How do we know what the allowed GSO projection rules are?

Choose some rules. and then construct correlation fn. via OPE and plumbing fixture.

Check for X-ing symmetry and modular inv.

Passing these $\Rightarrow$ we have a good CFT but not necessarily a good string theory.

Check for the absence of tachyons, absence of vacuum instability etc.

Result: Only 2 consistent string theories in $D=10$: IIA, IIB.

Correlation fn.s in the R-sector can in principle be found using the bosonization rules ($S_\alpha : e^{\frac{i}{2}(\pm\phi_1 \pm \cdots \pm \phi_5)}$)

In practice they can be computed from a few OPE & analyticity

$$\psi^\mu(z)\, e^{-\frac{\phi}{2}} S_\alpha(w) \simeq (z-w)^{-1/2} \frac{i}{2} \Gamma^\mu_{\alpha\beta}\, e^{-\frac{\phi}{2}} S^\beta(w)$$

$$\psi^\mu(z)\, e^{-\frac{\phi}{2}} S^\alpha(w) \simeq (z-w)^{-1/2} \frac{i}{2} (\Gamma^\mu)^{\alpha\beta} e^{-\frac{\phi}{2}} S_\beta(w)$$

$$e^{-\frac{3\phi}{2}} S^\alpha(z)\, e^{-\frac{\phi}{2}} S_\beta(w) \simeq (z-w)^{-2} \delta^\alpha_\beta\, e^{-2\phi}(w)$$

$$e^{-\frac{\phi}{2}} S_\alpha(z)\, e^{-\frac{\phi}{2}} S_\beta(w) \simeq (z-w)^{-1}\, i\, \Gamma^\mu_{\alpha\beta}\, e^{-\phi} \psi_\mu(w)$$

+ anti-hol.

Γ^μ's are 16×16 symmetric matrices satisfying: $\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu} I_{16}$. $\hookrightarrow$ identity.

In the product to two Γ^μ's, upper spinor index is contracted with lower spinor index.

$$(\Gamma^\mu \Gamma^\nu)^\alpha{}_\beta = (\Gamma^\mu)^{\alpha\gamma} \Gamma^\nu_{\gamma\beta}.$$

Ex. $e^{-\frac{\phi}{2}} S_\alpha$, $e^{-\frac{3\phi}{2}} S^\beta$ have conformal weights $(0, 1)$.

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Off-shell state in string theory: GSO even states $|V\rangle$ in the CFT satisfying:

① $L_0^-|V\rangle = 0$, $b_0^-|V\rangle = 0$, $L_0^\pm = L_0 \pm \bar{L}_0$, $b_0^\pm = b_0 \pm \bar{b}_0$

② Picture no. -1 in NS sector
$-1/2$ in R sector.

– both in the hol. and anti-hol. sector.

Physical (on-shell) state: $Q_B|V\rangle = 0$, total gh. no. 2

$|V\rangle$ and $|\tilde{V}\rangle$ are equivalent if $|\tilde{V}\rangle - |V\rangle = Q_B|\Lambda\rangle$ for some $|\Lambda\rangle$ satisfying $b_0^-|\Lambda\rangle = 0$, $L_0^-|\Lambda\rangle = 0$.

Examples of physical states:

NSNS: Ex. check that the following state is physical:

$$\zeta_{\mu\nu}\, c\, \bar{c}\, e^{-\phi} e^{-\bar{\phi}} \psi^{\mu} \bar{\psi}^{\nu} e^{ik\cdot X}(0)|0\rangle$$

$$k^2=0, \quad k^{\mu}\zeta_{\mu\nu} = 0 = k^{\nu}\zeta_{\mu\nu}$$

graviton, dilaton, 2-form fields.

$$\zeta_{\mu\nu} \equiv \zeta_{\mu\nu} + k_{\mu}\xi_{\nu} + k_{\nu}\bar{\xi}_{\mu}, \quad k\cdot\xi = 0, \quad k\cdot\bar{\xi} = 0.$$

Ex. 2. Check that there are no tachyons.

$c\bar{c}\, e^{ik\cdot X}(0)|0\rangle \to$ is not present since it has picture no. $(0,0)$.

A physical state in the NSR sector:

$$\xi^{\mu}_{\alpha}\, c\bar{c}\, e^{-\frac{\phi}{2}} S_\alpha\, e^{-\bar{\phi}}\, \bar{\psi}^{\mu}\, e^{ik\cdot X}(0)|0\rangle$$

<u>Ex.</u> $k^2=0,\quad k_\mu \xi^{\mu}_{\alpha}=0,\quad k_\nu \Gamma^{\nu}_{\alpha\beta}\, \xi^{\beta}_{\mu}=0 \quad \left(Q_B|V\rangle = 0\right)$

$\Downarrow$

A spin $\frac{3}{2}$ and spin $\frac{1}{2}$ state

- gravitino + dilatino

Another set of massless spin $3/2$ + spin $1/2$ state from the RNS.

- same chirality in IIB.
- opposite " in IIA

these theories describe $N=2$ supersymmetric theories in $D=10$.

Type IIB / type IIA supergravity coupled to an infinite tower of massive states. $\Rightarrow$ No UV divergences

Heterotic string theory

Holomorphic sector: Like superstring

Anti-hol- sector: Like bosonic string

(no $\bar\beta, \bar\gamma$, $\bar\xi, \bar\eta, \bar\phi$, $\bar\psi^\mu$).

What about D (No of X^μ's) $\rightarrow$ cannot be 10 and 26 at the same time.

We take 10 X^μ's: $X^0, \ldots X^9$.

Add a CFT of central charge (16, 0).

$\Rightarrow$ $\bar T^X + \bar T_{CFT}$ has central charge $\bar c = 26$

CFT of central charge (16,0) are quite restrictive.

~ Only CFT's consistent with X-ing symmetry and modular inv.

⇒ $E_8 \times E_8$ heterotic and SO(32) heterotic string.

CFT's have dim (1,0) virasoro primary operators satisfying

$$\bar{J}^a(\bar{z})\,\bar{J}^b(\bar{w}) \simeq \frac{1}{2(\bar{z}-\bar{w})^2} + i f^{ab}{}_c \frac{1}{\bar{z}-\bar{w}} \bar{J}^c(\bar{w})$$

↳ structure consts. of $E_8 \times E_8$ or SO(32)

$$\xi^a_\mu\, c\,\bar{c}\, e^{-\phi}\psi^\mu\, \bar{J}^a e^{ik\cdot X}(0)|0\rangle$$

→ phys. if $k^2=0$, $k^\mu \xi^a_\mu = 0$

∃ massless gauge fields.

Heterotic string theory describes $N=1$ supergravity in $D=10$ coupled to $E_8 \times E_8$ or $SO(32)$ super Yang-Mills and ∞ tower of massive states.

For simplicity we'll focus on heterotic string theory $\rightsquigarrow$ involves less writing

- only two sectors NS, R instead of four.

Generalization to type II is straightforward.

Instead of string theory, let us analyze the CFT a bit more.

How to build states with picture no. q?

Begin with $|q,k\rangle = e^{q\phi} e^{ik\cdot X_{(0)}}(0)$

Apply $\beta_n, \gamma_n, b_n, c_n, \tilde{b}_n, \tilde{c}_n$ & matter ops.

$$\beta(z) = \sum \beta_n z^{-n-\frac{3}{2}}, \quad \gamma(z) = \sum_n \gamma_n z^{-n+\frac{1}{2}}$$

Ex. $\beta(z) e^{q\phi}(0) \sim z^q$, $\gamma(z) e^{q\phi}(0) \sim z^{-q}$

$\Rightarrow$ Ex. $\beta_n|q,k\rangle = 0$ for $n \geq -q-\frac{1}{2}$, $\gamma_n|q,k\rangle = 0$ for $n \geq q+\frac{3}{2}$.

$$\begin{cases} \beta_n|q,k\rangle = 0 & \text{for } n \geq -q-\frac{1}{2} \\ \gamma_n|q,k\rangle = 0 & \text{for } n \geq q+\frac{3}{2}. \end{cases}$$

Ex. n is $\mathbb{Z}+\frac{1}{2}$ for $q \in \mathbb{Z}$
n is $\mathbb{Z}$ for $q \in \mathbb{Z}+\frac{1}{2}$

Unless $\boxed{q = -\frac{1}{2}, -1,}$ $-\frac{3}{2}$, there is at least one $n>0$ s.t. $\beta_n|q,k\rangle \neq 0$ or $\gamma_n|q,k\rangle \neq 0$.

Example: $q=0$. $\beta_n|0,k\rangle = 0$ for $n \geq -\frac{1}{2}$

$\gamma_n|0,k\rangle = 0$ for $n \geq \frac{3}{2}$

$\gamma_{\frac{1}{2}}|0,k\rangle \neq 0$ $\quad \underbrace{(\gamma_{1/2})^M|0,k\rangle \neq 0}_{\text{Ex.}}$ $\to$ Reduces L_0 e.v. by $\frac{1}{2}M$.

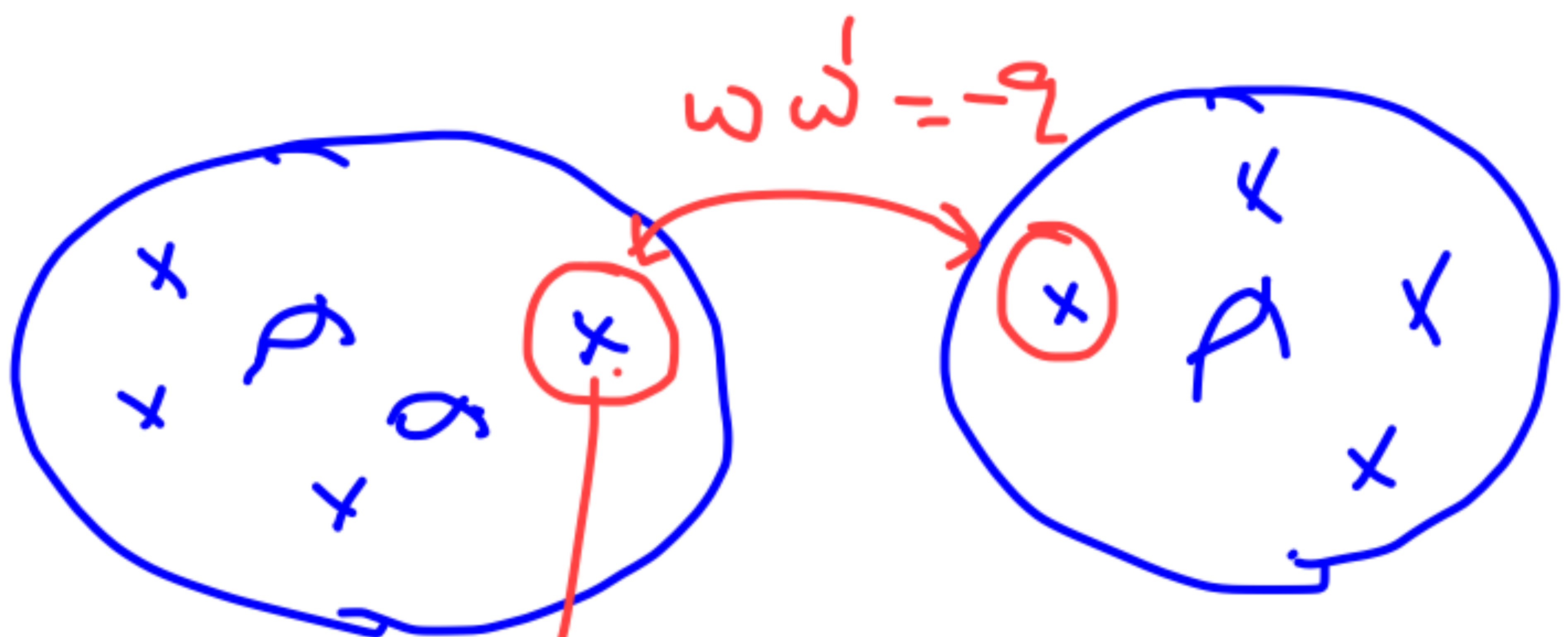
The spectrum of L_0 is unbounded from below.

Consider plumbing fixture: $ww' = -q$

$\Downarrow$ insert:

$$\Phi_r(0)\ \Phi_s(0)\ \langle \Phi_r^c | \Phi_s^c \rangle\ q^{h_s}\, \bar{q}^{\bar{h}_s}$$

Picture no. $\quad p \quad -2-p \quad -2-p \quad p$.



picture no. of this is determined by the rest of the vertex op & genus.

p is not determined by picture no. conr.

Naive guess: Sum over p

In actual practice: Fix p.

What value of p should we use?

Ans. Any value (formally)

We have unbounded L_0 ev - from below.

q^{h_s} h_s can become arbitrarily -ve.

→ make sense by analytic continuation.

$$1 + q^{-1} + q^{-2} + \cdots = \frac{1}{1-q^{-1}} = -\frac{q}{1-q}$$

$p=-1$ does not have this problem.

$p=-\frac{1}{2}, -\frac{3}{2}$ have mild problem due to β_0, γ_0

→ cleverly taken care of in string theory

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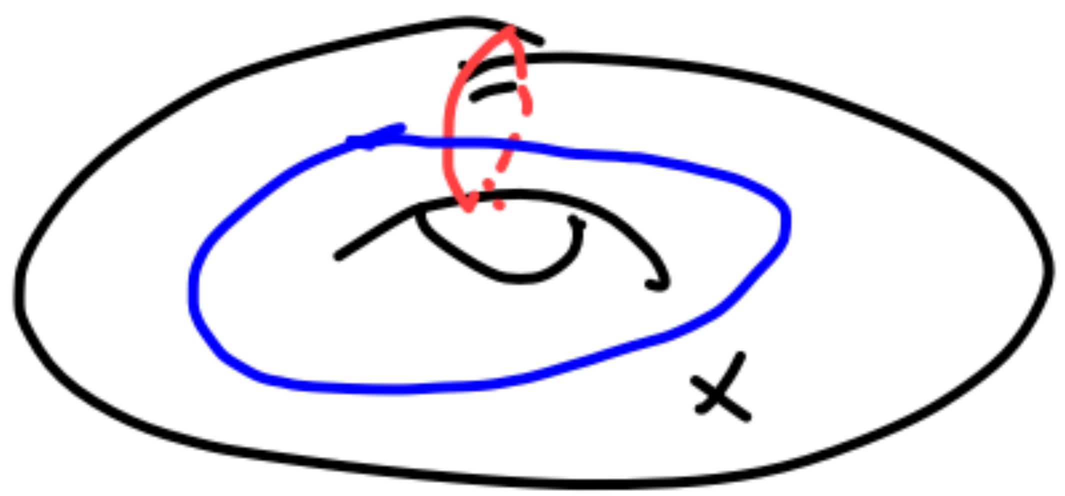
Another point about higher loop amplitudes.

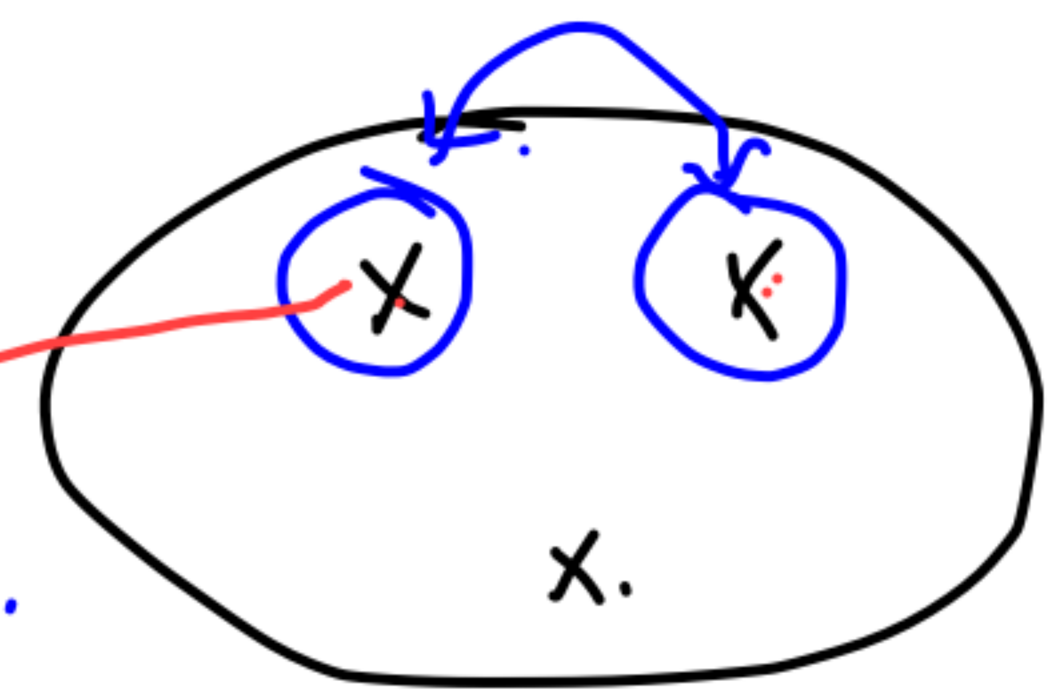
→ we have to sum over spin structure.

(periodic and anti-periodic b.c. on β, γ, ψ^μ along different non-contractible cycles)

Physical reason

→ plumbing fixture.





Either NS or R

⇒ anti-periodic or periodic b.c. along the $|w| = |q|^{1/2}$ circle.

$M_{g,m,n}$: Moduli space of Riemann surface of genus g with m NS and n R puncture.

S $\rightarrow$ include sum over spin structure.

$M_{g,m,n}$

String amplitudes: We need to integrate $\underbrace{\text{Something}}_{?}$ over $M_{g,m,n}$.

Two possible approaches.

(1) Make world-sheet SUSY as manifest as possible.

⇝ supermoduli space

– besides the usual bosonic coordinates of $M_{g,m,n}$, we have a set of grassmann odd variables.

→ supermoduli # depends on g, m, n.

(2) Carry out the integration over the grassmann odd variables explicitly.

→ can be done locally in patches of $\mathcal{M}_{g,m,n}$.

⇒ Leads to insertion of PCO's on the Riemann surface $\prod_{\alpha=1}^{K} X_0(y_\alpha)$

Choice of y_α ⇒ choice of gauge.

$\alpha : 1, 2, \dots, K.$

$$K + m\times(-1) + n\times\left(-\tfrac{1}{2}\right) = 2g-2 \Rightarrow K = 2g-2+m+\tfrac{n}{2}$$

$\uparrow$ # of odd supermoduli

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Construct the amplitude.

find the analog of $\Omega_p^{(g,m,n)}(V_1, \cdots V_m, W_1, \cdots W_n)$

① As in bosonic string theory,

introduce $(m+n)$ disks D_a with coord. w_a,

$2g-2+m+n$ spheres S_i with coord. z_i,

$3g-3+2(m+n)$ circles C_s.

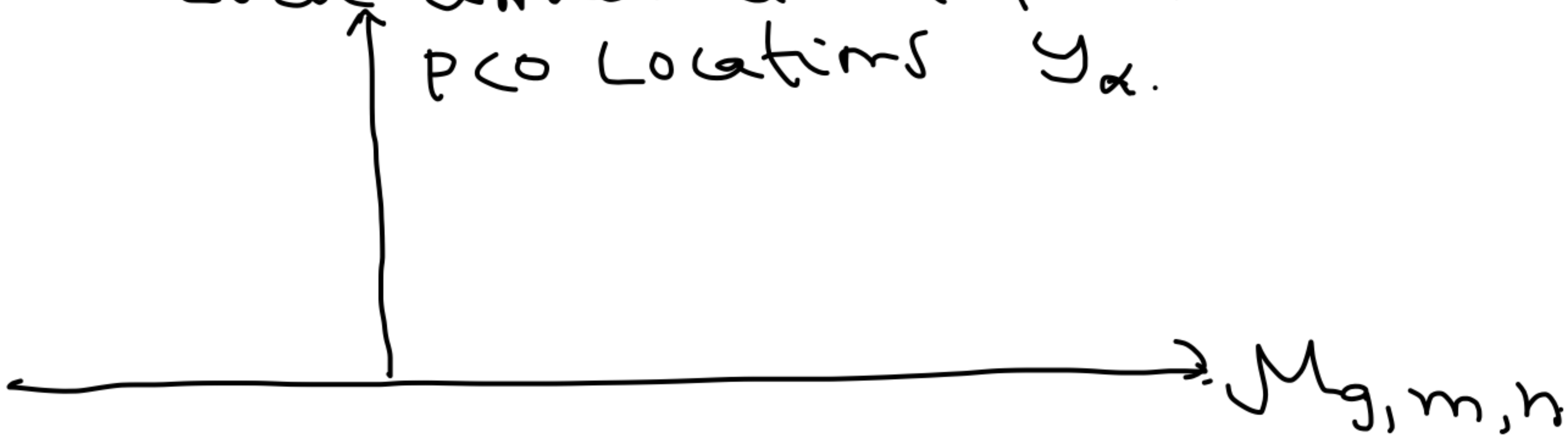
Transition fn. $\sigma_s = F_s(\tau_s)$

② Extra ingredient: Locations y_α of $2g-2+m+\frac{n}{2}$ PCO's. (in w_a coordinate if on D_a, in z_i coord. if on S_i)

(3) Introduce the fiber bundle $\mathcal{P}_{g,m,n}$.

Local coordinate at punctures

PCO locations y_α.



$\{t^m\}$: coordinates on $\mathcal{P}_{g,m,n}$

$$\sigma_s = F_s(\tau_s, \vec{t}\,), \quad y_\alpha(\vec{t}\,)$$

$$B_m = \sum_{s} \oint_{c_s} \frac{\partial F_s(\tau_s, \vec{t})}{\partial t^m} b(\sigma_s)\, d\sigma_s \quad \Big| \; \beta, \gamma. \;\; \to \xi, \eta, \phi.$$

$$+ \sum_s \oint_{c_s} \frac{\partial \overline{F_s(\tau_s, \vec{t})}}{\partial t^m} \bar{b}(\bar{\sigma}_s)\, d\bar{\sigma}_s$$

$$- \sum_\alpha \frac{1}{x(y_\alpha)} \frac{\partial y_\alpha(\vec{t})}{\partial t^m} \partial \xi(y_\alpha) \Rightarrow \text{Extra term.}$$

formal.

$$\Omega_p^{(g,m,n)}(V_1, \dots V_m, W_1, \dots W_n) = \sum_{m_1, \dots m_p} dt^{m_1} \wedge \dots dt^{m_p}$$

$$\left\langle B_{m^1} \cdots B_{m^p} \prod_{\beta=1}^{2g-2+m+\frac{n}{2}} x(y_\beta)\, V_1 \cdots V_m\, W_1 \cdots W_n \right\rangle_{\Sigma_{g,m,n}}$$

$\to$ well defined expression $\Big] \times (-2\pi i)^{-(3g-3+m+n)}$

$\Omega_p^{(g,m,n)}$ satisfies

$$\Omega_p^{(g,m,n)}(Q_B V_1, \cdots W_m) + \cdots$$

$$= (-1)^p \, d\Omega_{p-1}^{(g,m,n)}(V_1, \cdots W_m)$$

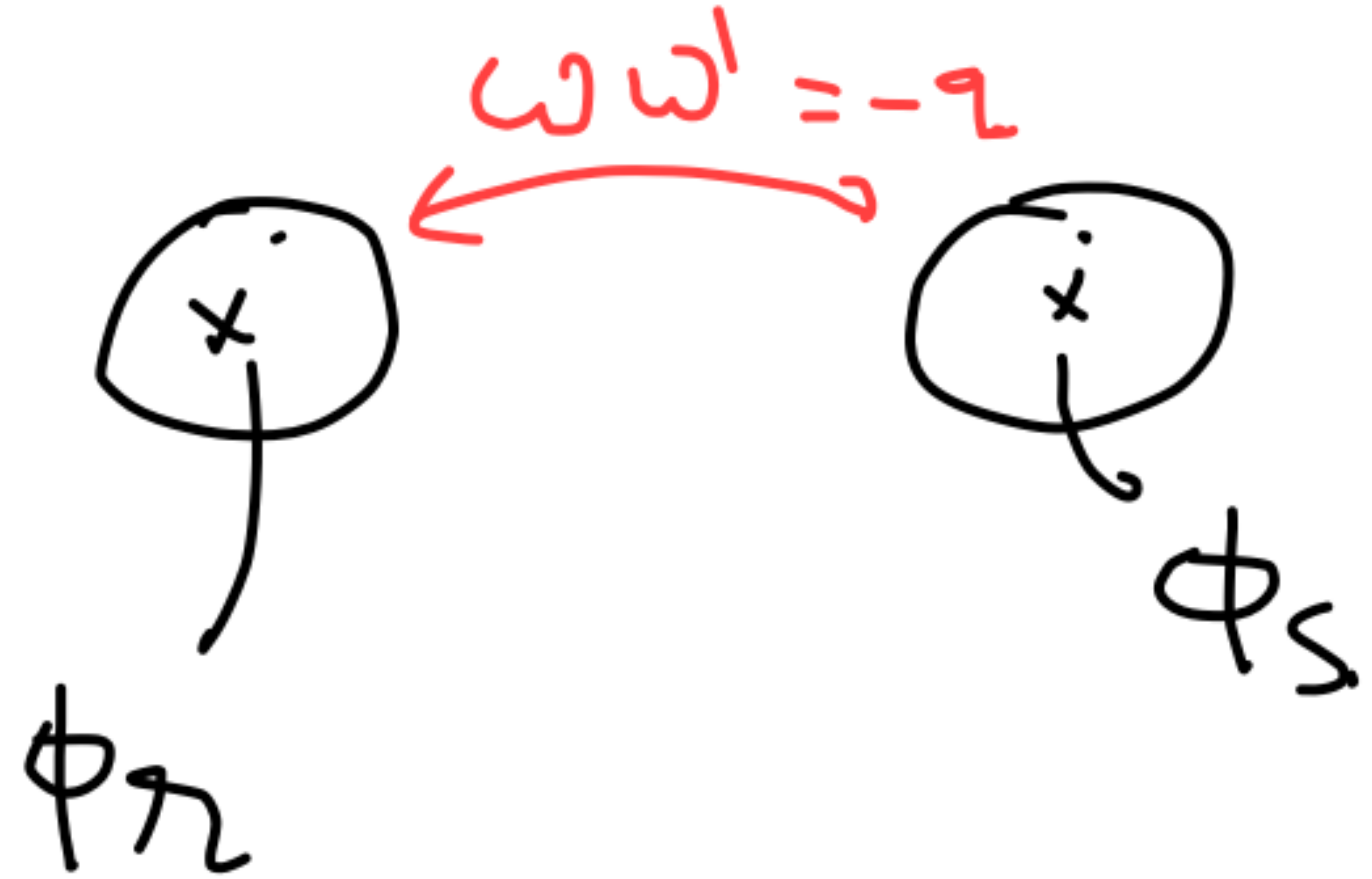
$$A(V_1, \cdots V_m, W_1, \cdots W_n)$$

$$= (g_s)^{3g-3+m+n} \int \Omega_{6g-6+2(m+n)}^{(g,m,n)}(V_1, \cdots V_m, W_1, \cdots W_n)$$

$$S_{g,m,n} \to \text{Section } \# \; \mathcal{P}_{g,m,n}$$

Once we have this, we can now express the amplitude as sum over Feynman diagrams.

Procedure: Same as in bosonic string theory, but with one difference.



$$\Rightarrow \int dq \wedge d\bar{q} \; \langle \phi_r^c | \, b_0 \bar{b}_0 \, | \phi_s^c \rangle \, q^{h_s} \bar{q}^{\bar{h}_s}$$

For NS punctures, ϕ_r, ϕ_s have $p = -1$ ✓

$\Rightarrow$ ϕ_r^c, ϕ_s^c have picture no. -1

For R puncture: ϕ_r, ϕ_s have $p = -\frac{1}{2}$

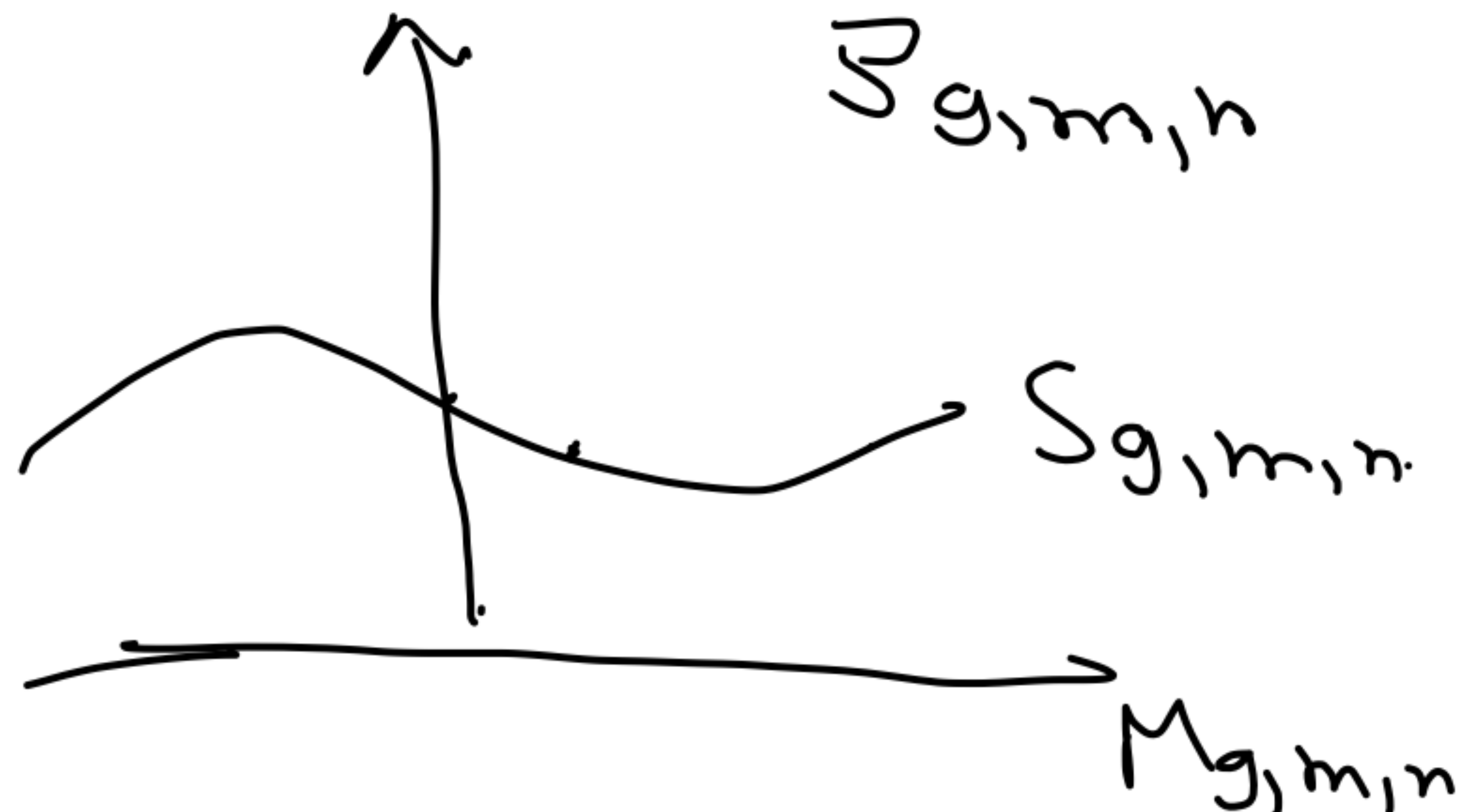
$\Rightarrow$ ϕ_r^c and ϕ_s^c have $p = -\frac{3}{2}$ $\Rightarrow$ $\langle \phi_r^c | b_0 \bar{b}_0 | \phi_s^c \rangle$ will vanish.

We avoid this by inserting one PCO inside $\langle \phi_r^c | \cdots | \phi_s^c \rangle$

We insert:

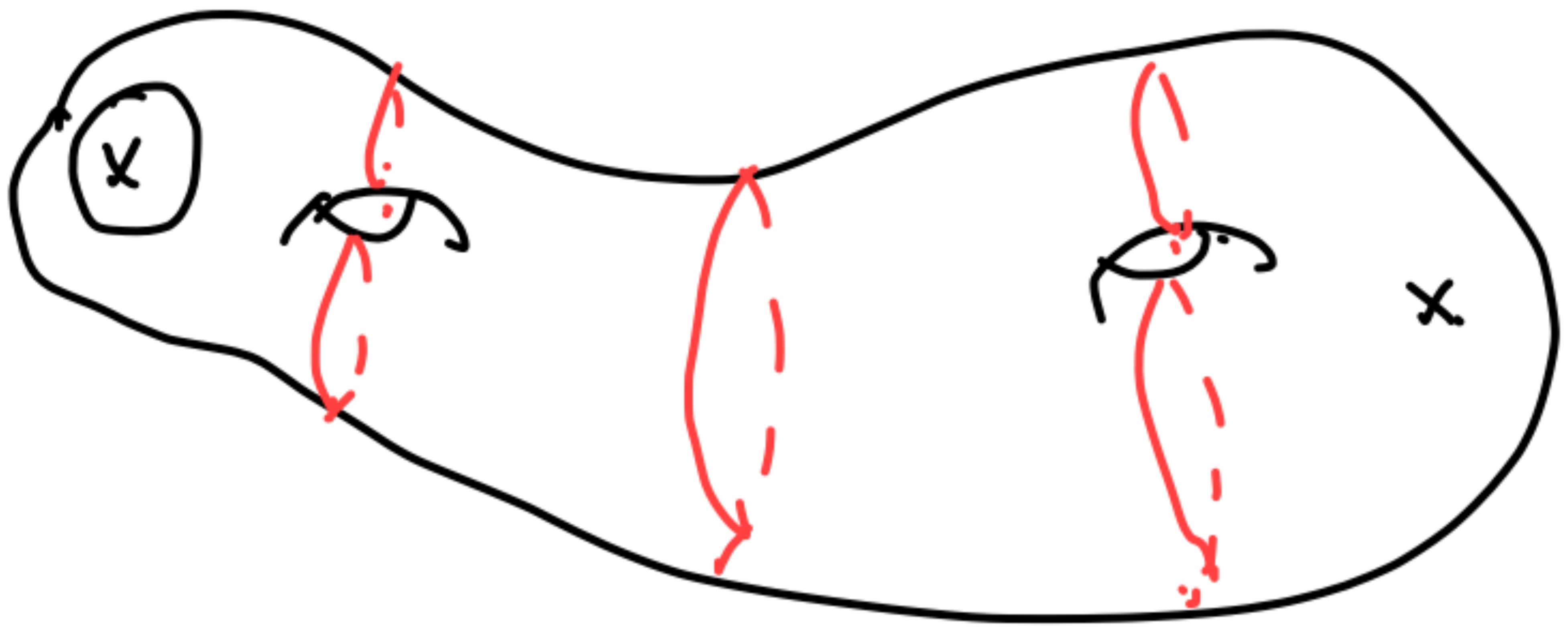
$$X_0 = \oint \frac{d\omega}{\omega} X(\omega)$$

→ Average of PCO insertions on a circle.



$S_{g,m,n}$: specifies local coords & PCO locations

One PCO insertion is average of all possible insertions on $|\omega| = |q|^{1/2}$ circle.



$(3g-3+m+n)$ circles C_s on $S_i \cap S_j$

$\sigma_s = F_s(\tau_s)$, $\boxed{\sigma_s = -\dfrac{q_s}{\tau_s}}$

$\sigma_s = \tau_s + a_s$ on $S_i \cap D_a$

$(3g-3+m+n)$ q_s $\Rightarrow$ $6g-6+2(m+n)$ real parameters.