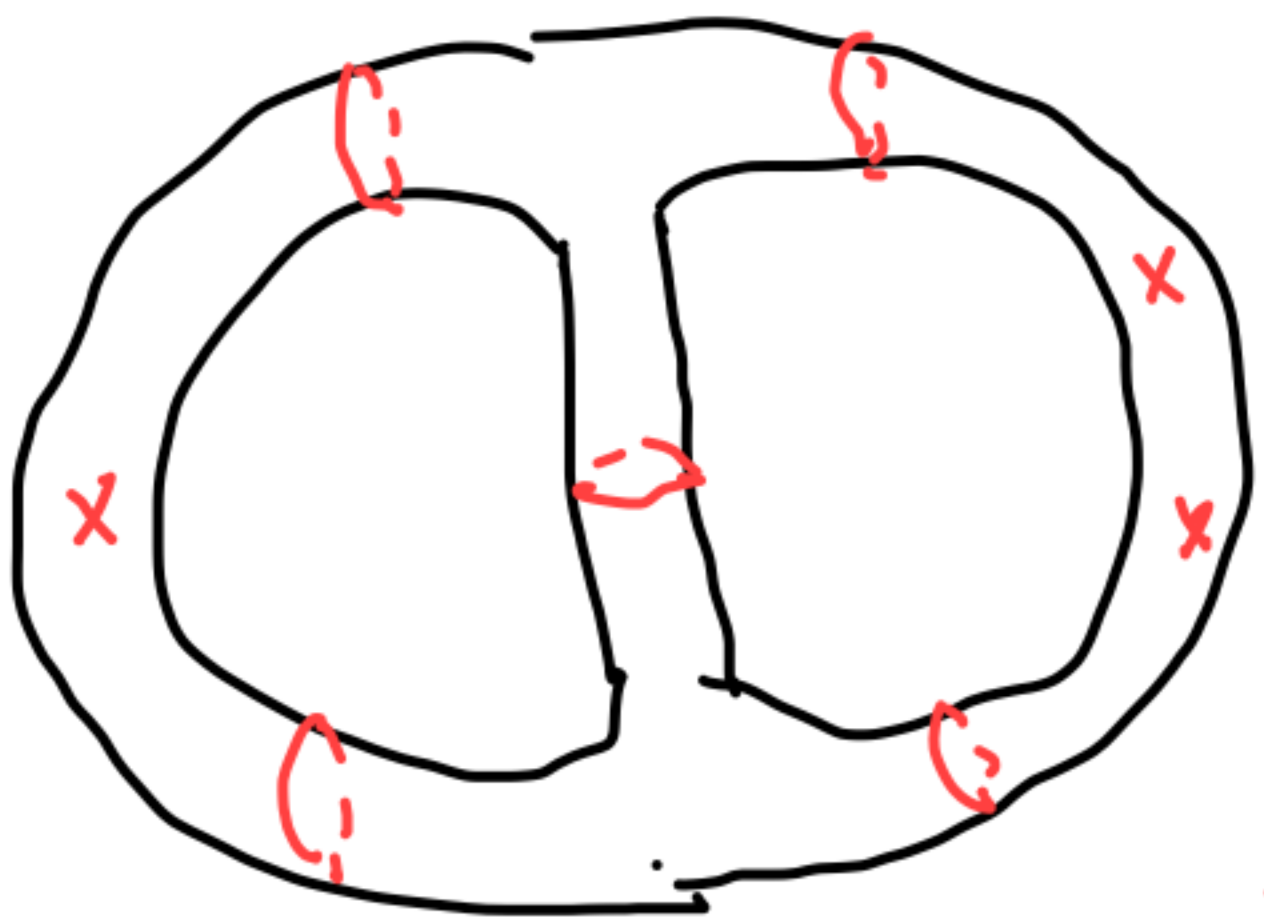


$\Rightarrow$ 4-pt fn. on torus.

Example 2. Two Loop, 3-point fn



⇓ Fatten



⇒ Genus 2, 3 point fn

What is the precise dictionary?

ⓐ What is computed? (QFT language)

ⓑ How is it computed using CFT correlators?

→ Off-shell Green's fn. with external tree level propagators removed.

will be called Amplitude

$\Rightarrow$ Green's fn.

$\Downarrow$ LSZ

S-matrix

Defn. of off-shell string state:

— A state $|V\rangle = V(0)|0\rangle$ in the CFT

satisfying:

$b_0^- |V\rangle = 0, \quad L_0^- |V\rangle = 0.$ not necessarily

$Q_B |V\rangle = 0$

$\Downarrow$ $\Downarrow$

How to compute on-shell condition

$A(V_1, \cdots, V_n)$?

Begin with a general description of Riemann surface of genus g with n-punctures:

→ can be regarded as a union of n disks, one around each puncture, and $2g-2+n$ spheres, each with 3 holes, joined along $3g-3+2n$ circles.

Disks: D_a, $a=1, \ldots n$, spheres: S_i $i=1, \ldots 2g-2+n$

Circles: C_s : $s=1, \ldots 3g-3+2n$. $\to S_i \cap S_j$ or $S_i \cap D_a$

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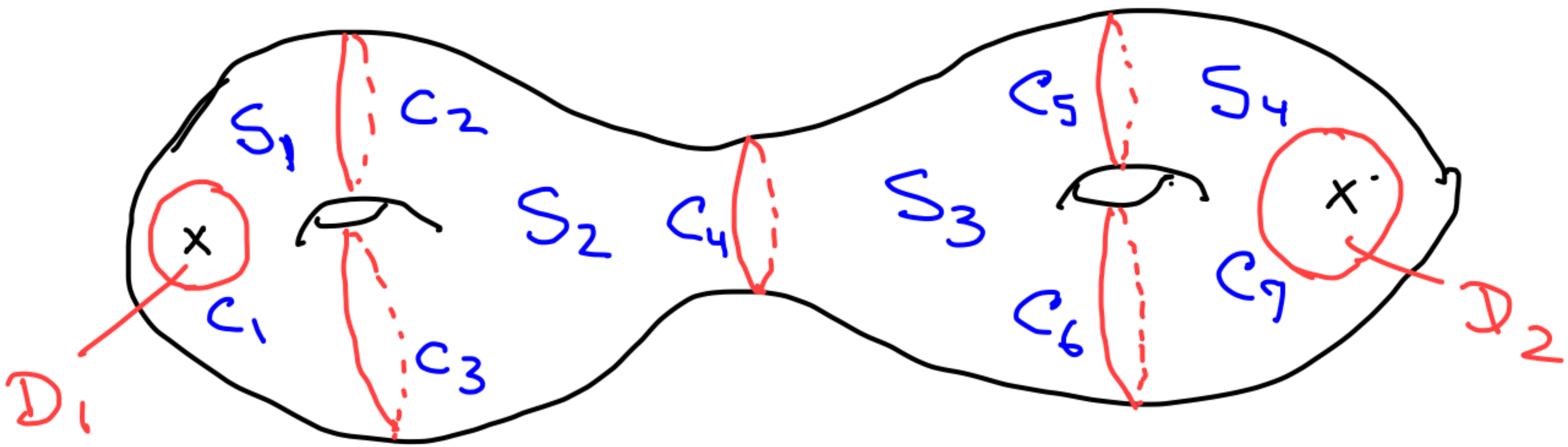
Example: $g = 2$, $n = 2$.



# of spheres $2g-2+n = 4$

# of circles $3g-3+2n = 7$

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Fix some orientation on each circle

C_s (arbitrary)

Introduce complex coordinate system:

w_a on D_a with $w_a=0$ being the puncture

z_i on S_i

σ_s: coordinate on the left of C_s (z_i or w_a)

τ_s: coordinate on the right of C_s (z_i or w_a)

$\tau_s = F_s(\sigma_s) \rightarrow$ specifies the Riemann surface.

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$\{F_s(\tau_s)\}$ is considered equivalent to $\{\tilde{F}_s(\tau_s)\}$ if they can be related by coord. tr. of the form:

$$z_i \to h_i(z_i), \quad w_a \to \tilde{h}_a(w_a), \quad \tilde{h}_a(0) \equiv 0$$

space of equivalence classes is finite dimensional & known as the moduli space of genus g Riemann surface with n punctures.

$\Downarrow$

$6g-6+2n$

$\mathcal{M}_{g,n}$

Introduce another space $\mathcal{P}_{g,n}$

Equivalence classes of $\{F_s(\tau_s)\}$

under coordinate trs. $z_i \to h_i(z_i)$

and $w_a \to e^{i\alpha_a} w_a$, α_a are

arbitrary constants.

Physically $\mathcal{P}_{g,n}$ contains information about $\mathcal{M}_{g,n}$ and choice of local coordinate at the puncture.

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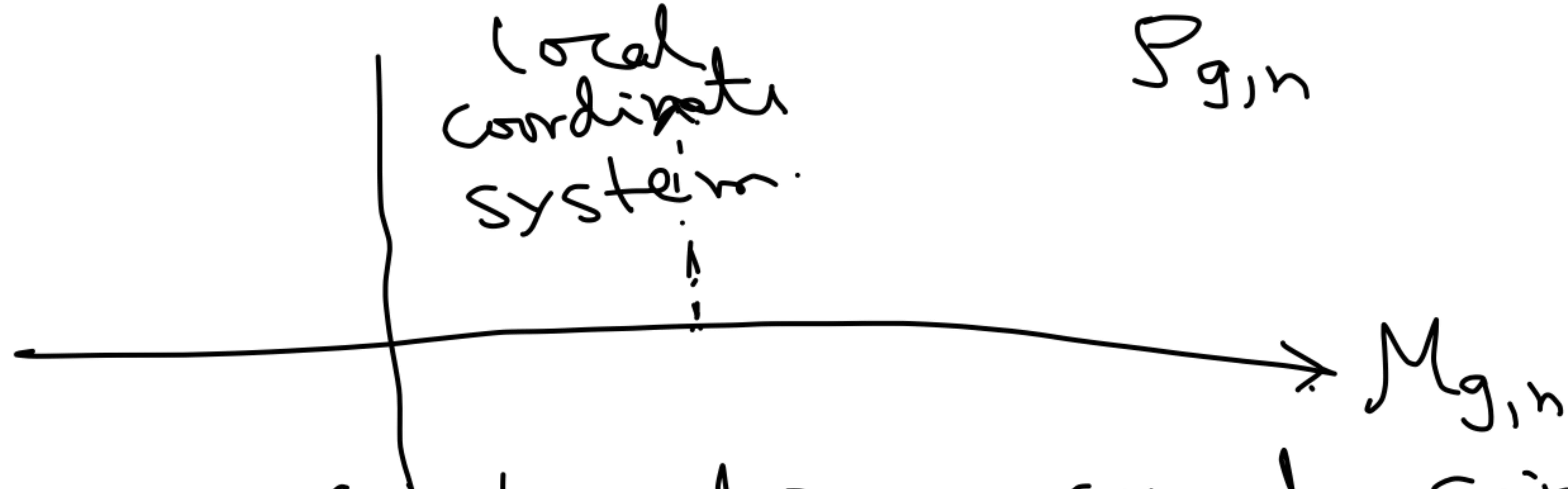
$\mathcal{P}_{g,n}$ can be regarded as a fiber bundle with base $\mathcal{M}_{g,n}$ and fiber containing choice of local coordinates.



$\mathcal{P}_{g,n}$ is infinite dimensional since it has info. about n independent f-s $\to$ choice of n local coordinates upto phase.

$\{t^m\} = \vec{t}$: coordinates on $\mathcal{P}_{g,n}$

∞ # of coordinates

Given $\vec{t}$, we have a given set of fn.s $F_s(\tau_s;\vec{t})$

not unique since it can be changed by $z_i \to h_i(z_i)$ or $w_a \to e^{i\alpha_a} w_a$.

Pick some representative F_s.

$$\sigma_s = F_s(\tau_s;\vec{t})$$

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Define a CFT "operator"

$$B_m = \sum_{s=1}^{3g-3+2n} \left[\oint_{C_s} \frac{\partial F_s(\tau_s, \vec{t}\,)}{\partial t^m} \, d\sigma_s \, b(\sigma_s) \right.$$

$$\left. + \oint_{C_s} \frac{\partial \overline{F_s(\tau_s, \vec{t}\,)}}{\partial t^m} \, d\bar{\sigma}_s \, \bar{b}(\bar{\sigma}_s) \right]$$

For a given set of off-shell string states $|V_1\rangle, \ldots, |V_n\rangle$ define a p form $\Omega_p^{(g,n)}(V_1, \ldots V_n)$ on $\mathcal{P}_{g,n}$ as follows:

$$\Omega_p^{(g,n)}(V_1, \cdots V_n) = \sum_{m_1, \cdots, m_p} \langle B_{m_1} \cdots B_{m_p} V_1 \cdots V_n \rangle_{\Sigma_{g,n}} (-2\pi i)^{-(3g-3+n)} dt^{m_1} \wedge \cdots \wedge dt^{m_p}$$

$w_1 = 0$ $w_n = 0$

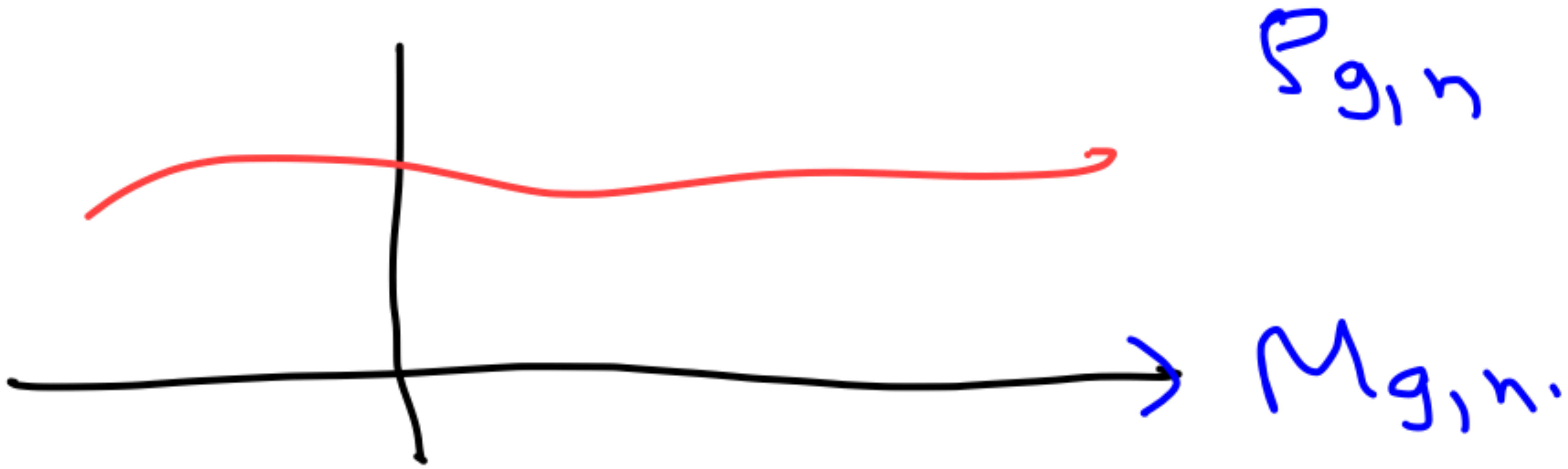
$\Sigma_{g,n}$: Genus g Riemann surface with n-puncture and given choice of local coords at punctures determined by the point at $\mathcal{P}_{g,n}$.

(1) This description is manifestly invariant under change of coordinate in $\mathcal{P}_{g,n}$ $t^i \to f^i(\vec{t})$

(2) If we want to pull back $\Omega_p^{(g,n)}$ on some p-dimensional subspace of $\mathcal{P}_{g,n}$ with intrinsic coordinates $u^1, \ldots u^p$, then the result is obtained simply by replacing t^m by u^i's.

Defn. of $A(V_1, \cdots V_n)$.

① choose a section $S_{g,n}$ of $\mathcal{S}_{g,n}$.



② $$A(V_1, \cdots, V_n) = (g_s)^{2g-2+n} \int_{S_{g,n}} \Omega^{(g,n)}_{6g-6+2n}(V_1, \cdots V_n).$$

Tree level S-matrix : iA

Loop level S-matrix : Requires LSZ

We'll not derive this formula but we'll check various consistency conditions.

Crucial identity (derived using CFT properties)

$$\Omega_p^{(g,n)}(Q_B V_1, V_2, \dots V_n) + (-1)^{V_1}\,\Omega_p^{(g,n)}(V_1, Q_B V_2, \dots V_n)$$
$$+ \dots + \Omega_p^{(g,n)}(V_1, \dots V_{n-1}, Q_B V_n)\,(-1)^{V_1}(-1)^{V_2}\cdots(-1)^{V_{n-1}}$$
$$= (-1)^p\, d\Omega_{p-1}^{(g,n)}(V_1, \dots, V_n)$$

$$Q_B V(z,\bar z) = \oint_z j_B(w)\, V(z,\bar z) + \oint_{\bar z} \bar j_B(\bar w)\, V(z,\bar z)$$

$$Q_B V(0)|0\rangle = Q_B|V\rangle$$

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Ex: $\{Q_B, b(z)\} = T(z)$, $\{Q_B, \bar{b}(\bar{z})\} = \bar{T}(\bar{z})$

$\oint \epsilon(z) T(z)\, dz$ generates $z \to z + \epsilon(z)$

Normalization

$\oint$ is defined such that $\oint \frac{dz}{z} = 1$

includes $\frac{1}{2\pi i}$ factor.

$$\oint \frac{d\bar{z}}{\bar{z}} = 1.$$

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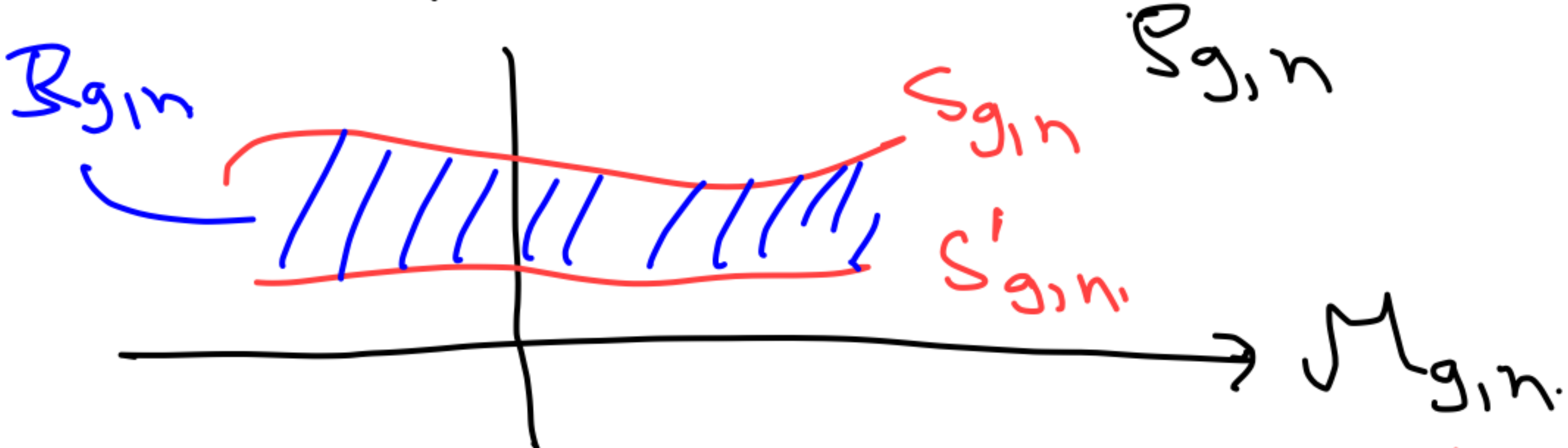
Consequences

① Dependence on the section.



This has to be supplemented by separate analysis of contribution from $\partial\mathcal{M}_{g,n}$.

$$\int_{S_{g,n}} \Omega^{(g,n)}_{6g-6+2n}(V_1,\dots,V_n) - \int_{S'_{g,n}} \Omega^{(g,n)}_{6g-6+2n}(V_1,\dots,V_n)$$

$$= \int_{R_{g,n}} d\Omega^{(g,n)}_{6g-6+2n}(V_1,\dots,V_n) = -\int_{R_{g,n}} \Big[\Omega^{(g,n)}_{6g-6+2n+1}(Q_B V_1, V_2,\dots V_n) + (-1)^{V_1}\,\Omega^{(g,n)}_{6g-6+2n+1}(V_1, Q_B V_2, V_3,\dots V_n) + \dots\Big]$$

$= 0$ if $Q_B V_i = 0 \ \forall i$

② $A(Q_B \Lambda, V_2, V_3, \cdots V_n)$ $\quad Q_B V_i = 0$ for $i = 2, \cdots, n$

$$= \int_{S_{g,n}} \Big[\Omega^{(g,n)}_{6g-6+2n} (Q_B \Lambda, V_2, V_3, \cdots V_n)$$

$$+ \underbrace{\Omega^{(g,n)}_{6g-6+2n} (\Lambda, Q_B V_2, \cdots V_n)(-1)^\Lambda}_{=0 \text{ anyway.}}$$

$$+ \cdots \Big]$$

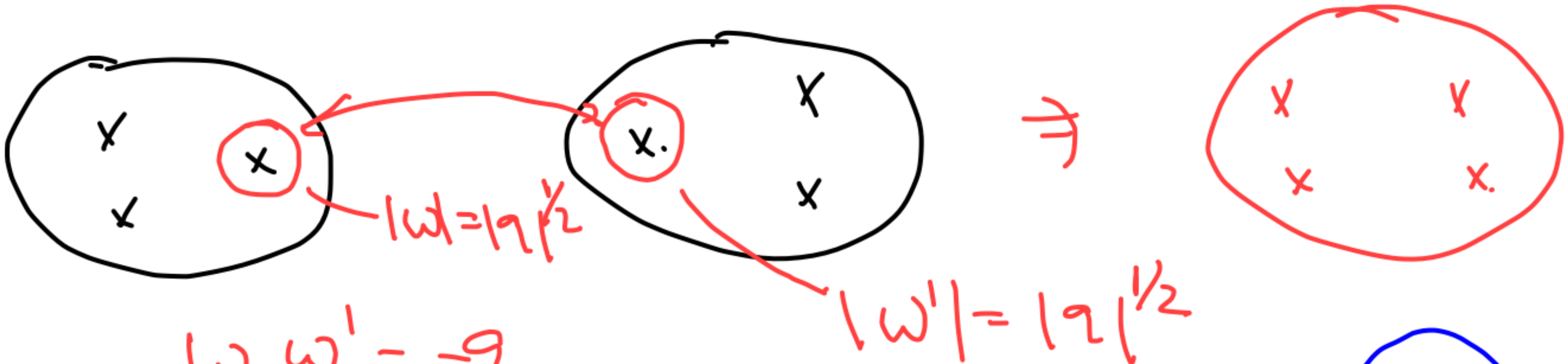
$$= \int_{S_{g,n}} d\Omega^{(g,n)}_{6g-6+2n-1} (\Lambda, V_2, \cdots V_n)$$

$= 0$ up to contributions from $\partial \mathcal{M}_{g,n}$

to be analyzed separately.

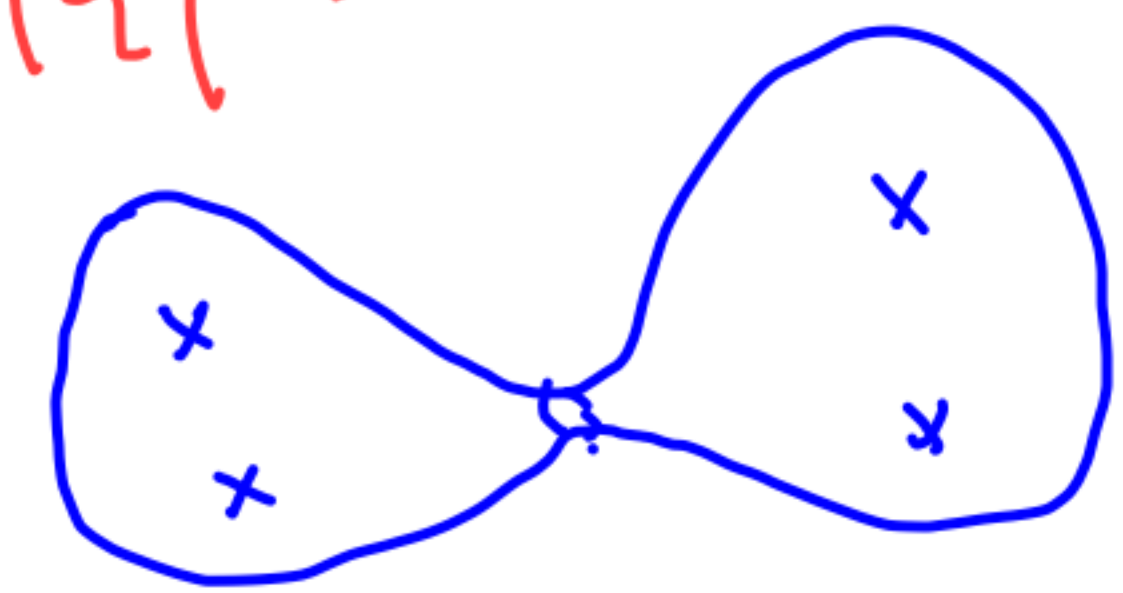
Pure gauge states decouple.

$\Rightarrow$ Proof of general coordinate invariance and other gauge invariances in the interacting theory.



$$w w' = -q$$

$q \to 0$ limit $\Rightarrow$ boundary of $\mathcal{M}_{0,4}$



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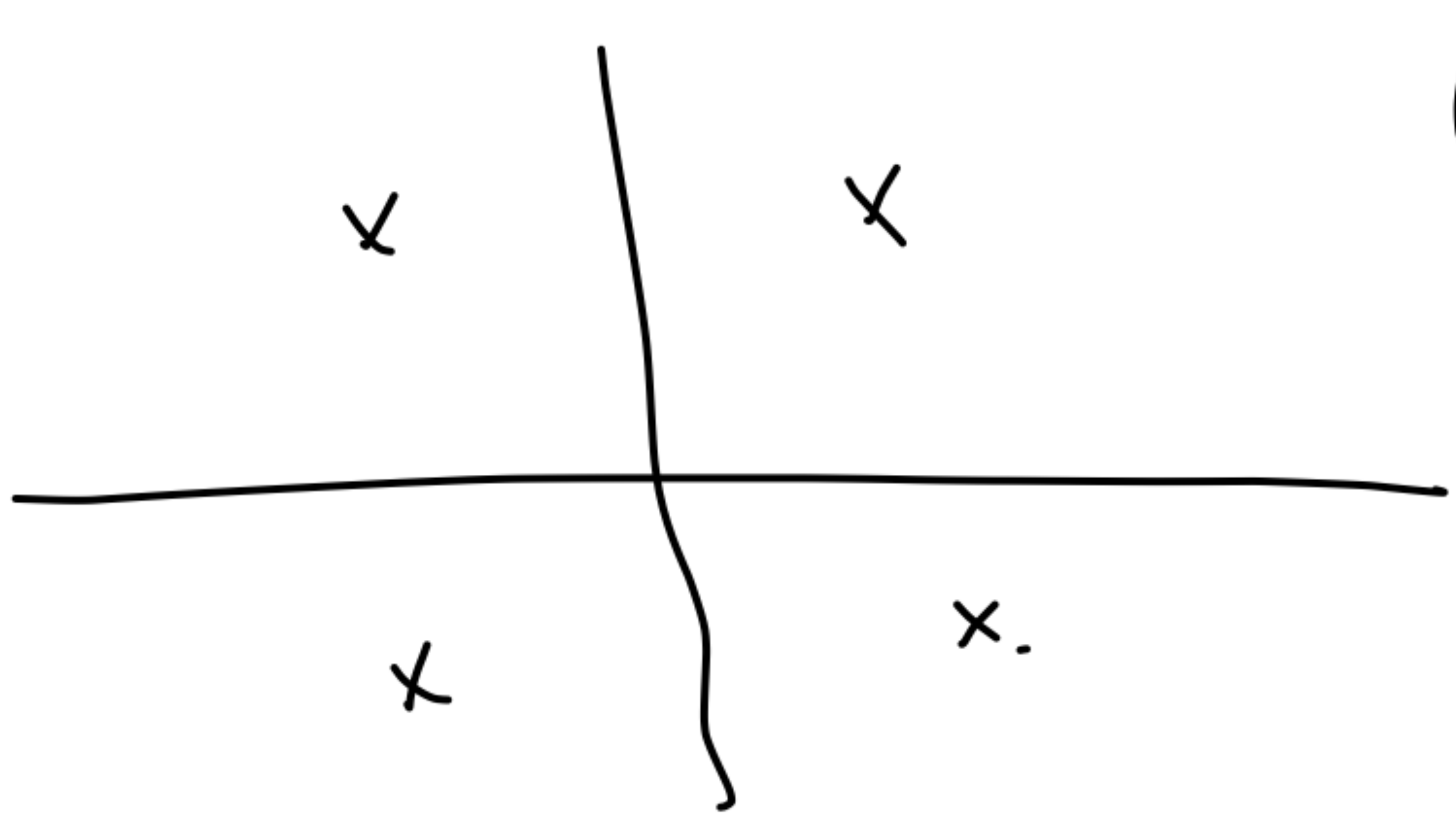
boundary →

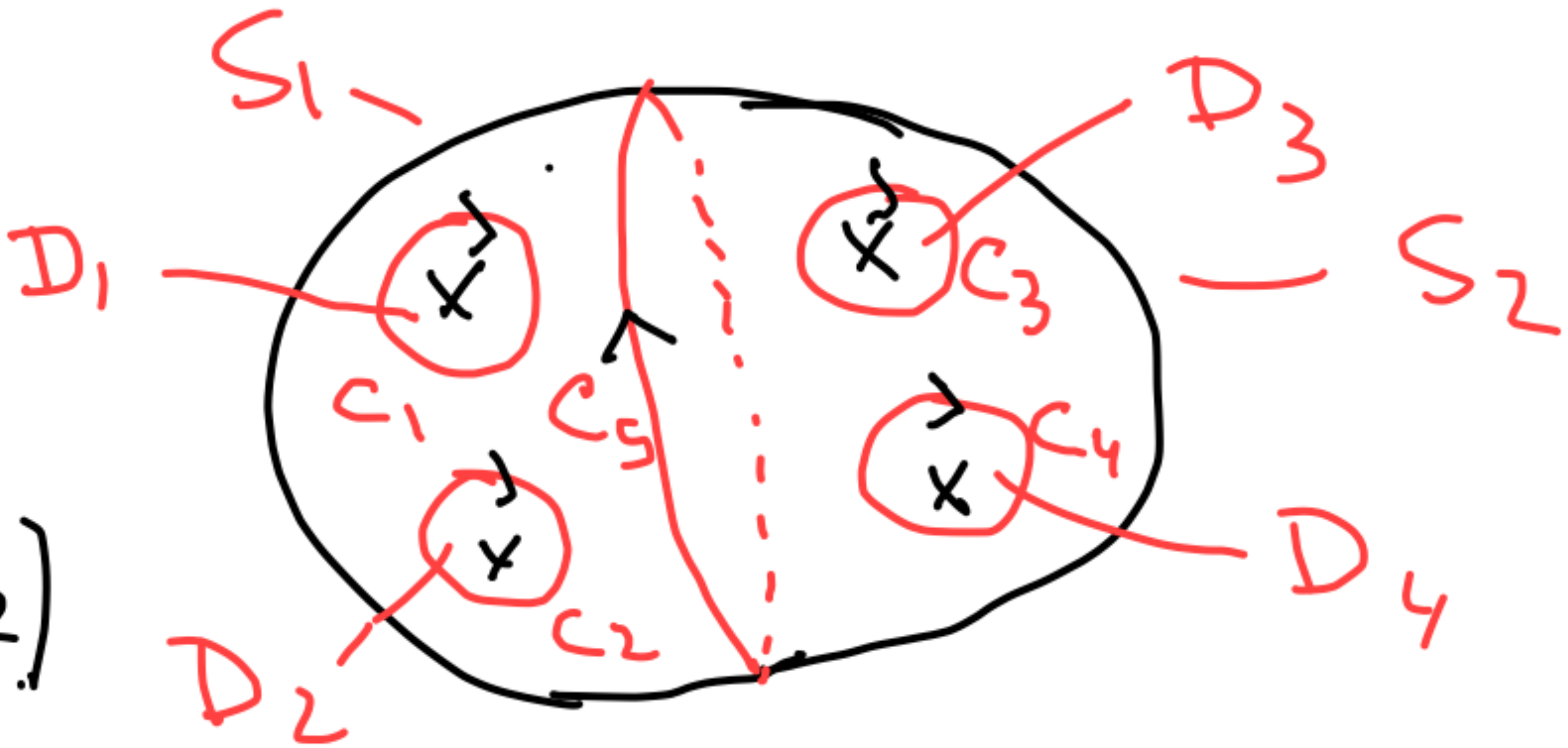


$\Rightarrow$ $q \to 0$ limit of the previous diagrams

We'll apply this to sphere four point fn: $M_{0,4}$

$\rightarrow$ $6g-6+2n = 2$ dim. moduli space.

On-shell amplitudes



C_5:
$$z_1 = F_5(z_2) = -q/z_2$$

On C_1:
$$z_1 = F_1(\omega_1)$$

On C_2
$$z_1 = F_2(\omega_2)$$

C_3: $z_2 = F_3(\omega_3)$

C_4: $z_2 = F_4(\omega_4)$