Black hole temperature in Euclidean formalism

Consider a static solution with metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + g^2(r)d\Sigma^2\,, \tag{1}$$

where Σ is some $D-2$ space and $f(r)$ vanishes at $r = r_0$ at least linearly, $f(r) = f_0(r - r_0) + O(r - r_0)^2$. We assume that $g(r)$ is such that the surface is regular.

• Show that $r = r_0$ is a null surface. To do this, change $r \to r_*$ so that

$$ds^2 = -f(r)(dt^2 - dr_*^2) + g^2(r)d\Sigma^2\,. \tag{2}$$

The (t, r_*) part of the metric is now conformal to 2D Minkowski space. What is then $r = r_0$ mapped into? Is this a null surface?

It follows that $r = r_0$ is a Killing horizon. Now analytically continue $t = -i\tau$.

• Obtain the horizon temperature $T_H = \beta^{-1}$ from the condition that the periodicity $\tau \sim \tau + \beta$ makes the τ, r part of metric regular (*i.e.,* free of conical singularities). To this effect, impose the condition that the ratio of the proper circumferential length of the orbits of the vector ∂_τ, given by $C(r) = \int d\tau\, \|\partial_\tau\| = \beta\sqrt{f(r)}$, to their proper radius $R(r) = \int dr/\sqrt{f(r)}$, in the limit $r \to r_0$, equals 2π, *i.e.,*

$$\lim_{r\to r_0} \frac{C(r)}{R(r)} = \lim_{r\to r_0} \frac{dC(r)}{dR(r)} = 2\pi\,. \tag{3}$$

Show that this implies

$$\beta = \frac{4\pi}{|f'(r_0)|}\,. \tag{4}$$

Observe that we could rescale $t \to \lambda t$ and T_H would rescale too. Fixing this ambiguity is typically done by choosing a particular asymptotic static frame, which fixes the normalization of the asymptotic timelike Killing vector.

By performing a simple change of coordinates, find the form of the temperature for a metric of the form

$$ds^2 = -F(\bar{r})dt^2 + \frac{d\bar{r}^2}{G(\bar{r})} + \bar{r}^2 d\Omega^2\,, \tag{5}$$

with a horizon $\bar{r} = \bar{r}_H$ where $F(\bar{r}_H) = 0$. Is this more or less general than the previous form?

Optional: Show that the Euclidean prescription for the temperature of a general stationary horizon is the same as obtained from $T_H = \kappa/(2\pi)$ where κ is the surface gravity for a Killing horizon generated by a vector k

$$k^\mu \nabla_\mu k_\nu = \kappa k_\nu \tag{6}$$

(i.e., the non-affinity of the Killing generator of the horizon). *Hint:* show first that the surface gravity can be equivalently written as

$$\kappa = \sqrt{\nabla_\mu |k| \nabla^\mu |k|} \tag{7}$$

in the limit where $|k| \to 0$, where $|k|$ is the norm of k.

Black Hole thermodynamics from Euclidean Quantum Gravity

In the semiclassical approximation to the Euclidean Quantum Gravity Path Integral, the thermodynamical partition function and free energy are given by

$$Z[\beta] = e^{-\beta F} \approx e^{-I_E[g^{(\mathrm{cl})}]} \tag{1}$$

where $I_E[g^{(\mathrm{cl})}]$ is the Euclidean action of a classical solution $g^{(\mathrm{cl})}$ with periodic boundary conditions in imaginary time. For geometries that are asymptotically flat or asymptotically Anti-deSitter this action is actually infinite and needs renormalization. A conventional way to do it is by subtracting the action $I_E[g^0]$ of a reference background spacetime g^0 *with the same asymptotics*. We shall use this method to compute the free energy of the Schwarzschild black hole and of a black brane in Anti-deSitter space in five dimensions (of interest in the AdS/CFT correspondence).

a. Schwarzschild black hole thermodynamics

For the Euclidean Schwarzschild solution

$$ds^2 = \left(1 - \frac{2GM}{r}\right) d\tau^2 + \frac{dr^2}{1 - 2GM/r} + r^2 d\Omega_2 \tag{2}$$

the natural background to compare it to is the (Euclidean) Minkowski solution, which can be regarded as the ground state for asymptotically flat spacetimes.

The Euclidean action is

$$I_E = -\frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{g}R - \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^3x\sqrt{h}K\,. \tag{3}$$

For vacuum solutions, $R_{\mu\nu} = 0$, so $R = 0$ and the Einstein-Hilbert term in the gravitational action vanishes. The Gibbons-Hawking-York boundary term can contribute, so

$$I_E[g^{(\mathrm{cl})}] - I_E[g^0] = -\frac{1}{8\pi G}\int_{\partial\mathcal{M}} \sqrt{h}(K - K^0)\,, \tag{4}$$

where the integral is taken at the boundary of spacetime $\partial\mathcal{M}$. One must make sure that the geometries induced on $\partial\mathcal{M}$ by the black hole spacetime and by the Minkowski background are the same. However, the extrinsic curvatures of these boundary geometries are different (since they are embedded in different spaces), and this is what gives rise to a non-zero value of the renormalized action.[1]

• To regularize the calculation, take the boundary hypersurface $\partial\mathcal{M}$ to be at a large but finite constant radius $r = R_b$, and henceforth expand all quantities in powers of $1/R_b$ up to next-to-leading-order. The boundary metrics are

$$\begin{aligned} ds^2(\partial\mathcal{M}) &= h_{\mu\nu}dx^\mu dx^\nu = \left(1 - \frac{2GM}{R_b}\right) d\tau^2 + R_b^2 d\Omega_2\,, \\ ds_0^2(\partial\mathcal{M}) &= h^0_{\mu\nu}dx^\mu dx^\nu = d\tau_0^2 + R_b^2 d\Omega_2\,. \end{aligned} \tag{5}$$

[1] Indeed, a related difference in extrinsic curvatures is the geometric meaning of the mass of an asymptotically flat spacetime.

Since the angular parts of the metrics are already equal, it only remains to make the length of the circles of Euclidean time the same for both geometries,

$$\int_0^{\beta} d\tau \sqrt{h_{\tau\tau}} = \int_0^{\beta_0} d\tau_0 \sqrt{h^0_{\tau\tau}}\,. \tag{6}$$

This fixes the value of the τ-periodicity β_0 of the background, for a given value of β for the black hole.

• Next, compute the integrands in (4), using $\sqrt{h}K = n^\mu \partial_\mu \sqrt{h}$, where n is the outward radial unit normal to the boundary. After performing the integrals in (4), the limit $R_b \to \infty$ should yield a finite result. The free energy, relative to the Minkowski ground state, is $I_E[g^{(\mathrm{cl})}] - I_E[g^0] = \beta F$.

• Expressing F as a function of the temperature, use conventional thermodynamics

$$E = \frac{\partial(\beta F)}{\partial \beta}\,, \qquad S = \left(\beta \frac{\partial}{\partial \beta} - 1\right)(\beta F)\,, \tag{7}$$

to obtain the energy and entropy. Check that these agree with the expected results.

b. AdS black brane thermodynamics

Consider the five-dimensional Euclidean geometry

$$ds^2 = \frac{r^2}{\ell^2}\left(f(r)d\tau^2 + \sum_{i=1}^{3}(dx^i)^2\right) + \frac{\ell^2}{r^2}\frac{dr^2}{f(r)}\,, \qquad f(r) = 1 - \frac{r_H^4}{r^4}\,, \tag{8}$$

which is a solution to the Einstein-Anti-deSitter equations

$$R_{\mu\nu} = -\frac{4}{\ell^2} g_{\mu\nu} \tag{9}$$

derived from the Euclidean action

$$I_E = -\frac{1}{16\pi G}\int_{\mathcal{M}} d^5x\sqrt{g}\left(R + \frac{12}{\ell^2}\right) - \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^4x \sqrt{h}K\,. \tag{10}$$

Here r_H is a constant that normally we would expect to correspond to the mass. The negative cosmological constant is denoted through the 'AdS radius' ℓ. The Lorentzian version of this geometry (with real time $t = -i\tau$) is a black hole geometry with an event horizon at the largest positive real root of $f(r)$, $r = r_H$. The radial direction extends from $r = r_H$ to ∞.

The horizon extends along the three planar directions x^i, and therefore this solution is known as a planar black hole, or black 3-brane. In order to regularize the volume along these directions, we identify them periodically $x^i \sim x^i + V^{1/3}$, in such a way that $\int dx^1 dx^2 dx^3 = V$. Given the translational invariance along the x^i directions, quantities like the total mass (or energy) $M = E$, entropy S, free energy F etc, will be extensive, *i.e.,* proportional to V. Therefore we can just as well talk about energy density $\rho = E/V$, entropy density $s = S/V$, and free energy density $f = F/V$. We want to compute these using the Euclidean formalism explained above.

• First, obtain the temperature of the black brane. You can do this using the results of the previous exercise.

• The free energy is obtained from the Euclidean action (10) for the solution as in (1). Even for finite V, this action is infinite due to the radial integration extending to $r \to \infty$. In order to obtain a finite 'renormalized' result, we will first use the method of "background subtraction", i.e., appropriately subtracting the free energy of a reference background, which we take to be empty AdS space: this is the solution obtained setting $r_H = 0$ (*i.e.*, $f = 1$) in (8). Again, this requires that you introduce a large-radius regulator R_b, and match the two geometries on the surface $r = R_b$; in particular, you must match the length of the Euclidean time circles as in (6).

You must be careful with the bulk terms in (10): the scalar curvature is not zero. Hint: In order to obtain it, you only need to know that these are solutions of the Einstein-AdS equations.

• If you have done the calculation of bulk and boundary terms correctly, you must have obtained that

$$f = -\frac{\pi^3 \ell^3}{16G} T^4 \,, \tag{11}$$

where T is the temperature of the black brane. Using this result, you can compute the energy density and entropy density. Check that the latter is equal to $1/(4G)$ times the 'area density' of the horizon.

• Compute the specific heat of the black brane. Is it positive or negative?

•Now compute the renormalization of the action using the method of *counterterm subtraction.*

For this, we add to the action (10) the counterterms

$$I_c = \frac{1}{8\pi G} \int_{\partial \mathcal{M}} d^4x \sqrt{h} \left(\frac{3}{\ell} + \frac{\ell}{4} \mathcal{R} \right) \,, \tag{12}$$

where $\mathcal{R}$ is the scalar curvature of the boundary metric. Since for the AdS black brane the boundary metric is flat, this term will vanish, but we include it since it is needed for general curved boundaries. The other term is non-zero and divergent, and is chosen to cancel the leading divergences of the EH-GHY action. Applying it to the black brane, you should recover exactly the same result for the action as before.

There are several things to observe about this result (just read, you are not asked to do anything else):

(i) In conventional thermodynamics, the free energy density of a thermal gas gives its pressure as $P = -f$. Therefore we can assign a pressure to this black brane, which satisfies the conventional thermodynamic relation

$$E + PV = TS \,. \tag{13}$$

(ii) Observe that $\rho \propto T^4$, $s \propto T^3$. These are the same relations satisfied by a gas of photons, and in fact, by any scale-invariant thermal system in $3+1$ dimensions. The prefactor of T^4

in ρ is a measure of the number of local degrees of freedom (*e.g.,* photon polarizations) of the system, which is proportional to the 'Stefan-Boltzmann constant'.

For such a gas,

$$P = \frac{\rho}{3}\,, \tag{14}$$

which may be familiar to you as the equation of state for a 'radiation perfect fluid' used in cosmology.

The fact that the thermodynamics of a five-dimensional black brane in AdS takes the same form as the thermodynamics of a conformally-invariant four-dimensional theory (without gravity) is one of the most basic features of the AdS/CFT correspondence. The Stefan-Boltzmann constant computed gravitationally as above is a measure of the microscopic number of local degrees of freedom of the black brane.

You may want to extend this analysis to an arbitrary dimension ≥ 3.

c. AdS black hole thermodynamics

The same theory (10) has black hole solutions with spherical horizons, with metric

$$ds^2 = \left(1 - \frac{\mu}{r^2} + \frac{r^2}{\ell^2}\right) d\tau^2 + \frac{dr^2}{1 - \frac{\mu}{r^2} + \frac{r^2}{\ell^2}} + r^2 d\Omega_3^2\,. \tag{15}$$

Here μ is a parameter for the mass of the black hole (in this exercise you will find how the mass M is related to μ). When $\mu = 0$ the geometry is that of empty AdS_5 in so-called *global coordinates* (whereas, in the previous exercise, when $r_H = 0$ we obtain empty AdS_5 in *Poincaré coordinates*). When $\mu \neq 0$, in the limit $\ell \to \infty$ we recover the Schwarzschild-Tangherlini solution in five dimensions.[2]

When $\mu > 0$ this metric has a black hole horizon at the radius $r = r_+$ where $g_{\tau\tau} = 0$. This is a quartic equation that one can solve explicitly, but it will often be more convenient to solve for μ as

$$\mu = \frac{r_+^2(r_+^2 + \ell^2)}{\ell^2} \tag{16}$$

and then use r_+ as a parameter in the solution instead of μ.

Begin by first determining the temperature of the black hole in terms of r_+. You must find

$$T = \frac{2r_+^2 + \ell^2}{2\pi\ell^2 r_+}\,. \tag{17}$$

Following the steps of the previous exercises, and performing counterterm subtraction, compute the Euclidean action and then the free energy of the solution. You must find that[3]

$$F = -\frac{\pi}{8G}\frac{r_+^2}{\ell^2}\left(r_+^2 - \ell^2\right) + E_0\,, \tag{18}$$

[2]You may also want to verify that if you rescale $r \to \lambda r$, $t \to t/\lambda$, $\mu \to \lambda^4 \mu$, and take $\lambda \to \infty$, then you recover (8) after replacing the large sphere $\lambda^2 d\Omega_3$ (by taking stereographic coordinates) with the flat $\mathbf{R^3}$ metric $\sum_{i=1}^3 (dx^i)^2$. How do you interpret this limit?

[3]The area of the unit S^3 is $2\pi^2$.

where

$$E_0 = \frac{3\pi}{32G}\ell^2 \tag{19}$$

is a vacuum, β-independent contribution. This is interpreted in dual terms as the Casimir energy of the CFT on a spatial S^3.

Derive from here that the energy (mass) of the black hole is

$$M = \frac{3\pi}{8G}\mu + E_0\,, \tag{20}$$

and verify that the entropy reproduces correctly the Bekenstein-Hawking area law.

With these results you can now explore the thermodynamics of this system. You can first observe that for any given temperature above $T_{min} = \sqrt{2}/\pi\ell$ there exist two different black holes, one larger than the other, while below T_{min} there is no black hole phase and only a thermal gas of radiation is possible. The specific heat of the black hole solutions changes between the two black hole phases, from being negative for $r_+/\ell < 1/\sqrt{2}$ (small μ) to being positive for $r_+/\ell > 1/\sqrt{2}$ (large μ). Large black holes inside the 'AdS box' have positive specific heat and therefore are thermodynamically stable (locally).

Going further, you can plot F as a function of β (hint: do a parametric plot, with F and β given in terms of the parameter r_+). Bearing in mind that the free energy of empty AdS$_5$ is E_0, which is β-independent, you will then see that F takes the 'swallow tail' shape characteristic of first order phase transitions. Recall that in the canonical ensemble, a system will tend to minimize its free energy. A first order phase transition happens when the phase of lowest F changes from one configuration to another (the free energy is continuous across the transition, but the entropy is not: latent heat is released or absorbed, which you may compute). In the present system, if we start at low temperature T (large β), the preferred phase is initially empty (thermal) AdS space (with $F = E_0$), but as the temperature is raised (β decreases) there arrives a moment when the preferred phase is the Schwarzschild-AdS solution (with $F < E_0$).

This phase transition, which occurs when $r_+ = \ell$ and $T = T_{HP} = 3/(2\pi\ell)$, is known as the *Hawking-Page transition*. It is a very important phenomenon in AdS/CFT, where it is interpreted as a confinement/deconfinement transition in the dual quantum field theory.

Observe that $T_{HP} > T_{min}$, so there are some 'large' black holes (with $1/\sqrt{2} < r_+/\ell < 1$) that are locally thermodynamically stable (i.e., have positive specific heat) but are not globally thermodynamically stable since there is another phase (thermal AdS) with lower free energy. Small black holes are never thermodynamically preferred.

Normalizability in AdS

Consider a spacetime that is asymptotic to AdS_{d+1} (with unit radius) in the form

$$ds^2 \to \frac{dz^2 - dt^2 + d\mathbf{x}_{d-1}^2}{z^2} + \dots \tag{1}$$

$$= -r^2 dt^2 + \frac{dr^2}{r^2} + r^2 d\mathbf{x}_{d-1}^2 + \dots , \tag{2}$$

the two forms of the geometry being related by $z = 1/r$; we can equivalently work with one or the other, with asymptotic infinity being at $z \to 0$ or $r \to \infty$. We have taken the boundary to be flat, but it will become clear that the analysis remains valid if it is spatially spherical.

Take a massive scalar field that satisfies the Klein-Gordon equation

$$\Box\Phi - m^2\Phi = 0\,. \tag{3}$$

The Klein-Gordon product for two such fields is

$$(\Psi, \Phi) = \int_\Sigma d^d x \sqrt{h}\, n^\mu \left(\Psi^* \partial_\mu \Phi - \Phi^* \partial_\mu \Psi\right) , \tag{4}$$

where Σ is a spacelike section of the spacetime, n^μ its normal, and $\sqrt{h}$ is the volume element in it. This product remains constant in time for solutions of the Klein-Gordon equation—if you have not seen this before, you can easily verify it (the product is the space integral of the Wronskian)—and it defines a conserved norm for these fields.

In (1), take Σ to be a surface at constant t, so that, asymptotically,

$$n^t = \frac{1}{r}\,, \qquad \sqrt{h} = r^{d-2} \tag{5}$$

and for simplicity consider fields of the form

$$\Phi \sim e^{-i\omega t}\phi(r) \tag{6}$$

(we suppress the spatial dependence on $\mathbf{x}$ just to keep notation lighter).

• Find that

$$(\Phi, \Phi) \sim \int dr\, r^{d-3}\, \phi^2(r)\,. \tag{7}$$

Then, if asymptotically we have

$$\phi(r) \sim \frac{1}{r^\Delta} \tag{8}$$

show that normalizability requires

$$\Delta \geq \frac{d}{2} - 1\,. \tag{9}$$

This is the same as the unitarity bound for scalar operators of a CFT_d.

The asymptotic solution for a massive field in AdS has two independent modes, with leading terms of the form

$$\phi \sim \frac{\phi^{(0)}}{r^{d-\Delta}} + \cdots + \frac{\phi^{(d)}}{r^{\Delta}} + \ldots \tag{10}$$

where

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2} \tag{11}$$

(if you have never obtained this result before, you should do it now).

Conclude that

- The mode $\phi^{(d)}$ is always normalizable
- The mode $\phi^{(0)}$ is normalizable only if the mass lies in the range

$$m_{BF}^2 \leq m^2 \leq 1 + m_{BF}^2 \tag{12}$$

  where $m_{BF}^2 = -d^2/4$ is the mass of the Breitenlohner-Freedman bound.